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Introduction To Variational Relations : D. T. Luc

Lecture 1 : 10 · 5 · 2010

I. Projection:

H : Hilbert space

$K \subseteq H$ be a closed convex set.

Hilbert Projection Theorem:

$\forall\ a \in H\ \exists$ a unique $\bar{x} \in K$ such that

$$\|a - \bar{x}\| \leq \|a - x\| \quad \forall\ x \in K.$$

In the form of variational inequality (VI) :

1) — $\bar{x} = \mathrm{proj}_K\, a \iff \langle \bar{x} - a,\ x - \bar{x} \rangle \geq 0 \quad \forall\ x \in K$ — (1)

Define $F : H \longrightarrow H$ as

$$F(x) = x - a.$$

Then (1) becomes finding $\bar{x} \in K$ such that

$$\langle F(\bar{x}),\ x - \bar{x} \rangle \geq 0 \quad \forall\ x \in K.$$

This is nothing but VI.

Consider the problem : $\min\limits_{x \in K} f(x)$.

By optimality condition at the point of minima $\bar{x} \in K$, we have,

$-\nabla f(\bar{x}) \in N_K(\bar{x})$ ie $\langle \nabla f(\bar{x}),\ x - \bar{x} \rangle \geq 0 \quad \forall\ x \in K$

which is a VI with $F = \nabla f$.

( $N_K(\bar{x})$ = normal cone to set K at $\bar{x}$

$= \{ x^* \in H' : \langle x^*, x - \bar{x} \rangle \leq 0 \quad \forall\ x \in K \}$ ).

Hilbert Projection Theorem

$\Rightarrow$ Riesz Representation Theorem :

$\forall\ \phi \in H'\ \exists$ a unique $a \in H$ such that

$$\phi(x) = \langle a, x \rangle \quad \forall\ x \in H$$

⇒ Stampacchia's Theorem:

$A(u,v)$ : bilinear form, continuous, coercive

$K \subseteq H$ nonempty closed convex and [any] $\psi \in H'$ $\exists$ a unique $\bar{u} \in K$ such that

$$A(\bar{u}, v-\bar{u}) - \psi(v-\bar{u}) \geq 0 \quad \forall\, v \in K.$$

II. Generalizations :

- $VI(K,F)$ : Find $\bar{x} \in K$ such that

$$\langle F(\bar{x}), x-\bar{x} \rangle \geq 0 \quad \forall\, x \in K$$

As discussed the optimality condition for differentiable function $f$, if $f$ is nonsmooth the optimality condition now becomes

$$0 \in \partial f(\bar{x}) + N_K(\bar{x})$$

ie, there exists $\xi \in \partial f(\bar{x})$ such that

$$\langle \xi, x-\bar{x} \rangle \geq 0 \quad \forall\, x \in K.$$

Therefore, for a set-valued mapping $F$, $VI(K,F)$ is :

Find $\bar{x} \in K$ such that

$$\langle \xi, x-\bar{x} \rangle \geq 0 \quad \forall\, x \in K,\ \forall\, \xi \in F(\bar{x}).$$

- Mixed VI : $\langle F(\bar{x}), x-\bar{x} \rangle + h(x) - h(\bar{x}) \geq 0 \quad \forall\, x \in K$

- Generalized VI : $\langle F(\bar{x}), g(x) - g(\bar{x}) \rangle + h(x) - h(\bar{x}) \geq 0 \quad \forall\, x \in K$

- Equilibrium problem : $\psi(x, \bar{x}) \geq 0 \quad \forall\, x \in K$

- Set-valued variational inclusion :

For set-valued maps $F, G$, find $\bar{x} \in K$ such that

$$F(\bar{x}, x) \subseteq G(\bar{x}, x) \quad \forall\, x \in K.$$

Inclusion may be replaced by empty / nonempty inclusion.

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## III. Abstract Models:

$A, B, C$ are nonempty sets

consider set-valued maps: $S : A \rightrightarrows A$

$T_1 : A \rightrightarrows B$

$T_2 : A \times B \rightrightarrows C$.

$R$: relation linking $a \in A$, $b \in B$, $c \in C$.

$\therefore$ Variational Relation (VR) is:

Find $\bar{a} \in A$ such that

1. $\bar{a} \in S(\bar{a})$
2. $R(\bar{a}, b, c)$ holds $\forall\, b \in T_1(\bar{a})$

$\forall\, c \in T_2(\bar{a}, b) \equiv c \in \bigcup_{b \in T_1(\bar{a})} T_2(\bar{a}, b)$

**eg1:** $A = B = C = X$ (any given set)

$S(x) = \{y \in X : g_i(y) \le 0,\ i \in I,\ h_j(y) = 0,\ j \in J\}$

$= S$ (independent of $x$)

$T_1(x) = S(x) = S$

$T_2(x, y) = \{y\}$

$R(x, y, z)$ holds $\iff f(z) - f(x) \ge 0$ where $f : X \to \mathbb{R}$.

$\therefore$ (VR) is:

Find $\bar{x} \in X$ such that

1. $\bar{x} \in S(\bar{x}) = S$ ie $\bar{x}$ is feasible
2. $R(\bar{x}, y, z)$ holds $\forall\, y \in T_1(\bar{x}) = S$ ie $y$ is feasible

$\forall\, z \in T_2(\bar{x}, y) = \{y\}$ ie $z = y$.

$\Rightarrow f(y) - f(\bar{x}) \ge 0 \quad \forall\, y \in S$. (optimality)

**eg2:** $A = B = C = X$

$S(x) = X$, $T_1(x) = X$, $T_2(x, y) = \{y\}$.

$R(x, y, z)$ holds $\iff \psi(x, z) \ge 0$ where $\psi : X \times X \to \mathbb{R}$.

$\therefore$ (VR) $\equiv$ Find $\bar{x} \in X$ such that

$\psi(\bar{x}, y) \ge 0 \quad \forall\, y \in X$.

eg 3 : $A = B = C = K$

$S(x) = K$, $\quad T_1(x) = K$, $\quad T_2(x, y) = \{y\}$

$R(x, y, z)$ holds $\Leftrightarrow \langle F(x), z - x \rangle + h(z) - h(\bar{x}) \geq 0$

$\therefore$ (VR) $\equiv$ Find $\bar{x} \in K$ such that

$$\langle F(\bar{x}), y - \bar{x} \rangle + h(y) - h(\bar{x}) \geq 0 \quad \forall\, y \in K$$

(Mixed VI).

V. Existence :

Assume $S, T_1, T_2$ have nonempty values.

Criterion 1 : $\bar{a}$ is a solution $\Leftrightarrow \bar{a} \in \bigcap_{b \in B} P(b)$

where

$$P(b) = [A \setminus T_1^{-1}(b)) \cup \{a \in A : a \in S(a),\ R(a, b, c) \text{ holds } \forall\, c \in T_2(a, b)\}$$

Criterion 2 : $\bar{a}$ is a solution $\Leftrightarrow B \setminus P^{-1}(a) = \phi$.

In particular, if $A = B$, then (VR) has a solution under the following conditions :

) $a \longmapsto A \setminus P^{-1}(a)$ has a fixed point if it is non-empty valued $\forall a$

) $T_1(a) \subseteq S(a)$

) $R(a, a, c)$ holds $\forall\, c \in T_2(a, a)$ if $a \in S(a)$.

Particular case (KKM relations) : (Here $A = B$)

$R$ is KKM if $\forall\, a_1, a_2, \ldots, a_k \in A$,

$\forall\, a \in co\{a_1, a_2, \ldots, a_k\}$,

$\exists\, i$ such that

$R(a, a_i, c)$ holds $\forall\, c \in T_2(a, a_i)$.

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<u>Thm</u>: (Sufficient condition) Assume:

(i) $A = B$ non empty convex compact

(ii) $P$ has closed values

(iii) $co\,(T_1(a)) \subseteq S(a) \quad \forall a \in A$

(iv) $R$ is KKM

Then (VR) has a solution.

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$R$ is a relation on $A \times B \times C$ defined as:

- $\phi(a, b, c) \geq 0$ where $\phi: A \times B \times C \longrightarrow \mathbb{R}$ (inequality)
- $F(a, b, c) \subseteq G(a, b, c)$ (inclusion)

eg: Let $T: X \rightrightarrows Y$, $\quad \phi: K \times Y \longrightarrow \mathbb{R}$ (coupling function)

$K \subseteq X$

The problem may be formulated as:

Find $\bar{x} \in K$ such that

$$\phi(y - \bar{x}, \xi) \geq 0 \quad \forall \xi \in T(\bar{x}), \quad \forall y \in K.$$

If $Y = X'$, $\quad \phi(x, \xi) = \xi(x) \quad \forall \xi \in X', \; x \in X$

If $X = H$, $\quad Y = X' = H$, $\quad \phi(x, \xi) = \langle \xi, x \rangle$.

eg: <u>(For KKM relation)</u>

Find $\bar{x} \in K \subseteq X$ such that

$$x^*(y - \bar{x}) \geq 0 \quad \forall y \in K, \; \forall x^* \in T(\bar{x}) \text{ where } T: X \rightrightarrows X'$$

Its (VR) form is:

$A = B = K$, $\quad C = X'$,

$S(x) = K$, $\quad T_1(x) = K$, $\quad T_2(x, y) = T(x)$

$R(x, y, \xi)$ holds $\Leftrightarrow \xi(y - x) \geq 0$.

**Monotonicity :**

i) Monotone : $y^*(y-x) + x^*(x-y) \geq 0 \quad \forall x^* \in T(x), \ \forall y^* \in T(y)$

ii) Quasimonotone : $\max\{x^*(y-x), y^*(x-y)\} \geq 0$

iii) Pseudomonotone : $x^*(y-x) \geq 0$ for some $x^* \in T(x)$

$$\Rightarrow y^*(y-x) \geq 0 \quad \forall y^* \in T(y)$$

iv) Proper quasimonotone : $\forall x_1, \ldots, x_k \in K$

$\forall x \in co\{x_1, \ldots, x_k\}$,

there exist $i$ such that

$$\xi(x_i - x) \geq 0 \quad \forall \xi \in T(x_i).$$

(i) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Leftrightarrow$ $R$ is KKM.

**Finite Solvability :**

. VR model and finite solvability :

(VR) $\equiv$ Find $\bar{a} \in A$ such that

1. $\bar{a} \in S(\bar{a})$
2. $R(\bar{a}, b, c)$ holds $\forall b \in T_1(\bar{a})$

$\forall c \in T_2(\bar{a}, b)$.

Criterion 1 : $\bar{a}$ is a solution $\Leftrightarrow$ $\bar{a} \in \bigcap_{b \in B} P(b)$

Def$^n$ : (VR) is finitely solvable if for every finite subset $D \subseteq B$, there exists $a_D \in A$ such that

$\forall b \in D$ either $b \notin T_1(a_D)$ or $\begin{cases} a_D \in S(a_D) \\ R(a_D, b, c) \text{ holds } \\ \qquad \forall c \in T_2(a_D, b) \end{cases}$

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Solvability of (VR) $\underset{\Leftarrow}{\overset{\text{(trivial)}}{\Rightarrow}}$ Finite solvability
(need additional cond's)

eg: (For FS) $\min f(x)$

subject to $g_i(x) \le 0$, $i \in I$

$h_j(x) = 0$, $j \in J$

$x \in X$.

(VR) $\equiv$ $A = B = C = X$

$S(x) = S$ (feasible set) (independent of $x$)

$T_1(x) = S$

$T_2(x,y) = \{y\}$

$R(x,y,z)$ holds $\Leftrightarrow$ $f(z) - f(x) \ge 0$

(FS) : $y_1, \ldots, y_k \in X$, $\exists\, a_D \in X$ such that

either $y_i \notin T_1(a_D) = S$ ie $y_i$ is not feasible

or $\begin{cases} a_D \in S(a_D) = S \text{ ie } a_D \text{ is feasible and} \\ f(y_i) - f(a_D) \ge 0. \end{cases}$

In this example, (FS) holds.

II. <u>Intersectional closedness:</u>

Let $F: \Lambda \rightrightarrows X$

$$\bigcap_{\lambda \in \Lambda} cl(F(\lambda)) \ne \phi \Rightarrow \bigcap_{\lambda \in \Lambda} F(\lambda) \ne \phi$$

<u>Def^n</u>: $F$ is intersectionally closed if

$$\bigcap_{\lambda \in \Lambda} cl(F(\lambda)) = cl \bigcap_{\lambda \in \Lambda} F(\lambda).$$

eg: $C_i \subseteq \mathbb{R}^n$ be nonempty convex. By Rockafellar (1970), if $\bigcap_{i \in I} ri\, C_i \ne \phi$, then

$$\bigcap_{i \in I} cl\, C_i = cl \bigcap_{i \in I} C_i$$

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g) $f_i : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{\pm\infty\}$ convex functions

$$\bigcap_{i \in I} cl\,(epi\, f_i) = cl \bigcap_{i \in I} epi\, f_i = cl\ epi\,(\sup_{i \in I} f_i)$$

$$= epi\,(\sup_{i \in I} cl\, f_i)$$

[ $cl\, f$ : lsc hull of $f$. $epi\,(cl\, f) = cl\ epi\, f$ ].

Sufficient Conditions for I.C :

$F(\lambda)$ is closed $\forall \lambda$

$F(\cdot)$ is outer continuous ie,

$$\limsup_{\lambda \to \lambda_0} F(\lambda) \subseteq F(\lambda_0) \quad \forall \lambda_0 \in \Lambda$$

$\limsup_{\lambda \to \lambda_0} F(\lambda) = \{ x \in X \ ; \ \exists\ \lambda_n \longrightarrow \lambda_0,\ x_n \in F(\lambda_n)$ such that $x_n \longrightarrow x \}$.

$F(\cdot)$ is transfer closed, ie

$$\bigcap_{\lambda \in \Lambda} cl\, F(\lambda) = \bigcap_{\lambda \in \Lambda} F(\lambda)$$

Particular case:

$F : \Lambda \rightrightarrows X$

$F(\lambda) = \{ x \in X : \psi(\lambda, x) \leq 0 \}$ where $\psi : \Lambda \times X \longrightarrow \mathbb{R}$

$\bigcap_{\lambda \in \Lambda} F(\lambda) = \{ x \in X : \ \psi(\lambda, x) \leq 0 \quad \forall \lambda \in \Lambda \}$

Sufficient conditions:

i) $\phi(\lambda, \cdot)$ is lsc in $x \Rightarrow$ level set of $\phi(\lambda, \cdot)$ is closed

$\Rightarrow \bigcap$ " " " " " "

i) $\phi(\lambda, \cdot)$ is transfer closed in $x$.

(ie if $\phi(\lambda, x) > 0$ for some $\lambda$, $x$

$\Rightarrow \exists\ \lambda' \in \Lambda$, a neighborhood of $x$, $N(x)$ such that

$$\phi(\lambda', x') > 0 \quad \forall x' \in N(x)$$

If $\phi$ is continuous in $x$, then holds true for $\lambda = \lambda'$. )

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(iii) $\forall \bar{x} \in X$, $\sup_{\lambda} \phi(\lambda, x) > 0 \;\forall\, x$ close to $\bar{x}$, then $\exists\, \lambda' \in \Lambda$ and a neighborhood of $\bar{x}$, $N(\bar{x})$ such that $\phi(\lambda', x') > 0 \quad \forall\, x' \in N(\bar{x})$.

III. FS Principle :

Thm: A is compact.

R is IC (ie the map $b \longmapsto P(b)$ is IC).

(VR) is solvable $\Leftrightarrow$ it is finitely solvable.

Pf: ($\Rightarrow$) Trivial.

($\Leftarrow$) Apply finite intersection property.

Defn: Finite Intersection Property (F.I.P).

Consider the family $\{A_i : i \in I\}$.

$$\bigcap_{\text{finite } i} A_i \neq \phi \Rightarrow \bigcap_{i \in I} A_i \neq \phi.$$

Applications:

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Weierstrass Thm.:

consider the problem $(\bar{P})$
$$\min f(x)$$
$$\text{st} \quad g_i(x) \leq 0, \; i \in I$$
$$h_j(x) = 0, \; j \in J$$

(VR) form: $A = B = C = X$

$S(x) = S$ (feasible set)

$T_1(x) = S$

$T_2(x, y) = \{y\}$

$R(x, y, z)$ holds $\Leftrightarrow f(z) - f(x) \geq 0$.

Here, (VR) is finitely solvable.

$\Rightarrow$ Problem $(\bar{P})$ is solvable if (IC) is satisfied.

(ie $b \longmapsto P(b)$ is IC).

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$P : B \rightrightarrows A$ where $P(b) = (A \setminus T_1^{-1}(b))$
$\cup \{ x \in X : x \in S(x),$
$f(b) - f(x) \geq 0 \}$.

(IC) holds $\Leftrightarrow$ $x \longmapsto \{ y \in X : y$ is feasible,
$f(x) - f(y) \geq 0 \}$ is (IC).

According to Weierstrass Thm. :
If $X$ is compact and $f : X \longrightarrow \mathbb{R}$ is lsc, then $f$ has a minimum in $X$.

Def$^n$: $f$ is IC if $\forall x_0 \in X$, $\exists$ a neighbourhood of $x_0$, $N(x_0)$, such that $\forall x \in N(x_0)$ $\exists \alpha > \inf f$, $f(x) > \alpha$
$\Rightarrow \exists \beta > \inf f$, $\exists N' \subseteq N(x_0)$ such that
$f(x) > \beta \quad \forall x \in N'$.

2. <u>Saddle point</u> : $f : X \times Z \longrightarrow \mathbb{R}$
$(\bar{x}, \bar{z}) \in X \times Z$ is a saddle point of $f$ if
$f(\bar{x}, z) \leq f(x, \bar{z}) \quad \forall (x, z) \in X \times Z$.

<u>Von Neumann Thm.</u> : If $f$ is continuous convex-concave, and $X$, $Z$ are non empty compact convex, then $f$ has a saddle point.

(VR) : $A = B = X \times Z$
$S(x, z) = X \times Z = T_1(x, z)$.
No $C$, No $T_2$.
$R((x, z), (x', z'))$ holds $\Leftrightarrow$ $f(x, z') \leq f(x', z)$.

(FS) : $(x_1, z_1), \ldots, (x_n, z_n) \in X \times Z$.
$\exists x_0 \in X$, $z_0 \in Z$ such that
$f(x_0, z_i) \leq f(x_i, z_0) \quad \forall i$.

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(FS) is true if $f$ is quasiconvex in $x$
quasiconcave in $z$.

$$(IC) \Leftrightarrow (x, z) \longmapsto \{(x', z') \in X \times Z : f(x', z) - f(x, z') \leq 0\}$$
is (IC).

# Parametric VR:

Let $\Lambda$ be a topological (parameter) space.

I. Model:

Corresponding to a fixed $\lambda \in \Lambda$, we have (VR) as
$(VR)^\lambda$ : Find $\bar{a} \in A^\lambda$ such that

1. $\bar{a} \in S^\lambda(\bar{a})$
2. $R^\lambda(\bar{a}, b, c)$ holds $\forall\ b \in T_1^\lambda(\bar{a})$
$\forall\ c \in T_2^\lambda(\bar{a}, b)$.

where $S^\lambda : A^\lambda \rightrightarrows A^\lambda$
$T_1^\lambda : A^\lambda \rightrightarrows B^\lambda$
$T_2^\lambda : A^\lambda \times B^\lambda \rightrightarrows C^\lambda$.
$R^\lambda$ is a relation on $A^\lambda \times B^\lambda \times C^\lambda$.

Let $\Sigma^\lambda$ be the solution set of $(VR)^\lambda$.
Let $A^\lambda, B^\lambda, C^\lambda, S^\lambda, T_1^\lambda, T_2^\lambda$ be fixed.

II. Continuity of set-valued maps:

$F : \Lambda \rightrightarrows X$

(i) Superior / outer limit:

$$\limsup_{\substack{\lambda \to \lambda_0 \\ \lambda \neq \lambda_0}} F(\lambda) = \{x \in X : \exists\ \lambda_k \xrightarrow{\neq} \lambda_0,\ x_k \in F(\lambda_k) \to x\}.$$

ii) Inferior / inner limit:

$$\liminf_{\lambda \to \lambda_0} F(\lambda) = \{x \in X : \forall\ \lambda_k \xrightarrow{\neq} \lambda_0,\ \exists\ x_k \in F(\lambda_k) \to x\}.$$

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iii) F is closed if its graph is closed in $\Lambda \times X$.

iv) F is outer-continuous at $\lambda_0$ if

$$\limsup_{\lambda \underset{\neq}{\to} \lambda_0} F(\lambda) \subseteq F(\lambda_0)$$

v) F is inner-continuous (lsc) at $\lambda_0$ if

$$\liminf_{\lambda \underset{\neq}{\to} \lambda_0} F(\lambda) \supseteq F(\lambda_0)$$

vi) F is usc at $\lambda_0$ if $\forall$ open set $V \supseteq F(\lambda_0)$, $\exists$ open set $U \ni \lambda_0$ such that

$$F(\lambda) \subseteq V \quad \forall \lambda \in U.$$

Properties: $F, G : \Lambda \rightrightarrows X$.

1. F and G are outer-continuous $\Rightarrow$ $F \cap G$ is also outer-continuous.
$((F \cap G)(\lambda) = F(\lambda) \cap G(\lambda))$.

2. F and G are inner-continuous $\not\Rightarrow$ $F \cap G$ is inner-continuous

New concepts:

i) Superior open limit:

$$\operatorname{limsupo}_{\lambda \underset{\neq}{\to} \lambda_0} F(\lambda) = \{ x \in X : \exists \lambda_k \xrightarrow{\neq} \lambda_0, \ \exists V \ni x \text{ such that } V \subseteq F(\lambda_k) \}.$$

ii) Inferior open limit:

$$\operatorname{liminfo}_{\lambda \to \lambda_0} F(\lambda) = \{ x \in X : \exists U \ni \lambda_0, \ \exists V \ni x \text{ such that } V \subseteq F(\lambda) \ \forall \lambda \in U \}.$$

iii) F is outer-open if $\operatorname{limsupo} F(\lambda) \subseteq F(\lambda_0)$.

iv) F is inner-open if $\operatorname{liminfo} F(\lambda) \supseteq F(\lambda_0)$.

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(iii) and (iv) are specifically for maps.

Single-valued functions are neither outer-open nor inner-open.

Properties:

1. $F$ is inner open $\Leftrightarrow$ $F^c$ is outer-continuous.
   ($F^c(\lambda) = X \setminus F(\lambda)$).
2. $F$ is outer-open $\Leftrightarrow$ $F^c$ is inner-continuous.
3. $F$ is inner-open $\Rightarrow$ $F$ is inner continuous.
4. $\left.\begin{array}{l} F \text{ is inner-open} \\ G \text{ is inner-continuous} \end{array}\right\} \Rightarrow F \cap G$ is inner continuous.

III. Stability of $\Sigma^\lambda$:

A: non empty compact $\quad R^\lambda : \lambda \in \Lambda$.

$S : A \rightrightarrows A$

$T_1 : A \rightrightarrows B$

$T_2 : A \times B \rightrightarrows C$.

Thm: $\Sigma^\lambda$ is outer-continuous at $\lambda_0$ under the following conditions:

(i) $S(\cdot)$ is closed.

(ii) $T_1(\cdot)$ is inner-continuous

(iii) $T_2(\cdot,\cdot)$ is inner-continuous

(iv) $R^\cdot(\cdot,\cdot,\cdot)$ is closed at $\lambda_0$.

$$
\begin{aligned}
(\lambda_k \longrightarrow \lambda_0, \quad & a_k \in S(a_k) \longrightarrow a \\
& b_k \in T_1(a_k) \longrightarrow b \\
& c_k \in T_2(a_k, b_k) \longrightarrow c \\
& R^{\lambda_k}(a_k, b_k, c_k) \text{ holds} \Rightarrow R^{\lambda_0}(a, b, c) \text{ holds})
\end{aligned}
$$

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Pf: To prove: $\left.\begin{array}{l}\lambda_k \longrightarrow \lambda_0 \\ a_k \in \Sigma^{\lambda_k} \longrightarrow a\end{array}\right\} \Rightarrow a \in \Sigma^{\lambda_0}.$

- $a_k \in S(a_k) \Rightarrow$ by (i), $a \in S(a)$
- $b \in T_1(a)$, $c \in T_2(a,b) \Rightarrow b_k \in T_1(a_k) \longrightarrow b$

  $c_k \in T_2(a_k, b_k) \longrightarrow c.$
- $R^{\lambda_k}(a_k, b_k, c_k)$ holds $\Rightarrow$ by (iv), $R^{\lambda_0}(a,b,c)$ holds.

$\Rightarrow a \in \Sigma^{\lambda_0}.$

Thm 2: $\Sigma^{\lambda}$ is inner-continuous at $\lambda_0$ if

(i) $S(\cdot)$ is inner-open.

(ii) $T_1(\cdot)$ has compact values

(iii) $T_2(a,\cdot)$ is closed $\forall a \in A$

(iv) $R^{\lambda_0}(a,b,c)$ does not hold whenever $R^{\lambda_k}(a, b_k, c_k)$ does not hold

for some $b_k \in T_1(a)$, $c_k \in T_2(a, b_k)$

with $b_k \longrightarrow b$, $c_k \longrightarrow c$.

V. Application:

Parametric Equilibrium Problem:

$X \subseteq \mathbb{R}^n$, $\phi_\lambda : X \times X \longrightarrow \mathbb{R}$, $\lambda \in \Lambda$.

$(VI)^\lambda$: Find $\bar{x} \in X$ such that

$$\phi_\lambda(\bar{x}, y) \geq 0 \quad \forall y \in X.$$

$(VR)^\lambda$: $A = B = C = X$

$S(x) = X = T_1(x)$

$T_2(x,y) = \{y\}$

$R^\lambda(x,y)$ holds $\Leftrightarrow \phi_\lambda(x,y) \geq 0.$