Existence and uniqueness results for variational inequalities

Lecture 1

Basic results

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Outline of lecture 1

I- Why V.I.?

II- Set-valued or not?

  a- The monotone case

  b- The quasimonotone case

II- First existence results

  a- The linear case

  b- The finite dimensional case

Notations

- X a Banach space
- X^* its topological dual (w^*-top.)
- $\langle \cdot, \cdot \rangle$ the duality product

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Stampacchia variational inequality (strong):

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Find $\bar{x} \in C$ such that there exists $\bar{x}^* \in T(\bar{x})$ for which

$$\langle \bar{x}^*, y - \bar{x} \rangle \geq 0, \qquad \forall\, y \in C.$$

Notation : $S(T, C)$ set of solutions ($\subset C$).

I

Why variational inequalities?

First motivation = Optimality conditions

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ and $C \subseteq \operatorname{dom} f$ be a convex subset.

$$(P) \qquad \text{find } \bar{x} \in C \ : \ f(\bar{x}) = \inf_{x \in C} f(x)$$

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Necessary condition: f is lsc + ...

if $\bar{x}$ is a solution of (P) then

$$\bar{x} \in S(\partial f(\bar{x}), C).$$

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Perfect case: f convex

$f : X \to \mathbb{R} \cup \{+\infty\}$ a proper convex function

C a nonempty convex subset of X, $\bar{x} \in C$ + C.Q.

Then

$$f(\bar{x}) = \inf_{x \in C} f(x) \quad \Longleftrightarrow \quad \bar{x} \in S(\partial f, C)$$

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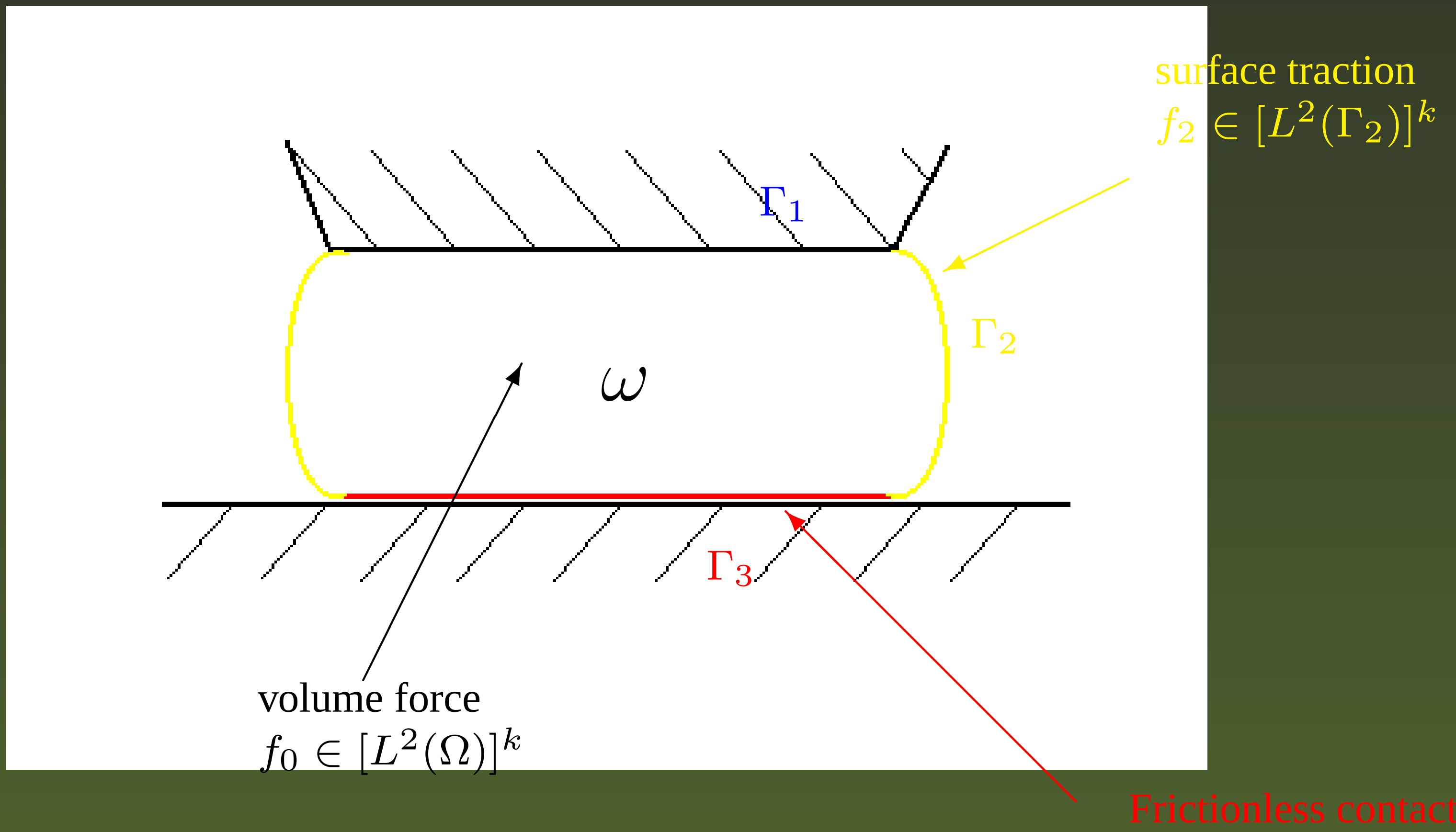
What about quasiconvex case? see the third lecture

Signorini's frictionless contact problem



Notations

Functional spaces:

$$\mathbb{S}^k = \{\sigma = (\sigma_{ij})_{ij} \in \mathbb{R}^{k\times k} \ : \ \sigma_{ij} = \sigma_{ji}\} = \mathbb{R}_s^{k\times k}$$

$$W = \{v \in H^1(\Omega)^k \ : \ v = 0 \text{ sur } \Gamma_1\}$$

$$Q = \{q = (q_{ij}) \in L^2(\Omega)^{k\times k} \ : \ q_{ij} = q_{ji},\ 1 \leq i,j \leq k\} = L^2(\Omega)_s^{k\times k}$$

$$W_2 = \{v \in W \ : \ v_\nu \leq 0 \text{ a.e. on } \Gamma_3\}$$

Deformation operator: $\varepsilon : H^1(\Omega)^k \longrightarrow Q$

$$\varepsilon_{ij}(u) = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right], \qquad 1 \leq i,j \leq k.$$

Scalar product: $\langle p, q\rangle_Q = \int_\Omega p_{ij}(x) q_{ij}(x)\,dx$ et $\langle u, v\rangle_W = \langle \varepsilon(u), \varepsilon(v)\rangle_Q$

Elasticity operator: $\mathcal{F} : \Omega \times \mathbb{S}^k \longrightarrow \mathbb{S}^k$

Stress function : $\sigma : H^1(\Omega)^k \longrightarrow Q$ defined by

$$\begin{array}{rccc} \sigma(u) : & \Omega & \longrightarrow & \mathbb{S}^k \\ & x & \longmapsto & \mathcal{F}(x, \varepsilon(u)(x)). \end{array}$$

Formulations of the problem:

Find a displacement field $u : \Omega \longrightarrow \mathbb{R}$ such that

$$\begin{array}{rll} -\text{Div}\,\sigma(u) = f_0 & \text{on} & \Omega \\ u = 0 & \text{on} & \Gamma_1 \\ \sigma(u)\nu = f_2 & \text{on} & \Gamma_2 \\ u_\nu \leq 0,\ \sigma(u)_\nu \leq 0,\ \sigma(u)_\nu u_\nu = 0,\ \sigma(u)_\tau = 0 & \text{on} & \Gamma_3 \end{array}$$

Variational formulation:

Find $u \in W_2$ such that

$$\langle \sigma(u), \varepsilon(v) - \varepsilon(u) \rangle_Q \geq \langle f, v - u \rangle_W , \quad \forall\, v \in W_2$$

where f is an element of W defined by

$$\langle f, v \rangle_W = \int_\Omega f_0.v\, dx + \int_{\Gamma_2} f_2.v\, da.$$

II - Set-valued or not?

Stampacchia variational inequality (strong):

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Two particular cases:

a- The monotone case

b- The quasimonotone case

II - a

The monotone case

Let $T : X \to 2^{X^*}$ be a map

T is monotone iff $\forall\, x, y \in X, \forall\, x^* \in T(x)$ and $\forall\, y^* \in T(x)$

$$\langle y^* - x^*, y - x \rangle \geq 0.$$

Dense single-valuedness results

Theorem 1 (Zarantonello 1973)

Let X be a separable Banach space and let $T : X \to 2^{X^*}$ be a monotone operator. Then

- *the set of all $x \in \operatorname{dom}(T)$ for which $T(x)$ is not single-valued has empty interior;*
- *if X is finite-dimensional then this set has Lebesgue measure zero.*

Dense single-valuedness results

Theorem 2 (M. Lassonde '09)
Let $T : Z \to 2^{X^*}$ be minimal w -cusco from a Baire space Z into the dual of a Banach space X. If there is a subset $C \subset X^*$ with the $w*$ -RNP such that the set $\{z \in Z : T(z) \cap C \neq \emptyset\}$ is dense in Z, then there is a dense G_δ subset D of Z such that, at each point of D, T is single-valued and upper semicontinuous.

- Maximal monotone operators $T : Z \to 2^{X^*}$ with nonempty values, are typical examples of minimal w -cusco mappings
- A nonempty bounded subset A of the dual space X^* is said to be $w*$ -dentable provided for every $\varepsilon > 0$ there exists a w^* -open half-space V in X^* such that $A \cap V \neq \emptyset$ and $diam(A \cap V) < \varepsilon$.
- A subset A of X^* is said to have the $w*$ Radon-Nikodim Property provided every nonempty bounded subset of A is w^*-dentable.

Pointwise Single-valuedness

Theorem 3 (Kenderov 1975)
Let X be a Banach space and $T : X \to 2^{X^*}$ be a monotone set-valued map.
If T is lower semicontinuous at a point x_0,
then T is single-valued at x_0.

T is lsc at x_0 if, for every open V such that $V \cap T(x_0) \neq \emptyset$, there exists a neigh. U of x_0 such that $V \cap T(x) \neq \emptyset$, for any $x \in U$.

Local Single-valuedness

Theorem 4 (Dontchev and Hager 1994)
Let $T : X \to 2^{X^*}$ *be a monotone set-valued map. If* T *is Lipschitz-like around* $(x_0, y) \in \operatorname{gr} T$,
then T *is single-valued in a neigh. of* x_0.

Local Single-valuedness

Theorem 5 (Dontchev and Hager 1994)
Let $T : X \to 2^{X^*}$ be a monotone set-valued map. If T is Lipschitz-like around $(x_0, y) \in \operatorname{gr} T$, then T is single-valued in a neigh. of x_0.

- $T : X \to 2^Y$ is *Lipschitz-like around* $(x, y) \in \operatorname{gr} T$ if it exist a neighb. U of x, a neighb. V of y and $l > 0$ such that

$$T(u) \cap V \subset T(u') + l\|u' - u\|\overline{B}_Y(0,1), \quad \forall\, u, u' \in U$$

Local Single-valuedness

Theorem 6 (Dontchev and Hager 1994)
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  - also called *Aubin property* or *pseudo-Lipschitzianity*.

Local Single-valuedness

Theorem 7 (Dontchev and Hager 1994)
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- $\Rightarrow T$ is nonempty valued on U.

Local Single-valuedness

Theorem 8 (Dontchev and Hager 1994)
Let $T : X \to 2^{X^*}$ *be a monotone set-valued map. If* T *is Lipschitz-like around* $(x_0, y) \in \operatorname{gr} T$, *then* T *is single-valued in a neigh. of* x_0.

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- If T is single-valued

  T is Lipschitz-like around $(x, T(x)) \Leftrightarrow T$ is loc. Lipschitz at x

II - b

The quasimonotone case

II - b

The quasimonotone case

or What about the quasimonotone case?

Quasimonotonicity

Let $T : X \to 2^{X^*}$ be a set-valued map

- T is quasimonotone iff

$$\exists\, x^* \in T(x) \; : \; \langle x^*, y - x\rangle > 0 \Rightarrow \langle y^*, y - x\rangle \geq 0, \; \forall\, y^* \in T(y).$$

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Examples:

A function $f : X \to \mathbb{R} \cup \{+\infty\}$ is said to be *quasiconvex* on K if, for all x, $y \in K$ and all $t \in [0, 1]$

$$f(tx + (1 - t)y) \leq \max\{f(x), f(y)\}.$$

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$$S_\lambda = \{x \in X\ :\ f(x) \leq \lambda\} \text{ is convex.}$$

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- f differentiable

  f is quasiconvex iff f' is quasimonotone

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- f is quasiconvex iff ∂f is quasimonotone
- the normal operator $N_f : X \to 2^{X^*}$ defined by

$$x \quad \mapsto \quad N_f(x) = N(S_{f(x)}, x)$$

is quasimonotone.

Links with other monotonicities

- T is monotone iff $\forall\, x, y \in X, \forall\, x^* \in T(x)$ and $\forall\, y^* \in T(x)$
$$\langle y^* - x^*, y - x\rangle \geq 0.$$
- T is pseudomonotone iff

$$\exists\, x^* \in T(x) \ : \ \langle x^*, y - x\rangle \geq 0 \Rightarrow \langle y^*, y - x\rangle \geq 0,\ \forall\, y^* \in T(y).$$

- T is quasimonotone iff

$$\exists\, x^* \in T(x) \ : \ \langle x^*, y - x\rangle > 0 \Rightarrow \langle y^*, y - x\rangle \geq 0,\ \forall\, y^* \in T(y).$$

monotone

$\Downarrow$

pseudomonotone

$\Downarrow$

quasimonotone

No hope in the quasimonotone case

No similar result (single-valued property) can be obtained in the quasimonotone case!!.

Consider $T : \mathbb{R} \to 2^{\mathbb{R}}$ defined by

$$T(x) = \mathbb{R}_{++}.$$

then

- T is pseudomonotone
- T is Lipschitz-like at each point of its graph

  (U, V any neigh., l any positive real)
- T is multivalued!!!!

Definition 9 *A set-valued map $T : K \to 2^{X^*}$ is said to be*

- single-directional *at $x \in \operatorname{dom} T$ if, $T(x) \subseteq \mathbb{R}_+\{x^*\}$ for some $x^* \in T(x)$.*
- locally single-directional *at $x \in \operatorname{dom} T$ if it exists a neigh. U of x such that, for any $x' \in U$, $T(x')$ is single-directional at x'.*
- *strictly single-directional* at x if $T(x) \subseteq]0, +\infty[\{x^*\}$ for some $x^* \neq 0$.
- *locally strictly single-directional* at x if T is strictly single-directional at any point of a neigh. of x

Quasimonotone lsc case

Proposition 10 *Let* $T : X \to 2^{X^*}$ *be a set-valued map, lower semicontinuous at* $x \in X$*. Then*

i) *if* T *is quasimonotone, then* T *is single-directional at* x*;*

ii) *if* T *is pseudomonotone, then* T *is strictly single-directional or trivial at* x*.*

Recovering Kenderov'73 (monotone lsc)

Proposition 11 (D.A., J.-N. Corvellec & M. Lassonde '94)
A map $T : X \to 2^{X^*}$ is monotone if and only if, for any $\alpha^* \in X^*$, the map $T + \{\alpha^*\}$ is quasimonotone

Recovering Kenderov'73 (monotone lsc)

Proposition 13 (D.A., J.-N. Corvellec & M. Lassonde '94) *A map $T : X \to 2^{X^*}$ is monotone if and only if, for any $\alpha^* \in X^*$, the map $T + \{\alpha^*\}$ is quasimonotone*

Proposition 14 *Let $T : X \to 2^{X^*}$ be a set-valued map and x be a point of its domain* dom T *be such that, <u>for any $\alpha^* \in X^*$</u>, the map $T + \alpha^*$ is single-directional at x, then T is single-valued at x.*

Recovering Kenderov'73 (monotone lsc)

Hyp : T is monotone and lsc at x.

$\Rightarrow \forall\, \alpha^* \in X^*, T + \{\alpha^*\}$ is quasimonotone lsc at x

$\Rightarrow \forall\, \alpha^* \in X^*, T + \{\alpha^*\}$ is single-directional at x

$\Rightarrow T$ is single-valued at x.

Recovering Kenderov'73 (monotone lsc)

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Theorem 16 (Kenderov 1975)
Let X be a Banach space and $T : X \to 2^{X^*}$ be a monotone set-valued map.
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then T is single-valued at x_0.