

CIMPA-UNESCO-VIETNAM School

Variational Inequalities and Related Problems

Introduction to Variational Inequalities and Basic Tools

by Marc Lassonde

Université Antilles-Guyane

Institute of Mathematics, Hanoi

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BASIC INTERSECTION THEOREMS

Sperner (1928): In \mathbb{R}^n with unit vectors $\{e_i \mid i \in I\}$, let n closed sets $\{A_i \mid i \in I\}$ satisfy:

The union of the n sets contains the simplex $\text{conv}\{e_i \mid i \in I\}$ and each set A_i contains the \hat{i} -face $\text{conv}\{e_j \mid j \in I, j \neq i\}$.

Then all the sets have a point in common.

Knaster-Kuratowski-Mazurkiewicz (1929): In \mathbb{R}^n with unit vectors $\{e_i \mid i \in I\}$, let n closed sets $\{A_i \mid i \in I\}$ satisfy:

For every $J \subset I$ the J -face $\text{conv}\{e_j \mid j \in J\}$ is contained in the union $\bigcup\{A_j \mid j \in J\}$.

Then all the sets have a point in common.

Klee (1951): In a topological vector space, let n closed convex sets satisfy:

The union of the n sets is convex and the intersection of every $n - 1$ of them is nonempty.

Then all the sets have a point in common.

BASIC INTERSECTION THEOREMS REVISITED

(replacing \mathbb{R}^n with unit vectors $\{e_i \mid i \in I\}$ by a TVS with an arbitrary family of (not necessarily distinct) points $\{x_i \mid i \in I\}$)

Basic conditions implying that n sets have a point in common:

Sperner/Aleksandrov-Pasynkov (1957): *In a TVS, n closed sets $\{A_i \mid i \in I\}$ with n points $\{x_i \mid i \in I\}$ such that:*

The union of the n sets contains the convex hull $\text{conv}\{x_i \mid i \in I\}$ and each set A_i contains the convex hull $\text{conv}\{x_j \mid j \in I, j \neq i\}$.

KKM/Ky Fan (1961): *In a TVS, n closed sets $\{A_i \mid i \in I\}$ with n points $\{x_i \mid i \in I\}$ such that:*

For every $J \subset I$ the convex hull $\text{conv}\{x_j \mid j \in J\}$ is contained in $\bigcup\{A_j \mid j \in J\}$.

Klee (1951): *In a TVS, n closed convex sets with:*

The union of the n sets is convex and the intersection of every $n - 1$ of them is nonempty.

COMMENTS

- Passage from the simplex version to the TVS version is direct
- $\text{KKM} \Rightarrow \text{Sperner} \Rightarrow \text{Klee}$
- For convex sets, all the results are equivalent:
 $\text{KKM for convex sets} \Leftrightarrow \text{Sperner for convex sets} \Leftrightarrow \text{Klee}$
- For convex sets, proofs are easy
- For convex sets, KKM condition is equiv. to nonempty intersection:

John (2001): In a TVS, let n closed convex sets $\{A_i \mid i \in I\}$ with n points $\{x_i \mid i \in I\}$. TFAE:

- (1) For every $J \subset I$, $\text{conv}\{x_j \mid j \in J\} \subset \bigcup\{A_j \mid j \in J\}$;
- (2) For every $J \subset I$, $\text{conv}\{x_j \mid j \in J\} \cap \bigcap\{A_j \mid j \in J\} \neq \emptyset$.

- For arbitrary (closed) sets, no elementary proof of KKM Theorem is known; original proof relies on Sperner combinatorial lemma
- All the results hold for open sets as well

APPLICATIONS OF THE CONVEX VERSIONS: MINIMAX (IN)EQUALITIES

Ky Fan minimax inequality (1972) (special case): *Let K be a nonempty convex compact set in a TVS and let $\varphi : K \times K \rightarrow \mathbb{R}$ such that:*

- (1) *For each $x \in K$, the function $y \mapsto \varphi(x, y)$ is quasi-concave,*
- (2) *For each $y \in K$, the function $x \mapsto \varphi(x, y)$ is quasi-convex lsc,*
- (3) *For each $x \in K$, $\varphi(x, x) \leq 0$.*

Then, there exists $\bar{x} \in K$ such that $\varphi(\bar{x}, y) \leq 0$ for all $y \in K$, i.e.,

$$\min_{x \in K} \sup_{y \in K} \varphi(x, y) \leq 0.$$

Sion minimax equality (1958): *Let C and D be two nonempty convex sets in TVS', one of them being compact, and let $f : C \times D \rightarrow \mathbb{R}$ such that $f(\cdot, y)$ is quasi-convex lsc and $f(x, \cdot)$ is quasi-concave usc.*
Then:

$$\inf_{x \in C} \sup_{y \in D} f(x, y) = \sup_{y \in D} \inf_{x \in C} f(x, y).$$

PROOF OF SION MINIMAX EQUALITY

Assume D is compact. We always have:

$$\alpha = \max_{y \in D} \inf_{x \in C} f(x, y) \leq \beta = \inf_{x \in C} \max_{y \in D} f(x, y).$$

Suppose for a contradiction there is $\lambda \in \mathbb{R}$ s.t. $\alpha < \lambda < \beta$. Consider

$$F(x) = \{y \in D \mid f(x, y) > \lambda\}, \quad F^*(y) = \{x \in C \mid f(x, y) \leq \lambda\}.$$

• $F(x)$ is convex and $\overline{F(x)} \subset \{y \in D : f(x, y) \geq \lambda\}$. Moreover, $\alpha < \lambda \Rightarrow \forall y \in D \exists x \in C : y \notin \overline{F(x)}$, that is: $\bigcap \{\overline{F(x)} \mid x \in C\} = \emptyset$, hence, since D is compact, there is a finite $A \subset C$ s.t. $\bigcap \{\overline{F(x)} \mid x \in A\} = \emptyset$.

• Similarly, the sets $F^*(y)$ are convex and closed, and $\bigcap \{F^*(y) \mid y \in D\} = \emptyset$. A fortiori, $\bigcap \{F^*(y) \cap [A] \mid y \in D\} = \emptyset$, hence, since $[A]$ is compact, there is a finite $B \subset D$ s.t. $\bigcap \{F^*(y) \cap [A] \mid y \in B\} = \emptyset$.

Thus, we have found finite subsets $A \subset C$ and $B \subset D$ s.t.

(\star) $\bigcap \{\overline{F(x)} \cap [B] \mid x \in A\} = \emptyset$ and ($\star\star$) $\bigcap \{F^*(y) \cap [A] \mid y \in B\} = \emptyset$,
and we may assume that A and B are minimal w.r.t. (\star) and ($\star\star$).

Each set in the finite family $\mathcal{F} = \{\overline{F(x)} \cap [B] \mid x \in A\}$ is convex and closed, and every proper subfamily of \mathcal{F} has a nonempty intersection. Hence, by Klee theorem, the union $\bigcup \{\overline{F(x)} \cap [B] \mid x \in A\}$ of the sets in \mathcal{F} is not convex, so $[B]$ is not contained in $\bigcup \{F(x) \mid x \in A\}$. Thus, there is $y_0 \in [B]$ s.t. $y_0 \notin F(x)$ for all $x \in A$, that is, $A \subset F^*(y_0)$.

Proceeding the same way with the family $\{F^*(y) \cap [A] \mid y \in B\}$, we find $x_0 \in [A]$ s.t. $x_0 \notin F^*(y)$ for all $y \in B$, that is $B \subset F(x_0)$.

Here is the contradiction. On the one hand, the convex set $F^*(y_0)$ contains A , hence also $x_0 \in [A]$; on the other hand, the convex set $F(x_0)$ contains B , hence also $y_0 \in [B]$. Thus, we would have $y_0 \in F(x_0)$ and $x_0 \in F^*(y_0)$, which is impossible.

APPLICATIONS OF THE CONVEX VERSIONS: VARIATIONAL INEQUALITIES

Definitions. Let X be a TVS, X^* its topological dual and $T : X \rightrightarrows X^*$ a set-valued operator. We say that T is *monotone* if

$$\forall (x, x^*) \in gr(T), \forall (y, y^*) \in gr(T) : \langle y^* - x^*, y - x \rangle \geq 0$$

and *quasi-monotone* if

$$\forall (x, x^*) \in gr(T), \forall (y, y^*) \in gr(T) : \langle x^*, y - x \rangle > 0 \Rightarrow \langle y^*, y - x \rangle \geq 0.$$

Minty (1962)-Debrunner-Flor (1964) (special case): Let $K \subset X$ be a nonempty convex compact set in a TVS and let $T : X \rightrightarrows X^*$ be monotone with $\text{dom} T \subset K$. Then, there exists $\bar{y} \in K$ such that

$$\langle x^*, x - \bar{y} \rangle \geq 0, \quad \forall (x, x^*) \in gr(T).$$

Stampacchia (1964)-Aussel-Hadjisavvas (2004): Let $K \subset X$ be a nonempty convex compact set in a TVS and let $T : X \rightrightarrows X^*$ be quasi-monotone with $\text{dom} T = K$. Moreover, assume that T is hemicontinuous from X into (X^*, w^*) with convex w^* -compact values. Then, there exists $\bar{y} \in K$ et $\bar{y}^* \in T\bar{y}$ such that

$$\langle \bar{y}^*, x - \bar{y} \rangle \geq 0, \quad \forall x \in K.$$

COMPLEMENTS : MONOTONE, KKM, AND MINTY VI

Given $T : X \rightrightarrows X^*$, define $\Gamma_T : \text{dom } T \rightrightarrows X$ by

$$\Gamma_T(x) = \{y \in X \mid \langle x^*, x - y \rangle \geq 0, \forall x^* \in T(x)\}.$$

Say that $\Gamma : D \subset X \rightrightarrows X$ is *KKM* provided

$$\text{For every finite } A \subset D, \text{ conv } A \subset \bigcup \{ \Gamma(x) \mid x \in A \}.$$

Fact 1: T monotone $\Rightarrow \Gamma_T$ KKM $\Rightarrow T$ quasi-monotone.

Fact 2: T monotone $\Leftrightarrow \forall y^* \in X^*, T - y^*$ quasi-monotone.

Theorem: Let X be a LCTVS and $T : X \rightrightarrows X^*$. TFAE:

- (1) T monotone;
- (2) $\forall y^* \in X^*, \Gamma_{T-y^*}$ KKM;
- (3) $\forall K \subset \text{dom } T$ nonempty convex compact, $\forall y^* \in X^* \exists \bar{y} \in K :$
 $\langle x^* - y^*, x - \bar{y} \rangle \geq 0, \quad \forall (x, x^*) \in \text{gr}(T) \cap K \times X^*.$

EQUIVALENT FORMULATIONS OF THE KKM THEOREM: FIXED POINT AND MINIMAX

Fan (1961)-Browder (1968): Let K be a nonempty convex compact set in a TVS and let $\psi : K \rightrightarrows K$ a set-valued map verifying:

- (1) For each $x \in K$, the set $\psi(x)$ is convex,
- (2) For each $y \in K$, the set $\psi^{-1}(y)$ is open in K .

Then, there exists $\bar{x} \in K$ such that $\psi(\bar{x}) = \emptyset$ or $\bar{x} \in \psi(\bar{x})$.

Ky Fan minimax inequality (1972): Let K be a nonempty convex compact set in a TVS and let $\varphi : K \times K \rightarrow \mathbb{R}$ such that:

- (1) For each $x \in K$, the function $y \mapsto \varphi(x, y)$ is quasi-concave,
- (2) For each $y \in K$, the function $x \mapsto \varphi(x, y)$ is lsc,
- (3) For each $x \in K$, $\varphi(x, x) \leq 0$.

Then, there exists $\bar{x} \in K$ such that $\varphi(\bar{x}, y) \leq 0$ for all $y \in K$, i.e.,

$$\min_{x \in K} \sup_{y \in K} \varphi(x, y) \leq 0.$$

Brouwer (1909): Every continuous self-map $f : K \rightarrow K$ of a nonempty convex compact subset of \mathbb{R}^n possesses a fixed point.

EQUIVALENT FORMULATIONS OF THE KKM THEOREM: VARIATIONAL INEQUALITIES

Debrunner-Flor (1964)-Browder (1968): *Let $K \subset X$ be a nonempty convex compact set in a TVS, $T : X \rightrightarrows X^*$ be monotone with $\text{dom} T \subset K$ and $u : K \rightarrow X^*$ be continuous. Then, there exists $\bar{y} \in K$ such that*

$$\langle u(\bar{y}) + x^*, x - \bar{y} \rangle \geq 0, \quad \forall (x, x^*) \in \text{gr}(T).$$

Hartman-Stampacchia (1966)-Browder (1968): *Let $K \subset X$ be a nonempty convex compact set in a TVS and let $T : X \rightrightarrows X^*$ be monotone with $\text{dom} T = K$ and $u : K \rightarrow X^*$ be continuous. Moreover, assume that T is hemicontinuous from X into (X^*, w^*) with convex w^* -compact values. Then, there exists $\bar{y} \in K$ et $\bar{y}^* \in T\bar{y}$ such that*

$$\langle u(\bar{y}) + \bar{y}^*, x - \bar{y} \rangle \geq 0, \quad \forall x \in K.$$

EQUIVALENT FORMULATIONS OF THE KKM THEOREM: TOPOLOGICAL KLEE-TYPE INTERSECTION THEOREM

Definition. A topological space X is said to be *contractible* provided there exists a continuous map $f : X \times [0, 1] \rightarrow X$ such that $f(1, \cdot)$ is the identity map and $f(0, \cdot)$ is the constant map.

Horvath-Lassonde (1997): *In a TVS, let n closed convex sets such that:*

The union of the n sets is contractible and the intersection of every $n - 1$ of them is nonempty.

Then all the sets have a point in common.

Proof of the equivalence.

Topological Klee-type Theorem

\Rightarrow *The n -sphere is not contractible*

\Rightarrow Brouwer's Fixed Point Theorem

\Rightarrow KKM Theorem

\Rightarrow Topological Klee-type Theorem !!

COMPETITIVE FAN-BROWDER FIXED POINT THEOREM

Notation. Let I be an arbitrary set of indices. Given a family $\{X_i \mid i \in I\}$ of topological spaces, we let $X = \prod_{i \in I} X_i$ be the topological product of the spaces and $\pi_i : X \rightarrow X_i$ be the projection of X onto X_i . We simply write $\pi_i(x) = x_i$.

Toussaint (1984): For each $i \in I$, assume:

- (1) X_i is a nonempty convex compact set in a TVS;
- (2) $\varphi_i : X \rightrightarrows X_i$ has convex values and open fibers.

Then, the following alternative holds:

- (A) There exist $i \in I$ and $\bar{x} \in X$ such that $\bar{x}_i \in \varphi_i(\bar{x})$, or,
- (B) There exists $\bar{x} \in X$ such that, for each $i \in I$, $\varphi_i(\bar{x}) = \emptyset$.

Remark. For finite dimensional spaces X_i and finite I , the first theorem of this type is due to Gale and Mas-Colell (1975) (see also Debreu (1950)).

PROOF OF THE COMPETITIVE FAN-BROWDER FIXED POINT THEOREM

Suppose (B) is not verified, that is, $X = \bigcup\{\text{dom } \varphi_i \mid i \in I\}$ where $\text{dom } \varphi_i = \{x \in X \mid \varphi_i(x) \neq \emptyset\}$ is open. Since X is compact, there exist a finite subset $J \subset I$ and, for each $j \in J$, a closed subset $A^j \subset \text{dom } \varphi_j$ so that $X = \bigcup\{A^j \mid j \in J\}$. Consider the map $\varphi : X \rightrightarrows X$ defined by

$$\varphi(x) = \{y \in X \mid y_j \in \varphi_j(x) \forall j \in J(x)\},$$

where $J(x) = \{j \in J \mid x \in A^j\}$ is finite and not empty. It is easily verified that φ has nonempty convex values and open fibers, so by Fan-Browder's Theorem there is $\bar{x} \in X$ such that $\bar{x} \in \varphi(\bar{x})$, in particular there is $j \in J$ such that $\bar{x}_j \in \varphi_j(\bar{x})$, that is, (A) is verified. ■

NASH EQUILIBRIA

Notation. For $x \in X$, $i \in I$ and $y_i \in X_i$, we denote by (x^i, y_i) the point in X with the same coordinates as x except the i -th which is replaced by y_i .

Nash (1950): For each $i \in I$, assume:

- (1) X_i is a nonempty convex compact set in a TVS;
- (2) $u_i : X \rightarrow \mathbb{R}$ is continuous and, for each $x \in X$, $y_i \mapsto u_i(x^i, y_i)$ is quasi-concave.

Then, there exists $\bar{x} \in X$ verifying:

$$\forall i \in I, \forall y_i \in X_i, \quad u_i(\bar{x}^i, y_i) \leq u_i(\bar{x}).$$

Proof. Apply the competitive Fan-Browder Theorem with $\varphi_i : X \rightrightarrows X_i$ defined by $\varphi_i(x) = \{y_i \in X_i \mid u_i(x^i, y_i) > u_i(x)\}$ for each $x \in X$. ■

APPROXIMATION OF USC MULTI-MAP AND KAKUTANI-FAN-GLICKSBERG THEOREM

Approximation: Let X be a paracompact space, Y a LCTVS and $\varphi : X \rightrightarrows Y$ a USC map with compact convex values. Then, for any neighborhood \mathcal{U} of $gr(\varphi)$ in $X \times Y$, there exists a map $\varphi' : X \rightrightarrows Y$ with open graph and convex values such that $gr(\varphi) \subset gr(\varphi') \subset \mathcal{U}$.

Kakutani (1941), Ky Fan (1952), Glicksberg (1952): Let K be a nonempty convex compact set in a LCTVS and $\varphi : K \rightrightarrows K$ a USC map with nonempty compact convex values. Then, there exists $\bar{x} \in K$ such that $\varphi(\bar{x}) = \emptyset$ or $\bar{x} \in \varphi(\bar{x})$.

Proof. Suppose φ has no fixed point, that is, $gr(\varphi)$ is contained in the open set $\Delta^c = \{(x, y) \mid x \neq y\}$. So there is $\varphi' : K \rightrightarrows K$ with open fibers and convex values such that $gr(\varphi) \subset gr(\varphi') \subset \Delta^c$. Since φ' has no fixed point, by Fan-Browder Theorem there is $\bar{x} \in K$ such that $\varphi'(\bar{x}) = \emptyset$, hence also $\varphi(\bar{x}) = \emptyset$. ■

CONSTRAINED KAKUTANI-FAN-GLICKSBERG FIXED POINT THEOREM

Let K be a nonempty convex compact set in a LCTVS X . Assume:

- (1) $\varphi : K \rightrightarrows K$ is USC with nonempty compact convex values;*
- (2) $\psi : K \rightrightarrows K$ has open fibers, convex values and no fixed point;*
- (3) $V = \{x \in K \mid \varphi(x) \cap \psi(x) \neq \emptyset\}$ is open.*

Then, there exists $\bar{x} \in \varphi(\bar{x})$ such that $\varphi(\bar{x}) \cap \psi(\bar{x}) = \emptyset$.

CONSTRAINED MINIMAX INEQUALITY

Let K be a nonempty convex compact set in a LCTVS X . Assume:

- (1) $\varphi : K \rightrightarrows K$ is USC with nonempty compact convex values;
- (2) $f : K \times K \rightarrow \mathbb{R}$ verifies

$$\left\{ \begin{array}{l} \forall y \in K, \quad x \mapsto f(x, y) \text{ is lsc,} \\ \forall x \in K, \quad y \mapsto f(x, y) \text{ is quasi-concave,} \\ \forall x \in K, \quad f(x, x) \leq 0; \end{array} \right.$$

- (3) The set $\left\{ x \in K \mid \sup_{y \in \varphi(x)} f(x, y) > 0 \right\}$ is open.

Then, there exists $\bar{x} \in K$ such that

$$\left\{ \begin{array}{l} \bar{x} \in \varphi(\bar{x}), \\ \sup_{y \in \varphi(\bar{x})} f(\bar{x}, y) \leq 0. \end{array} \right.$$

Note. In case $f(x, y) = \langle Ax, x - y \rangle$ with $A : K \rightarrow X^*$, the constrained minimax inequality is called *quasi-variational inequality*.

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