Godunov Schemes with Augmented Quantities

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Abstract: We develop a new type of Godunov scheme for conservation laws. To start with, we first consider the 1D scalar conservation law

$$u_t + f(u)_x = 0,$$

where $f(u)$ can be either linear or nonlinear. The solution satisfies the entropy condition

$$U(u)_t + F(u)_x \leq 0,$$

for any nonlinear convex $U(u)$ and $F(u) = \int U'(u)f'(u)du$. The new scheme computes both the cell-average approximations to u and an $U(u)$,

$$u_j^n \simeq \frac{1}{\Delta x}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u(x,t_n)dx,\ U_j^n \simeq \frac{1}{\Delta x}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} U(u(x,t_n))dx,$$

where U_j^n is an augmented quantity. The novelty of the scheme is in its reconstruction. The solution is reconstructed in each cell as a step function,

$$u_R^n(x) = u_j^n \begin{cases} -h_j^n, & x_{j-\frac{1}{2}} < x \leq x_j, \\ +h_j^n, & x_j < x \leq x_{j+\frac{1}{2}}. \end{cases}$$

with h_j^n solved from

$$\frac{1}{\triangle x}\int_{x_{j+\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u_R^n(x)dx = U_j^n.$$

The scheme satisfies the entropy condition, and the numerical experiments show that it has second-order accuracy. The scheme is extended to the Euler system of fluid dynamics, in which besides the entropy the velocity and pressure are also chosen to be approximated by augmented quantities. The scheme for the Euler system satisfies the entropy condition, and the numerical examples show that it has second-order accuracy and performs better than a second-order ENO scheme.

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