

Optimization methods in portfolio management and option hedging *

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Abstract

These lecture notes give an introduction to modern, continuous-time portfolio management and option hedging. We present the stochastic control method to portfolio optimization, which covers Merton's pioneering work. The alternative martingale approach is also exposed with a nice application on option hedging with value at risk criterion.

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1 Introduction

Portfolio management is a fundamental aspect in economics and finance. It is an all natural and important activity in our society for households, pension fund managers, as well as for government debt managers. One has got a certain amount of money and tries to use it in such a way that one can draw the maximum possible utility from the results of the corresponding activities. This principle covers numerous and various situations of daily life. For example, imagine you are thinking of buying a house and are offered two different ones you can afford. One close to your office, but without a garden and close to a motor way, the other one with a nice landscape but requiring a long distance to work everyday. The decision about which one is more convenient for you (or has your preference) is in principle a portfolio problem. In a financial terminology, the problem of portfolio optimization of an investor trading in different assets is to choose an optimal investment, that is how many shares of which asset he should hold at any trading time, in order to maximize some subjective (depending on his preferences) criterion relying on his total wealth and/or consumption.

The earliest approach to solving a portfolio problem is the so-called mean-variance approach pioneered by H. Markowitz [9] in a one-period decision model. It still has great importance in real-life applications, and is widely applied in the risk management departments of banks. The main reasons for this is being the simplicity with which the algorithm can be implemented, and that it requires no special knowledge on probability (only expectation and covariances of random variables are enough to know). Markowitz was awarded the 1990 Nobel prize in economics for the importance of his contribution on the mean-variance approach.

However, the main drawback of this approach is the static nature of the problem : after the decision concerning the allocation of initial wealth to the different assets has been made at the beginning of the period, no further actions is allowed until the end of the period. Once the initial portfolio is chosen, the investor's job is complete and his only feasible action is to watch the prices move without the possibility to intervene. This is an extreme oversimplification of reality and totally ignores the highly volatile behaviour and dynamic nature of prices. In contrast, continuous-time models offer more satisfying solutions to these problems. By allowing for the possibility of trading at every time instant, the investor can react immediately if the situation dictates this. Furthermore, and as for the famous Black-Scholes formula in option pricing, the highly developed mathematical tools of stochastic control theory and stochastic calculus allow to find structure of the solution to the portfolio problem in a clearer, more explicit or tractable form than in the discrete-time case. Extensions of the mean-variance approach to a continuous-time framework has been largely studied in the literature, and we refer to the survey papers by Schweizer [16] and Pham [12]. One criticism of the mean-variance criterion is that risk is only measured in terms of the variance of the portfolio return. The symmetric form of the variance has the undesirable side effect of not only bounding possible losses, but also possible gains. To overcome this drawback, one should instead look at choice of suitable (non-symmetric) preferences, and this is usually done in terms of utility functions according to the von

Neumann Morgenstern theory [11].

The seminal work of Merton [10] is considered as a pioneering point for the continuous-time portfolio management. The author (Nobel prize in economics in 1997 together with M. Scholes) used stochastic control method to the asset allocation problem, and expressed optimal portfolio rule in terms of the solution of a second-order partial differential equation (PDE), the so-called Hamilton-Jacobi-Bellman equation. He was able to obtain explicit solutions for special examples. With the growing application of stochastic calculus to finance from the eighties, an alternative approach, the martingale method to portfolio optimization, was developed by Pliska [14], Karatzas et al. [7] and Cox and Huang [1] based on martingale theory and convex optimization.

This lecture notes are intended to give an introduction to modern continuous-time portfolio optimization. We present the two mentioned above main approaches for solving portfolio problem, and in particular Merton's one, and look at some other portfolio problems arising in option hedging.

The outline of these lectures is organized as follows. We recall in Section 2 some general principles on the axiomatic formulation of investor's preferences and the paradigm of expected utility. In Section 3, we highlight the main ideas of the two main approaches to solving portfolio problem in a simple binomial discrete-time setting. We develop in Section 4 some key tools in stochastic control approach in continuous-time, and apply to the Merton's problem, and to the more recent works for the computation of superhedging price of European options. In Section 5, we present the alternative martingale approach for continuous-time portfolio problem, and give some variant and recent application for the Value at Risk hedging. We conclude in Section 5 with some discussion on extensions studied in the literature.

2 Financial decision-making and preferences

In a financial market where investors are facing uncertainty, the return of an investment in assets is in general not known. For example, a stock yield depends on the resale price and the dividends. How to choose between several possible investments? In order to determine desirable strategies in an uncertain context, the preferences of the investor should be made explicit, and this is usually done in terms of expected utility criterion.

We suppose that the investor compares random returns whom he knows the probability distributions on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Under some axiomatic properties on the preferences of the individual, von Neumann and Morgenstern (1944) have shown that they may be represented by an expected utility criterion. More precisely, by denoting \succ the preference order on the set of random returns, we say that \succ satisfies the Von-Neumann Morgenstern criterion iff there exists some increasing function U from \mathbb{R} into \mathbb{R} , called utility function, such that :

$$X_1 \succ X_2 \iff \mathbb{E}[U(X_1)] > \mathbb{E}[U(X_2)].$$

The increasing property of the utility function means that the investor prefers more than less wealth. The choice of the utility function allows to precise the notions of risk aversion

and risk premium related to uncertainty :

• **Risk aversion and concavity of the utility function**

We consider an agent who dislikes risk : with respect to a random return X , he prefers to get with certainty the expectation $\mathbb{E}[X]$ of this return. This means that his utility function satisfies the Jensen's inequality

$$U(\mathbb{E}[X]) \geq \mathbb{E}[U(X)],$$

which holds true only for concave functions. Indeed, by choosing a random return X , which takes values x with probability $\lambda \in (0, 1)$, and x' with probability $1 - \lambda$, we have

$$U(\lambda x + (1 - \lambda)x') \geq \lambda U(x) + (1 - \lambda)U(x'),$$

which shows the concavity of the utility function U .

• **Degree of risk aversion and risk premium**

For a risk-averse agent with concave utility function U , we define the risk premium associated to a random portfolio return X as the positive amount $\pi = \pi(X)$ that he is ready to pay in order to get a certain gain. It is then defined by the equation

$$U(\mathbb{E}[X] - \pi) = \mathbb{E}[U(X)].$$

The quantity $\mathcal{E}(X) = \mathbb{E}[X] - \pi$ is called certainty equivalent of X and is smaller than the expectation of X .

Denote $\bar{X} = \mathbb{E}[X]$, and suppose that the portfolio return X is few risky so that we have the following approximation :

$$U(X) \approx U(\bar{X}) + (X - \bar{X})U'(\bar{X}) + \frac{1}{2}(X - \bar{X})^2U''(\bar{X}),$$

and so by taking expectation :

$$\mathbb{E}[U(X)] \approx U(\bar{X}) + \text{Var}(X)\frac{U''(\bar{X})}{2}.$$

We also have

$$U(\bar{X} - \pi) \approx U(\bar{X}) - \pi U'(\bar{X}),$$

which gives the approximation for the risk premium

$$\pi \approx -\frac{U''(\bar{X})}{2U'(\bar{X})}\text{Var}(X).$$

Hence, the certainty equivalent of X is given approximately by

$$\mathcal{E}(X) = \mathbb{E}[X] - \frac{1}{2}\alpha(\bar{X})\text{Var}(X),$$

where $\alpha(x)$ is defined as the local absolute risk aversion at the return level x :

$$\alpha(x) = -\frac{U''(x)}{U'(x)},$$

and is also called the Arrow-Pratt coefficient of absolute risk aversion of U at level x . We then see that in a first approximation, the variance of a portfolio return is a good indicator of its risk, and $\alpha(\bar{X})$ is the factor by which an economic agent with utility function U weights the risk. Moreover, the larger is the absolute risk aversion, the larger should be the expectation of the return in order to compensate its risk.

Let us write the random return X as $X = \bar{X}(1 + \varepsilon)$, where ε is interpreted as the relative payoff of the return X with respect to \bar{X} , and define the relative risk premium ρ of X by :

$$U(\bar{X}(1 - \rho)) = \mathbb{E}[U(X)] = \mathbb{E}[U(\bar{X}(1 + \varepsilon))].$$

The relative risk premium is interpreted as the proportion of return that the investor is ready to pay in order to get a certain gain. Then similarly as above, we obtain an approximation for ρ with :

$$\rho \approx \frac{1}{2}\gamma(\bar{X})\text{Var}(\varepsilon),$$

where

$$\gamma(x) = -\frac{xU''(x)}{U'(x)},$$

is the relative risk aversion at level x .

Example 2.1 The following classes of utility functions and their corresponding coefficients of risk aversion are standard examples.

- Constant Absolute Risk Aversion (CARA) : $\alpha(x)$ equals some constant $\alpha > 0$. Since $\alpha(x) = -(\ln U')'(x)$, it follows that $U(x) = a - be^{-\alpha x}$. By using an affine transformation, U can be normalized to

$$U(x) = 1 - e^{-\alpha x}.$$

- Constant Relative Risk Aversion (CRRA) : $\gamma(x)$ equals some constant $\gamma \in (0, 1]$. Up to affine transformations, we have

$$U(x) = \begin{cases} \ln x, & \text{for } \gamma = 1, \\ \frac{x^{1-\gamma}}{1-\gamma}, & \text{for } 0 < \gamma < 1 \end{cases}$$

So far, we have presented the classical theory of expected utility from the von Neumann-Morgenstern representation. However, it is well-known that in reality people may not behave according to this paradigm, as illustrated by the famous Allais paradox.

Example 2.2 (Allais paradox). The so-called *Allais paradox* questions the reality of the expected utility theory by considering the following situations. Let X_1 be a return with distribution

$$\mathbb{P}_{X_1} = 0,33\delta_{2500} + 0,66\delta_{2400} + 0,01\delta_0,$$

i.e. it yields 2500 dollars with probability 0,33, 2400 dollars with probability 0,66, and nothing with the remaining probability 0,01. On the other hand, let X_2 be the certain return that yields 2400 dollars for sure, i.e. $\mathbb{P}_{X_2} = \delta_{2400}$. When asked, most people prefer the sure amount X_2 , even though X_1 has the larger expected value, namely 2409 dollars. Next, consider the following two random returns Y_1 and Y_2 with probability distributions :

$$\mathbb{P}_{Y_1} = 0,34\delta_{2400} + 0,66\delta_0, \quad \mathbb{P}_{Y_2} = 0,33\delta_{2500} + 0,67\delta_0.$$

Here people tend to prefer the slightly riskier return Y_2 over Y_1 , in accordance with the expectations of Y_2 and Y_1 , which are respectively 825 and 816 dollars. This observation is due to Maurice Allais, and was confirmed by Kahnemann and Tversky with empirical tests where 82% of interviewees preferred X_2 over X_1 , while 83% chose Y_2 rather than Y_1 . This means that at least 65% chose both $X_2 \succ X_1$ and $Y_2 \succ Y_1$. As pointed by M. Allais, this simultaneous choice leads to a paradox in the sense that it is inconsistent with the von Neumann-Morgenstern representation. Indeed, if the agent is represented by some utility function U , and if both $X_2 \succ X_1$, $Y_2 \succ Y_1$ hold true, then by considering the returns Z_1 , Z_2 , with probability distribution $\mathbb{P}_{Z_1} = \frac{1}{2}(\mathbb{P}_{X_1} + \mathbb{P}_{Y_1})$, $\mathbb{P}_{Z_2} = \frac{1}{2}(\mathbb{P}_{X_2} + \mathbb{P}_{Y_2})$, we should have

$$\begin{aligned} \mathbb{E}[U(Z_2)] &= \frac{1}{2}\mathbb{E}[U(X_2)] + \frac{1}{2}\mathbb{E}[U(Y_2)] \\ &> \frac{1}{2}\mathbb{E}[U(X_1)] + \frac{1}{2}\mathbb{E}[U(Y_1)] = \mathbb{E}[U(Z_1)], \end{aligned}$$

and so $Z_2 \succ Z_1$. This is in contradiction with the fact that $\mathbb{P}_{Z_1} = \mathbb{P}_{Z_2}$.

In these notes, we shall focus on the expected utility criterion. We refer to the book by Föllmer and Schied [6] for a presentation of alternative preferences criteria.

3 Dynamic programming and martingale methods : an illustration via a simple example

We consider a simple discrete-time setting, and we present the main ideas behind the stochastic control and martingale approaches for solving portfolio optimization problems in finance. We follow the presentation of Korn [8].

Our market consists of a bond and a single stock in a two-period model $t = 0, 1, 2$. The bond and stock prices S_t^0, S_t are displayed according to a binomial model as follows :

Time t	0	1	2
S_t^0	1	1	1
S_t	1	$\frac{2}{1/2}$ (4/9)	$\frac{4}{1}$ (4/9)
		$\frac{1}{1/2}$ (5/9)	$\frac{1}{1}$ (5/9)
			$\frac{1}{1/4}$ (5/9)

This means that the bond price is assumed to be constant over time, whilst the stock price can only move to two different values at the next trading time. It can double or halve its value with probabilities 4/9 and 5/9 respectively.

The investor wants to maximize his expected utility from terminal wealth at time $t = 2$, which is completely determined by its initial capital x , and the strategy $\alpha = (\alpha_t)_{t=0,1}$ representing the fraction of wealth invested in stock at time $t = 0, 1$. Notice that $1 - \alpha$ represents the fraction of wealth invested in bond. The value α_t is chosen after observing the stock price S_t at time $t = 0, 1$. We denote $X_t^{x,\alpha}$ the wealth value at time t , which is then given by the equations :

$$X_1^{x,\alpha} = \left(\alpha_0 \frac{S_1}{S_0} + (1 - \alpha_0) \right) x, \quad X_2^{x,\alpha} = \left(\alpha_1 \frac{S_2}{S_1} + (1 - \alpha_1) \right) X_1^{x,\alpha}. \quad (3.1)$$

We suppose that the investor is characterized by an utility function $U(x) = \sqrt{x}$, and his goal is then to solve the optimization problem :

$$V_0(x) = \max_{\alpha} \mathbb{E} \left[U(X_2^{x,\alpha}) \right], \quad \text{s.t. } X_2^{x,\alpha} \geq 0. \quad (3.2)$$

3.1 Solution via the dynamic programming approach

The dynamic programming approach consists of solving the optimization problem (3.2), but starting at time $t = 1$ in either of the two possible states $S_1 = 2$ or $S_1 = 1/2$. After having solved these two particular problems, one is able to determine the optimal strategy at the starting time $t = 0$ with the help of the already computed optimal strategy α_1 for time $t = 1$. This two-step procedure is described as follows :

Step 1 : $t = 1$.

Consider first the sub-problem that arises if we are in the state $S_1 = 2$. Suppose that the current wealth X_1 is equal to a positive number y . Then, by choosing a strategy $\alpha_1 = a$, we can compute the expected utility from the corresponding terminal wealth $X_2^{y,a}$ as :

$$\begin{aligned} \mathbb{E} \left[U(X_2^{y,a}) \mid S_1 = 2, X_1 = y \right] &= \frac{4}{9} \sqrt{(2a + (1 - a))y} + \frac{5}{9} \sqrt{\left(\frac{1}{2}a + (1 - a)\right)y} \\ &= \left(\frac{4}{9} \sqrt{a + 1} + \frac{5}{9} \sqrt{1 - \frac{a}{2}} \right) \sqrt{y} =: f(a)U(y). \end{aligned}$$

Hence, the above relation shows that the optimal strategy a^* at time $t = 1$ and in the state $S_1 = 2$, can be found by maximizing $f(a)$ over $[-1, 2]$ (here the boundaries of this interval are determined by the requirement that $X_2 \geq 0$). A straightforward calculation shows that the unique maximizer is given by $a^* = 13/19$.

The situation in the state $S_1 = 1/2$ is analogous. As the probability of the events that the stock price doubles or halves are the same as in the state $S_1 = 2$, we obtain the same optimal strategy a^* . Thus, the optimal strategy at time $t = 1$ is independent of the state and of the wealth, and is given by

$$\alpha_1^* = \frac{13}{19}.$$

Moreover, the corresponding optimal expected utility is

$$V_1(y) = \mathbb{E} \left[U(X_2^{y,\alpha_1^*}) \mid X_1 = y \right] = f(\alpha_1^*)U(y) = \frac{19}{3\sqrt{38}} \sqrt{y}.$$

Step 2 : $t = 0$.

After having computed the optimal strategy α_1^* in every possible state at time $t = 1$, we are now able to compute the optimal strategy α_0 at the initial time $t = 0$. If we choose $\alpha_0 = a$, we can compute from (3.1)

$$\mathbb{E}\left[U(X_2^{x,(a,\alpha_1^*)})\right] = \mathbb{E}\left[U(X_2^{X_1^{x,a},\alpha_1^*})\right] = \mathbb{E}[V_1(X_1^{x,a})],$$

and so

$$\begin{aligned} V_0(x) &= \sup_a \mathbb{E}[V_1(X_1^{x,a})] \\ &= \sup_a f(\alpha_1^*)\mathbb{E}[U(X_1^{x,a})] = \sup_a f(\alpha_1^*)f(a)U(x), \end{aligned}$$

which gives again the maximizer

$$\alpha_0^* = \frac{13}{19}.$$

Therefore, the optimal strategy is given by

$$\alpha^* = (\alpha_0^*, \alpha_1^*) = \left(\frac{13}{19}, \frac{13}{19}\right).$$

3.2 Solution via the martingale approach

The main idea behind the martingale approach is a separation between the determination of the optimal terminal wealth and that of a corresponding portfolio process yielding it exactly. This relies on the representation of attainable claims by means of expectation under the unique risk neutral probability measure \mathbb{Q} in complete markets as in the Black-Scholes model or binomial model. In our simple two-period binomial model, it is easily checked that the unique risk neutral probability measure \mathbb{Q} , transforming the stock price into a martingale, is the one that always assign a probability of 1/3 to an up and 2/3 to a down move of the stock price. The corresponding tree for the stock price under \mathbb{Q} has the form

Time t	0	1	2
S_t	1	$\frac{2}{1/2}$ (1/3)	$\frac{4}{1/2}$ (1/3)
		$\frac{1}{1/2}$ (2/3)	$\frac{1}{1/2}$ (2/3)
			$\frac{1}{1/4}$ (2/3)

In our discrete-time setting, \mathcal{F}_2 is generated by the four different possible paths of the stock price, and so any claim, represented by a nonnegative \mathcal{F}_2 -measurable random variable H , is identified with a quadruplet $(h_1, h_2, h_3, h_4) \in \mathbb{R}_+^4$. We have then the following characterization of attainable claims. A nonnegative \mathcal{F}_2 -measurable random variable H is attainable by some portfolio wealth from initial capital x , if and only if its expectation under the risk-neutral probability measure (its price) is equal to x :

$$H = X_2^{x,\alpha} \quad \text{for some } \alpha \iff \mathbb{E}^{\mathbb{Q}}[H] = x.$$

In view of this characterization, the dynamic optimization problem (3.2) can be rewritten as a static optimization problem over terminal wealth :

$$V_0(x) = \sup_H \mathbb{E}[U(H)] \quad \text{s.t.} \quad \mathbb{E}^{\mathbb{Q}}[H] = x. \quad (3.3)$$

The resolution of the utility maximization problem is then achieved in two steps.

Step 1 : Computation of the optimal terminal wealth

Recalling the objective probabilities for the value function, the risk-neutral probability, and the four possible values h_1, \dots, h_4 for H , we rewrite the static optimization problem (3.3) as

$$\begin{aligned} \max_{h_i} \quad & \frac{16}{81} \sqrt{h_1} + \frac{20}{81} \sqrt{h_2} + \frac{20}{81} \sqrt{h_3} + \frac{25}{81} \sqrt{h_4} \\ \text{s.t.} \quad & h_i \geq 0, \quad \frac{1}{9}h_1 + \frac{2}{9}h_2 + \frac{2}{9}h_3 + \frac{4}{9}h_4 = x. \end{aligned}$$

This optimization problem in the four variables h_1, \dots, h_4 can be solved by the usual Lagrangian methods. The solution is given by

$$h_1^* = \left(\frac{32}{19}\right)^2 x, \quad h_2^* = h_3^* = \left(\frac{20}{19}\right)^2 x, \quad h_4^* = \left(\frac{25}{38}\right)^2 x.$$

Step 2 : Computation of the strategy generating the optimal wealth

Step 1 resulted in the optimal wealth $H^* = (h_1^*, \dots, h_4^*)$ at time $t = 2$ in every four possible states. We now compute the strategy $\alpha = (\alpha_0, \alpha_1)$ that will deliver this terminal wealth, i.e. such that $X_2^{x,\alpha} = H^*$. We denote α_1^u (resp. α_1^d) the value of the strategy α_1 decided at time $t = 1$ when $S_1 = 2$ (resp. $S_1 = 1/2$). From the equations (3.1) on the wealth, the relation $X_2^{x,\alpha} = H^*$ is explicitly written as :

$$\begin{aligned} \left(2\alpha_1^u + (1 - \alpha_1^u)\right) \left(2\alpha_0 + (1 - \alpha_0)\right) x &= h_1^* = \left(\frac{32}{19}\right)^2 x \\ \left(\frac{1}{2}\alpha_1^u + (1 - \alpha_1^u)\right) \left(2\alpha_0 + (1 - \alpha_0)\right) x &= h_2^* = \left(\frac{20}{19}\right)^2 x \\ \left(2\alpha_1^d + (1 - \alpha_1^d)\right) \left(\frac{1}{2}\alpha_0 + (1 - \alpha_0)\right) x &= h_3^* = \left(\frac{20}{19}\right)^2 x \\ \left(\frac{1}{2}\alpha_1^d + (1 - \alpha_1^d)\right) \left(\frac{1}{2}\alpha_0 + (1 - \alpha_0)\right) x &= h_4^* = \left(\frac{25}{38}\right)^2 x. \end{aligned}$$

A straightforward calculation shows that $\alpha_0 = 13/19$ and $\alpha_1^u = \alpha_1^d = 13/19$, and we retrieve the solution obtained by the dynamic programming approach.

4 Dynamic programming methods for portfolio optimization in continuous-time

The stochastic control and dynamic programming methods for continuous-time portfolio optimization was initiated by Merton [10]. He applied methodology and results from stochastic control theory to a financial context. Since this seminal paper, there is an important development of stochastic control motivated by applications in finance. We first give some classical background on dynamic programming method in a continuous-time diffusion framework, and then give some applications in portfolio management.

4.1 Dynamic programming and Hamilton-Jacobi-Bellman equation

We consider a dynamical system, characterized by the set of all quantitative variables giving an “exhaustive” description of the system. We suppose here that the state variables are in finite number, valued in \mathbb{R}^d . The system evolves in an uncertain context, and we denote $X_t(\omega)$ the state variable at time t in a scenario of the world $\omega \in \Omega$ measurable space equipped with a probability measure \mathbb{P} . The state of the system is influenced at any time by a control modelled as a process $(\alpha_t)_t$ whose value in $A \subset \mathbb{R}^m$ is decided at any time in function of the available information. In the sequel, we consider a model of controlled diffusion for X , governed by the following dynamics :

$$dX_s = b(X_s, \alpha_s)ds + \sigma(X_s, \alpha_s)dW_s. \quad (4.1)$$

Here W is a n -dimensional Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, b and σ are measurable functions on $\mathbb{R}^d \times A$, and $\alpha = (\alpha_t)_{t \geq 0}$ is an \mathbb{F} -adapted process controlling the state X_t via its drift and diffusion terms b and σ . We denote by \mathcal{A} the set of control processes $\alpha = (\alpha_t)$. We fix some finite horizon $T < \infty$, and the objective of the controller is to optimize, here to maximize to fix the ideas, over controls $\alpha \in \mathcal{A}$, an objective functional in the form ;

$$\mathbb{E} \left[\int_0^T f(X_t, \alpha_t)dt + g(X_T) \right] \rightarrow \text{maximize over } \alpha. \quad (4.2)$$

Here f and g and measurable functions on $\mathbb{R}^d \times A$ satisfying suitable integrability conditions ensuring that the above expectation is well-defined.

The dynamic programming method for solving (4.2) consists first by defining the value function associated to (4.2), that is the maximum value of the objective functional when varying the initial states, and then by deriving an analytic characterization of the value functions in terms of some partial differential equation, the so-called Hamilton-Jacobi-Bellman (HJB) equation. For any initial state $(t, x) \in [0, T] \times \mathbb{R}^d$, control $\alpha \in \mathcal{A}$, we denote $\{X_s^{t,x}, s \geq t\}$, the solution to (4.1) starting from x at time t (we omit the dependence in α for alleviating notations). We then define the value function associated to (4.2) by

$$v(t, x) = \sup_{\alpha \in \mathcal{A}} \mathbb{E} \left[\int_t^T f(X_s^{t,x}, \alpha_s)ds + g(X_T^{t,x}) \right], \quad (t, x) \in [0, T] \times \mathbb{R}^d,$$

so that the original maximization problem is given by $v(0, X_0)$. The derivation of the PDE satisfied by the value function relies on the dynamic programming principle (DPP), which formally states that it is never too late to behave optimally. This means that if we have optimally controlled the state from time θ ($t \leq \theta \leq T$) until T , then by optimizing the control from t until θ , this will provide an optimal decision over the whole interval $[t, T]$. Mathematically, this is written as

$$v(t, x) = \sup_{\alpha \in \mathcal{A}} \mathbb{E} \left[\int_t^\theta f(X_s^{t,x}, \alpha_s)ds + v(\theta, X_\theta^{t,x}) \right], \quad (4.3)$$

for all $(t, x) \in [0, T] \times \mathbb{R}^d$, and $\theta \in [t, T]$. Actually, the DPP (4.3) holds for all θ stopping time valued in $[t, T]$. Notice that this is the continuous-time version of the dynamic programming

relation written in paragraph 3.1. The expression in brackets on the r.h.s. of (4.3) is the utility gained from following the strategy α on $[t, \theta]$ (which results in the integral and in the value $X_\theta^{t,x}$ at time θ) and behaving optimally on the remaining interval $[\theta, T]$ (which results in $v(\theta, X_\theta^{t,x})$). The DPP or Bellman principle says that taking the supremum over the expected value of these expressions yields the value function.

We now derive the infinitesimal version of the DPP (4.3). In a first step, we do it formally by assuming all required smoothness properties, and suitable integrability conditions allowing to interchange limits. We apply Itô's formula to $v(s, X_s^{t,x})$ between $s = t$ and $s = \theta = t + h$ into (4.3), and we obtain :

$$0 = \sup_{\alpha \in \mathcal{A}} \mathbb{E} \left[\int_t^{t+h} f(X_s^{t,x}, \alpha_s) + \frac{\partial v}{\partial t}(s, X_s^{t,x}) + \mathcal{L}^{\alpha_s} v(s, X_s^{t,x}) ds + \int_t^{t+h} D_x v(s, X_s^{t,x})' \sigma(X_s^{t,x}, \alpha_s) dW_s \right],$$

where \mathcal{L}^a is the second-order differential operator

$$\mathcal{L}^a v = b(x, a) \cdot D_x v + \frac{1}{2} \text{tr}(\sigma \sigma'(x, a) D_x^2 v),$$

associated to the diffusion X . By assuming that the stochastic integral in this equation is a martingale (which requires some integrability conditions on the integrand), and by dividing by $h > 0$, we obtain :

$$0 = \sup_{\alpha \in \mathcal{A}} \mathbb{E} \left[\frac{1}{h} \int_t^{t+h} f(X_s^{t,x}, \alpha_s) + \frac{\partial v}{\partial t}(s, X_s^{t,x}) + \mathcal{L}^{\alpha_s} v(s, X_s^{t,x}) ds \right].$$

Taking the limit as h goes to zero, and by the mean-value theorem (assuming that interchanging limits is valid), we arrive at the so-called HJB equation :

$$0 = \sup_{a \in A} \left[\frac{\partial v}{\partial t}(t, x) + \mathcal{L}^a v(t, x) + f(x, a) \right], \quad (t, x) \in [0, T] \times \mathbb{R}^d. \quad (4.4)$$

This PDE is completed with the obvious boundary condition derived directly from the definition of the value function

$$v(T, x) = g(x), \quad x \in \mathbb{R}^d. \quad (4.5)$$

The formal derivation of the HJB equation is justified by a classical verification theorem in stochastic control theory, which states that a smooth solution to the HJB equation (when it exists) coincides with the value function, and that an optimal control can be constructed by looking at the values that yield the supremum in equation (4.4).

Theorem 4.1 (*Verification theorem*)

Suppose that $w \in C^0([0, T] \times \mathbb{R}^d) \cap C^{1,2}([0, T], \mathbb{R}^d)$ is a solution to (4.4)-(4.5), with suitable growth condition on x . Then

$$v(t, x) \leq w(t, x), \quad (t, x) \in [0, T] \times \mathbb{R}^d.$$

Moreover, if there exists a measurable function $\hat{a}(t, x)$ from $[0, T] \times \mathbb{R}^d$ into A , satisfying

$$\hat{a}(t, x) \in \arg \max_{a \in A} \left[\frac{\partial v}{\partial t}(t, x) + \mathcal{L}^a v(t, x) + f(x, a) \right],$$

and s.t. the stochastic differential equation

$$dX_s = b(X_s, \hat{a}(s, X_s))ds + \sigma(X_s, \hat{a}(s, X_s))dW_s,$$

admits an unique solution denoted $\{\hat{X}_s^{t,x}, s \geq t\}$ given an initial state (t, x) , then we have

$$w(t, x) = v(t, x) = \mathbb{E} \left[\int_t^T f(\hat{X}_s^{t,x}, \hat{\alpha}_s) ds + g(\hat{X}_T^{t,x}) \right], \quad (t, x) \in [0, T] \times \mathbb{R}^d,$$

where we set $\hat{\alpha}_s = \hat{a}(s, \hat{X}_s^{t,x})$, $s \geq t$. In particular, $\hat{\alpha}$ is an optimal Markovian control.

This theorem suggests the following method for solving the optimization problem (4.2).

- (i) Solve the HJB equation (4.4)-(4.5).
- (ii) Solve the argmaximum inside the HJB PDE (4.4) to obtain an optimal feedback control.

We state some further useful variants of our optimization problem and the corresponding HJB equations :

- **Infinite time horizon.** In this case, the value function is given by

$$v(x) = \sup_{\alpha \in A} \mathbb{E} \left[\int_0^\infty e^{-\beta s} f(X_s^x, \alpha_s) ds \right], \quad x \in \mathbb{R}^d,$$

where $\beta > 0$ is a discount factor large enough in order to ensure that the value function is finite. The HJB equation to this problem is

$$0 = \sup_{a \in A} \left[-\beta v + \mathcal{L}^a v(x) + f(x, a) \right], \quad x \in \mathbb{R}^d,$$

and we have a analogous verification theorem as in the finite horizon case.

- **Singular control.** In some cases, the supremum arising in the HJB equation may explode, typically when the set A of controls is bounded, and so the HJB equation in the form (4.4) is not well-defined. For example, consider the case where $A = \mathbb{R}_+$, $b(x, a) = 0$, $\sigma(x, a) = ax$, and $f(x, a) = 0$. Such an example is motivated by a financial problem as illustrated in paragraph 4.3. Then, we see that

$$\begin{aligned} & \sup_{a \in A} \left[\frac{\partial v}{\partial t}(t, x) + \mathcal{L}^a v(t, x) + f(x, a) \right] \\ &= \frac{\partial v}{\partial t}(t, x) + \sup_{a \geq 0} \left[\frac{1}{2} a^2 x^2 \frac{\partial^2 v}{\partial x^2}(t, x) \right] \\ &< \infty \quad \text{iff} \quad -\frac{\partial^2 v}{\partial x^2}(t, x) \geq 0, \end{aligned}$$

and in this case :

$$\sup_{a \in A} \left[\frac{\partial v}{\partial t}(t, x) + \mathcal{L}^a v(t, x) + f(x, a) \right] = \frac{\partial v}{\partial t}(t, x).$$

Then, the HJB equation for this singular control problem takes the form of a variational inequality

$$\min \left[-\frac{\partial v}{\partial t}(t, x), -\frac{\partial^2 v}{\partial x^2}(t, x) \right] = 0.$$

More generally, by denoting

$$H(x, p, M) = \sup_{a \in A} \left[b(x, a) \cdot p + \frac{1}{2} \text{tr}(\sigma \sigma'(x, a) M) \right],$$

the so-called Hamiltonian associated to the control problem (4.2), and assuming that the domain of H is in the form

$$\begin{aligned} \text{dom}(H) &:= \left\{ (x, p, M) \in \mathbb{R}^d \times \mathbb{R}^d \times S^d : H(x, p, M) < \infty \right\} \\ &= \left\{ (x, p, M) : G(x, p, M) \geq 0 \right\}, \end{aligned}$$

for some continuous function G , then the corresponding HJB variational inequality is

$$\min \left[-\frac{\partial v}{\partial t}(t, x) - H(x, D_x v(t, x), D_x^2 v(t, x)), G(x, D_x v(t, x), D_x^2 v(t, x)) \right] = 0.$$

We shall give in the next paragraph an example of singular control problem arising in finance.

Remark 4.1 The validity of the verification theorem, which gives sufficient condition for optimality, relies on the fact that there exists a smooth solution to the HJB equation, and that we are able to solve it explicitly. However, it is rare that we can find an explicit smooth solution to this highly nonlinear PDE (some examples are presented in the next paragraph), and moreover, there is not in general a smooth solution to the HJB equation. In recent years, the stochastic control method has been greatly improved by the notion of viscosity solutions to the HJB equation, which allows to overcome the a priori lack of regularity of the value function. The concept of viscosity solutions theory is beyond the scope of these lectures, and we refer the interested reader to Fleming and Soner [4], or more recently Pham [13].

4.2 Merton's portfolio selection problem

Consider an investor who invests at any time a proportion of his wealth in a stock. We suppose that the bond price grows at the constant interest rate r , and the stock price S evolves according to the Black-Scholes model :

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t,$$

where $\mu, \sigma > 0$ are constants, and W is a standard brownian motion. The nonnegative wealth process X controlled by the proportion α , valued in \mathbb{R} , invested in stock is then governed by the diffusion dynamics :

$$\begin{aligned} dX_t &= \alpha_t X_t \frac{dS_t}{S_t} + (1 - \alpha_t) X_t r dt \\ &= (\alpha_t (\mu - r) + r) X_t dt + \alpha_t \sigma X_t dW_t, \end{aligned}$$

and the objective of the investor is to maximize over portfolio strategy α his expected utility of terminal wealth at some finite horizon T . The value function for such control problem is then defined by

$$v(t, x) = \sup_{\alpha \in \mathcal{A}} \mathbb{E}[U(X_T^{t,x})], \quad (t, x) \in [0, T] \times \mathbb{R}_+, \quad (4.6)$$

where U is an increasing and concave function on \mathbb{R}_+ . According to the previous paragraph, the corresponding HJB equation for (4.6) is

$$\frac{\partial v}{\partial t}(t, x) + \sup_{a \in \mathbb{R}} [(a(\mu - r) + r)x \frac{\partial v}{\partial x}(t, x) + \frac{1}{2}a^2\sigma^2x^2 \frac{\partial^2 v}{\partial x^2}(t, x)] = 0, \quad (4.7)$$

for $(t, x) \in [0, T] \times \mathbb{R}_+$, together with the terminal condition

$$v(T, x) = U(x), \quad x \in \mathbb{R}_+. \quad (4.8)$$

Moreover, a candidate for the optimal control is obtained from the first-order condition for the maximum in the HJB equation (4.7) :

$$\hat{a}(t, x) = -\frac{\mu - r}{\sigma^2} \frac{\frac{\partial v}{\partial x}}{x \frac{\partial^2 v}{\partial x^2}}(t, x).$$

As the PDE (4.7) is highly-nonlinear, we cannot expect to get an explicit solution for a general utility function U . We then consider the special case of CRRA utility function

$$U(x) = x^\gamma, \quad 0 < \gamma < 1,$$

and we show that the HJB equation can be explicitly solved. Actually, we conjecture that the value function has the form

$$v(t, x) = \varphi(t)x^\gamma,$$

for some deterministic function φ to be determined. Substituting this form of v into (4.7)-(4.8) leads to an ordinary differential equation (ode) for φ :

$$\begin{aligned} \varphi'(t) + \lambda\varphi(t) &= 0 \\ \varphi(T) &= 1, \end{aligned}$$

where

$$\lambda = \sup_{a \in \mathbb{R}} [(a(\mu - r) + r)\gamma - \frac{1}{2}a^2\sigma^2\gamma(1 - \gamma)] = \frac{(\mu - r)^2}{2\sigma^2} \frac{\gamma}{1 - \gamma} + r\gamma.$$

The solution to this ode is equal to $\varphi(t) = e^{\lambda(T-t)}$, and so $v(t, x) = e^{\lambda(T-t)}x^\gamma$ satisfies the HJB equation (4.7)-(4.8). Moreover, by using

$$\hat{a}(t, x) = -\frac{\mu - r}{\sigma^2} \frac{\frac{\partial v}{\partial x}}{x \frac{\partial^2 v}{\partial x^2}}(t, x) = \frac{\mu - r}{\sigma^2(1 - \gamma)},$$

which is constant, the corresponding stochastic differential equation for the wealth process

$$dX_t = (\hat{\alpha}(\mu - r) + r)X_t dt + \hat{\alpha}\sigma X_t dW_t,$$

is a geometric brownian motion. We conclude with the verification theorem that the value function and the optimal control to Merton's portfolio selection problem with CRRA utility function are given by

$$v(t, x) = e^{\lambda(T-t)} x^\gamma, \quad \hat{\alpha}_t = \frac{\mu - r}{\sigma^2(1 - \gamma)}.$$

4.3 Super-replication cost in an uncertain volatility model

We consider a stock price with dynamics :

$$dX_t = \alpha_t X_t dW_t,$$

where W is a standard brownian motion, and (α_t) is an adapted process, representing the volatility of the stock. With respect to the Black-Scholes model, we do not assume that α is a known constant, and we only suppose that α may take all values in \mathbb{R}_+ . Given an option payoff $g(X_T)$ at maturity T , its superreplication cost is defined as the minimal initial capital, which allows to get a portfolio strategy leading to a terminal wealth dominating (superhedging) the payoff for all possible scenario of the volatility. In the case of constant volatility, the standard Black-Scholes theory states that the super-replication cost is equal to the unique arbitrage price $\mathbb{E}[g(X_T)]$. In our general uncertain volatility model, one shows that the super-replication cost is given by (see El Karoui and Quenez [3]) :

$$v_0 = \sup_{\alpha} \mathbb{E}[g(X_T)].$$

The computation of v_0 fits into the framework of stochastic control, and we define the corresponding value function

$$v(t, x) = \sup_{\alpha} \mathbb{E}[g(X_T^{t,x})], \quad (t, x) \in [0, T] \times \mathbb{R}_+. \quad (4.9)$$

The above stochastic control is singular due to the unboundedness of the control set. Indeed, the supremum in the HJB equation

$$\sup_{\alpha \in \mathbb{R}_+} \left[\frac{\partial v}{\partial t}(t, x) + \frac{1}{2} \alpha^2 x^2 \frac{\partial^2 v}{\partial x^2}(t, x) \right]$$

is finite if and only if

$$\frac{\partial^2 v}{\partial x^2}(t, x) \leq 0, \quad (t, x) \in [0, T] \times (0, \infty).$$

Assuming that v is smooth, this means that $v(t, \cdot)$ is concave on $(0, \infty)$ (actually, this concavity property can be proved rigorously by means of viscosity solutions argument). Moreover, by taking the particular zero control $\alpha = 0$ in (4.9), we immediately see that $v(t, x) \geq g(x)$. Therefore, by denoting \hat{g} as the concave envelope of g , i.e. the smallest

concave function, which is a majorant of g , we deduce that $v(t, \cdot) \geq \hat{g}$ on $(0, \infty)$. On the other hand, we have

$$\begin{aligned} v(t, x) &= \sup_{\alpha} \mathbb{E}[g(X_T^{t,x})] \leq \sup_{\alpha} \mathbb{E}[\hat{g}(X_T^{t,x})] \\ &\leq \sup_{\alpha} \hat{g}(\mathbb{E}[X_T^{t,x}]) = \hat{g}(x), \end{aligned}$$

where we used Jensen's inequality and the property that X is a martingale. We conclude that

$$\hat{v}(t, x) = \hat{g}(x), \quad (t, x) \in [0, T] \times (0, \infty).$$

For example, for a call option $g(x) = (x - K)_+$, we have $\hat{g}(x) = x$, while for a put option $g(x) = (K - x)_+$, we have $\hat{g}(x) = K$. Notice that when \hat{g} is not smooth, then $v = \hat{g}$ is also not smooth and so the classical verification theorem does not apply.

5 Martingale approach to continuous-time portfolio problem

5.1 Utility maximization

As illustrated in the discrete-time example of paragraph 3.2, the main idea of the martingale approach is a decomposition of the portfolio problem into a static optimization problem (determination of the optimal terminal wealth) and a representation problem (find a portfolio strategy that leads to this optimal terminal wealth).

We develop this method in the framework of a continuous-time complete market model. For simplicity of notation, we assume that there is one bond of price process $S_t^0 = 1$, i.e. zero interest rate, and one risky asset (stock) of price process S governed by

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dW_t, \quad 0 \leq t \leq T, \quad (5.1)$$

Here, T is a finite horizon, W is a standard brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with the natural filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ generated by W , and μ, σ are \mathbb{F} -adapted processes. This model is slightly more general than the Black-Scholes-Merton model since we do not assume that the coefficients μ and σ are constant, but only adapted with respect to the filtration generated by W , and satisfying suitable integrability conditions. A portfolio strategy for an investor is a \mathbb{F} -adapted process $\theta = (\theta_t)_t$ representing the number of shares invested in stock at any time; starting from an initial capital $x \geq 0$, the wealth of the investor is then given by

$$X_t^{x, \theta} = x + \int_0^t \theta_u dS_u, \quad 0 \leq t \leq T.$$

We denote by $\Theta(x)$ the set of admissible portfolio strategy θ , i.e. s.t. the corresponding wealth process is nonnegative : $X_t^{x, \theta} \geq 0, 0 \leq t \leq T$. Notice that the investment strategy may be equivalently described in terms of the proportion of wealth invested in stock, i.e. by the quotient $\alpha_t = \frac{\theta_t S_t}{X_t}$, as we did it in the previous paragraph for the Merton's portfolio selection problem.

Given an utility function U from \mathbb{R}_+ into \mathbb{R} satisfying conditions precised later, the utility maximization problem for the investor is

$$v(x) = \sup_{\theta \in \Theta(x)} \mathbb{E}[U(X_T^{x,\theta})], \quad x \geq 0. \quad (5.2)$$

The first step in the martingale approach is to reformulate this dynamic optimization problem into a static one by using a representation theorem for attainable claims. This is formulated precisely as follows. For the complete market model (5.1), there is a unique risk-neutral martingale measure \mathbb{Q} whose density process is given by

$$Z_t = \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \middle| \mathcal{F}_t\right] = \exp\left(-\int_0^t \frac{\mu_u}{\sigma_u} dW_u - \frac{1}{2} \int_0^t \left(\frac{\mu_u}{\sigma_u}\right)^2 du\right), \quad 0 \leq t \leq T. \quad (5.3)$$

Then, any nonnegative claim is attainable by a terminal wealth if and only if its price is equal to the initial capital of the wealth : for any $x \geq 0$, $H \in L_+^0(\mathcal{F}_T)$, the set of nonnegative \mathcal{F}_T -measurable random variables, we have the representation result

$$H = X_T^{x,\theta} \quad \text{for some } \theta \in \Theta(x) \iff \mathbb{E}^{\mathbb{Q}}[H] = \mathbb{E}[Z_T H] = x.$$

The dynamic optimization problem (5.2) is thus decomposed into :

(i) a static optimization problem

$$v(x) = \sup_{H \in \mathcal{H}(x)} \mathbb{E}[U(H)], \quad \mathcal{H}(x) := \{H \in L_+^0(\mathcal{F}_T) : \mathbb{E}[Z_T H] = x\}. \quad (\mathcal{S})$$

(ii) a representation problem

$$\begin{aligned} &\text{find a portfolio strategy } \theta^* \in \Theta(x) \quad \text{s.t.} \\ &X_T^{x,\theta^*} = H^*, \quad \text{where } H^* \text{ solves } (\mathcal{S}). \end{aligned} \quad (\mathcal{R})$$

We now focus on the solutions to these two sub-problems (\mathcal{S}) and (\mathcal{R}) .

• Solution to the static optimization problem (\mathcal{S})

Problem (\mathcal{S}) is a convex optimization problem with linear constraint, and may then be solved by Lagranger multiplier dual methods from convex analysis. The heuristics is the following. We consider the Lagrangian of problem (\mathcal{S}) by defining the function

$$\mathcal{L}(H, y) = \mathbb{E}[U(H)] - y(\mathbb{E}[Z_T H] - x), \quad H \in L_+^0(\mathcal{F}_T), \quad y \geq 0,$$

and we seek a zero of the gradient of $\mathcal{L}(H, y)$. This leads formally to the equations

$$\begin{aligned} \mathbb{E}[U'(H) - yZ_T] &= 0 \\ \mathbb{E}[Z_T H] - x &= 0. \end{aligned}$$

Hence, one looks at an H of the form

$$H = (U')^{-1}(yZ_T) := I(yZ_T)$$

with a positive y s.t.

$$\mathbb{E}^{\mathbb{Q}}[H] = \mathbb{E}[Z_T I(yZ_T)] = x,$$

and one would expect that such an H is solution to (\mathcal{S}) .

To make our heuristics considerations rigorous, we need to require some conditions on the utility function. We assume that U is C^1 , strictly concave on $(0, \infty)$, and satisfies the so-called Inada conditions

$$U'(0) := \lim_{x \rightarrow 0} U'(x) = \infty, \quad U'(\infty) = \lim_{x \rightarrow \infty} U'(x) = 0.$$

We denote by $I = (U')^{-1}$ the inverse of U' , which is then one to one strictly decreasing from $(0, \infty)$ into $(0, \infty)$. We have then the key relation

$$\sup_{x \geq 0} [U(x) - xy] = U(I(y)) - yI(y), \quad \forall y \geq 0. \quad (5.4)$$

Fix now some $x \geq 0$. Thus, for any $H \in \mathcal{H}(x)$, $y \geq 0$, we have from (5.4)

$$\mathbb{E}[U(H)] \leq \mathbb{E}[U(I(yZ_T))] - y(\mathbb{E}[Z_T I(yZ_T)] - x). \quad (5.5)$$

We shall assume that $\mathbb{E}[Z_T I(yZ_T)] < \infty$ for all $y \geq 0$. Then, since $I(0) = \infty$, and $I(\infty) = 0$, there exists an unique $y^* = y^*(x) > 0$ s.t.

$$\mathbb{E}[Z_T I(y^* Z_T)] = x. \quad (5.6)$$

Therefore, by setting

$$H^* = I(y^* Z_T) \in \mathcal{H}(x), \quad (5.7)$$

we deduce from (5.5) that

$$v(x) = \sup_{H \in \mathcal{H}(x)} \mathbb{E}[U(H)] = \mathbb{E}[U(H^*)],$$

which shows that H^* is solution to (\mathcal{S}) . In summary, the resolution to the static optimization problem consists simply of finding the unique risk-neutral martingale measure with density Z_T , determining the Lagrange multiplier y^* solution to (5.6) so that the solution to (\mathcal{S}) is given by (5.7).

- Computation of the optimal strategy in the representation problem (\mathcal{R})

Denoting by $H^* \in \mathcal{H}(x)$ the solution to problem (\mathcal{S}) , we now want to compute a portfolio strategy $\theta^* \in \Theta(x)$ s.t. the corresponding wealth process X^{x, θ^*} satisfies

$$X_T^{x, \theta^*} = H^* = I(y^* Z_T) \quad a.s.$$

Since a wealth process is a martingale under the risk-neutral martingale measure, the optimal wealth process is then given by

$$X_t^{x, \theta^*} = \mathbb{E}^{\mathbb{Q}}[I(y^* Z_T) | \mathcal{F}_t] =: \hat{M}_t \quad 0 \leq t \leq T,$$

and we notice in particular for $t = 0$, that $x = X_0^{x, \theta^*} = \mathbb{E}^{\mathbb{Q}}[I(y^* Z_T)] = \hat{M}_0$. The general method for determining the optimal portfolio strategy θ^* consists then of computing the \mathbb{Q} -martingale \hat{M}_t , and write it by derivation in terms of the Brownian motion $\hat{W} = W + \int \frac{\mu}{\sigma} dt$ under the risk-neutral martingale measure \mathbb{Q} :

$$d\hat{M}_t = \phi_t d\hat{W}_t, \quad (5.8)$$

and then to identify with the general form for the dynamics of the wealth process

$$dX_t^{x,\theta^*} = \theta_t^* dS_t = \theta_t^* S_t \sigma_t d\hat{W}_t, \quad (5.9)$$

so that

$$\theta_t^* = \frac{\phi_t}{\sigma_t S_t}, \quad 0 \leq t \leq T.$$

In some particular cases (see Examples below), the derivation (5.8) can be made explicit with Itô's formula. In the general case, the representation (5.8) is not explicit and involves Clark-Ocone formula.

Example 5.3 (Logarithmic utility)

We consider an utility function of the form

$$U(x) = \ln x.$$

One easily verify that $I(y) = 1/y$, $y > 0$. The optimal wealth process is then given by

$$\begin{aligned} X_t^{x,\theta^*} &= \mathbb{E}^{\mathbb{Q}} \left[\frac{1}{y^* Z_T} \middle| \mathcal{F}_t \right] = \mathbb{E} \left[\frac{Z_T}{Z_t} \frac{1}{y^* Z_T} \middle| \mathcal{F}_t \right] \\ &= \frac{1}{y^* Z_t} =: \hat{M}_t, \quad 0 \leq t \leq T, \end{aligned}$$

where y^* is s.t. $X_0^{x,\theta^*} = x$, and so $y^* = 1/x$. Hence, $X_t^{x,\theta^*} = x/Z_t$, $0 \leq t \leq T$. Now, recalling the expression (5.3) of Z , we have by Itô's formula :

$$d\hat{M}_t = \hat{M}_t \frac{\mu_t}{\sigma_t} d\hat{W}_t.$$

Therefore, by identifying with the dynamics (5.9) of the wealth process, we obtain the optimal portfolio strategy in terms of number of shares :

$$\theta_t^* = \frac{\mu_t X_t^{x,\theta^*}}{\sigma_t^2 S_t}, \quad 0 \leq t \leq T,$$

or equivalently in proportion :

$$\alpha_t^* = \frac{\theta_t^* S_t}{X_t^{x,\theta^*}} = \frac{\mu_t}{\sigma_t^2}, \quad 0 \leq t \leq T.$$

Example 5.4 (Power utility function and deterministic coefficients)

We consider an utility function of the form

$$U(x) = \frac{x^\gamma}{\gamma}, \quad 0 < \gamma < 1,$$

and we assume that the ratio $\lambda_t := \mu_t/\sigma_t$ is deterministic. This is a slight extension of the Merton's portfolio selection model as described in paragraph 4.2. One easily check that

$I(y) = y^{-\delta}$ with $\delta = 1/(1 - \gamma)$. The optimal wealth process is then given by

$$\begin{aligned} X_t^{x,\theta^*} &= \mathbb{E}^{\mathbb{Q}}\left[(y^*)^{-\delta} Z_T^{-\delta} | \mathcal{F}_t\right] = (y^*)^{-\delta} \mathbb{E}^{\mathbb{Q}}\left[\exp\left(\int_0^T \lambda_t \delta d\hat{W}_t - \frac{1}{2} \int_0^T \lambda_t^2 \delta^2 dt\right) | \mathcal{F}_t\right] \\ &= (y^*)^{-\delta} \exp\left(\frac{1}{2} \int_0^T \lambda_t^2 \delta(\delta - 1) dt\right) \mathbb{E}^{\mathbb{Q}}\left[\exp\left(\int_0^T \lambda_t \delta d\hat{W}_t - \frac{1}{2} \int_0^T \lambda_t^2 \delta^2 dt\right) | \mathcal{F}_t\right] \\ &= (y^*)^{-\delta} \exp\left(\frac{1}{2} \int_0^T \lambda_t^2 \delta(\delta - 1) dt\right) \exp\left(\int_0^t \lambda_u \delta d\hat{W}_u - \frac{1}{2} \int_0^t \lambda_u^2 \delta^2 du\right), \quad 0 \leq t \leq T. \end{aligned}$$

By writing that $X_0^{x,\theta^*} = x$, we get $x = (y^*)^{-\delta} \exp\left(\frac{1}{2} \int_0^T \lambda_t^2 \delta(\delta - 1) dt\right)$, and so

$$X_t^{x,\theta^*} = x \exp\left(\int_0^t \lambda_u \delta d\hat{W}_u - \frac{1}{2} \int_0^t \lambda_u^2 \delta^2 du\right) =: \hat{M}_t, \quad 0 \leq t \leq T.$$

By Itô's formula, we have

$$d\hat{M}_t = \hat{M}_t \lambda_t \delta d\hat{W}_t.$$

Therefore, by identifying with the dynamics (5.9) of the wealth process, we obtain the optimal portfolio strategy in terms of number of shares :

$$\theta_t^* = \frac{\lambda_t \delta \hat{M}_t}{\sigma_t S_t} = \frac{\mu_t}{(1 - \gamma) \sigma_t^2} \frac{X_t^{x,\theta^*}}{S_t}, \quad 0 \leq t \leq T,$$

or equivalently in proportion :

$$\alpha_t^* = \frac{\theta_t^* S_t}{X_t^{x,\theta^*}} = \frac{\mu_t}{(1 - \gamma) \sigma_t^2}, \quad 0 \leq t \leq T.$$

This generalizes the result obtained by Bellman method in paragraph 4.2 for constant coefficients.

5.2 Value at risk hedging criterion

In the context of complete market model as in the Black-Scholes model, the price of any contingent claim is equal to its replication cost, that is the initial capital allowing to construct a portfolio strategy whose terminal wealth replicates at maturity its payoff. This price is computed as the expectation of the (discounted) claim under the unique risk-neutral martingale measure. In particular, by putting up an initial capital at least equal to the cost of replication, we can stay on the safe side with a portfolio strategy whose terminal wealth superhedges the payoff of the option.

What if the investor is unwilling to put up the initial amount required by the replication? What is the maximal probability of a successful hedge the investor can achieve with a given smaller amount? Equivalently, one can ask how much initial capital an investor can save by accepting a certain shortfall probability, i.e. by being willing to take the risk of having to supply additional capital at maturity in e.g. 1% of the cases. This question corresponds to the familiar ‘‘Value at Risk’’ (VaR) concept, which is very popular among investors and

practioners. Just as in VaR a certain level of security (e.g. 99%) is chosen, and the investor is looking for the most efficient allocation of capital.

We formulate and solve this problem of VaR hedging criterion introduced by Föllmer and Leukert [5] in the framework and notations described in paragraph 5.1. Given a contingent claim represented by a nonnegative \mathcal{F}_T -measurable random variable H , and for some fixed initial capital x strictly smaller than the price of the claim, i.e. $x < \mathbb{E}^{\mathbb{Q}}[H]$, we are looking for a strategy $\theta \in \Theta(x)$ such that

$$\mathbb{P}[X_T^{x,\theta} \geq H] = \max \quad (5.10)$$

The set $\{X_T^{x,\theta} \geq H\}$ is called the success set associated to a strategy $\theta \in \Theta(x)$. As in the martingale approach for utility maximization, we show in a first step that the dynamical VaR problem (5.10) may be reduced to a static problem involving the construction of a success set of maximal probability.

Proposition 5.1 *Let $A^* \in \mathcal{F}_T$ be a solution of the problem*

$$\mathbb{P}[A] = \max \quad (5.11)$$

under the constraint

$$\mathbb{E}^{\mathbb{Q}}[H1_A] \leq x. \quad (5.12)$$

Then, the replicating strategy θ^ for the knock-out option*

$$H^* := H1_{A^*}$$

solves the VaR optimization problem (5.10), and the corresponding success set coincides almost surely with A^ .*

Proof. *Step 1.* Let $\theta \in \Theta(x)$ be an admissible strategy, and denote by $A = \{X_T^{x,\theta} \geq H\}$ its corresponding success set. Then, we have

$$X_T^{x,\theta} \geq H1_A \text{ a.s.}$$

since $X_T^{x,\theta} \geq 0$ by admissibility, and so by recalling that the wealth process is a \mathbb{Q} -martingale

$$\mathbb{E}^{\mathbb{Q}}[H1_A] \leq \mathbb{E}^{\mathbb{Q}}[X_T^{x,\theta}] = x.$$

Hence, A satisfies the constraint (5.12), which implies by (5.11)

$$\mathbb{P}[A] = \mathbb{P}[X_T^{x,\theta} \geq H] \leq \mathbb{P}[A^*].$$

Step 2. Let us consider the replicating strategy θ^* of $H^* = H1_{A^*}$ and notice that $\theta^* \in \Theta(x)$ since the corresponding weath process $X_t^{x,\theta^*} = \mathbb{E}^{\mathbb{Q}}[H^*|\mathcal{F}_t]$ is nonnegative. Its success set satisfies

$$\{X_T^{x,\theta^*} \geq H\} = \{H1_{A^*} \geq H\} \supseteq A^*.$$

On the other hand, Step 1 of the proof yields that

$$\mathbb{P}[X_T^{x,\theta^*} \geq H] \leq \mathbb{P}[A^*].$$

It follows that the two sets A^* and $\{X_T^{x,\theta^*} \geq H\}$ coincide up to \mathbb{P} -null sets. In particular, θ^* is an optimal strategy for (5.10). \square

The problem of constructing a maximal success set A^* is now solved by using the Neyman-Pearson lemma in statistical test theory. To this end, we introduce the measure \mathbb{Q}^* given by

$$\frac{d\mathbb{Q}^*}{d\mathbb{Q}} = \frac{H}{\mathbb{E}^{\mathbb{Q}}[H]}.$$

The static optimization problem of maximal success set is then rewritten as

$$\mathbb{P}[A] = \max$$

under the constraint

$$\mathbb{Q}^*[A] \leq \alpha := \frac{x}{\mathbb{E}^{\mathbb{Q}}[H]}.$$

This is known in statistical test theory as the Neyman-Pearson test of the null hypothesis \mathbb{Q}^* against the alternative hypothesis \mathbb{P} at the size α . The solution is constructed as follows. Define the level

$$c^* = \inf \left\{ c \geq 0 : \mathbb{Q}^* \left[\frac{d\mathbb{P}}{d\mathbb{Q}^*} > c \right] \leq \alpha \right\}, \quad (5.13)$$

and the set

$$A^* = \left\{ \frac{d\mathbb{P}}{d\mathbb{Q}^*} > c^* \right\} = \left\{ \frac{d\mathbb{P}}{d\mathbb{Q}} > a^* H \right\}, \quad \text{with } a^* = \frac{c^*}{\mathbb{E}^{\mathbb{Q}}[H]}. \quad (5.14)$$

If the set A^* satisfies

$$\mathbb{Q}^*[A^*] = \alpha, \quad \text{i.e. } \mathbb{E}^{\mathbb{Q}}[H1_{A^*}] = x, \quad (5.15)$$

then A^* is solution to the Neyman-Pearson test, i.e. (5.11)-(5.12). Indeed, let $A \in \mathcal{F}_T$ s.t. $\mathbb{Q}^*[A] \leq \alpha$. By definition of A^* , we then get

$$(1_{A^*} - 1_A) \left(\frac{d\mathbb{P}}{d\mathbb{Q}^*} - c^* \right) \geq 0, \quad \mathbb{Q}^* \text{ a.s.}$$

and so

$$\begin{aligned} \mathbb{P}[A^*] - \mathbb{P}[A] &= \mathbb{E}^{\mathbb{Q}^*} \left[(1_{A^*} - 1_A) \frac{d\mathbb{P}}{d\mathbb{Q}^*} \right] \\ &\geq \mathbb{E}^{\mathbb{Q}^*} [(1_{A^*} - 1_A)c^*] = c(\mathbb{Q}^*[A^*] - \mathbb{Q}^*[A]) = c(\alpha - \mathbb{Q}^*[A]) \geq 0, \end{aligned}$$

which shows that A^* solves the Neyman-Pearson test. From Proposition 5.1, we conclude that the optimal strategy of (5.10) is given by the replicating strategy of $H^* = H1_{A^*}$. Hence, the problem of VaR hedging is solved by hedging a suitable knock-out option.

Remark 5.1 The solution to problem (5.11)-(5.12) relies on the condition that the set A^* of (5.14) satisfies (5.15). This condition is clearly satisfied whenever the function $c \rightarrow \mathbb{Q}^* \left[\frac{d\mathbb{P}}{d\mathbb{Q}^*} > c \right]$ is continuous at c^* , i.e. $\mathbb{Q}^* \left[\frac{d\mathbb{P}}{d\mathbb{Q}^*} = c^* \right] = 0$. Since \mathbb{Q}^* is absolutely continuous with respect to \mathbb{P} , it suffices to check that

$$\mathbb{P} \left[\frac{d\mathbb{P}}{d\mathbb{Q}} = a^* H \right] = 0.$$

Remark 5.2 In formulation (5.10) of the VaR hedging criterion for a claim H , we are given some initial capital x (smaller than the price of the claim), and we are looking for the maximal success set

$$v(x) = \max_{\theta \in \Theta(x)} \mathbb{P}[X_T^{x,\theta} \geq H].$$

Equivalently, we may consider a given shortfall probability $\varepsilon \in (0, 1)$, and we are looking for the least amount of initial capital which allows us to stay on the safe side with probability $1 - \varepsilon$, i.e. we want to determine the minimal initial capital x s.t. there exists an admissible strategy $\theta \in \Theta(x)$ with

$$\mathbb{P}[X_T^{x,\theta} \geq H] \geq 1 - \varepsilon.$$

Then, this minimal capital is given by

$$x^* = \inf\{x \geq 0 : v(x) \geq 1 - \varepsilon\}.$$

Example 5.5 (VaR hedging in the Black-Scholes model)

We consider a geometric Brownian motion for the stock price

$$S_t = S_0 \exp \left(\left(\mu - \frac{1}{2} \sigma^2 \right) t + \sigma W_t \right),$$

and for simplicity, we set the interest rate equal to zero. We recall that the unique risk-neutral martingale measure \mathbb{Q} is given by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp \left(- \frac{\mu}{\sigma} W_T - \frac{1}{2} \left(\frac{\mu}{\sigma} \right)^2 T \right).$$

Notice that we can also write

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \text{const.} S_T^{-\mu/\sigma^2}. \tag{5.16}$$

A call option $H = (S_T - K)_+$ can be perfectly replicated from the initial capital given by the famous Black-Scholes formula

$$\mathbb{E}^{\mathbb{Q}}[H] = S_0 N(d_1) - K N(d_2),$$

where

$$d_1 = \frac{1}{\sigma \sqrt{T}} \ln(S_0/K) + \frac{1}{2} \sigma \sqrt{T}, \quad d_2 = d_1 - \sigma \sqrt{T},$$

and $N(\cdot)$ is the distribution function of the standard normal random variable.

Now, suppose we start from an initial capital x smaller than the Black-Scholes price, and we want to maximize the probability of success set. From the above result, the optimal strategy consists in replicating a knock-out option $H1_A$ where the success set A is of the form

$$A = \left\{ \frac{d\mathbb{P}}{d\mathbb{Q}} > \text{const.}H \right\}.$$

Due to (5.16), we can write

$$A = \{S_T^{\mu/\sigma^2} > a(S_T - K)_+\},$$

for some constant a chosen s.t.

$$\mathbb{E}^{\mathbb{Q}}[H1_A] = x. \tag{5.17}$$

We distinguish two cases.

- *Case (i) : $\mu \leq \sigma^2$.*

In this case, the success set takes the form

$$A = \{S_T < c\} = \{W_T < b\},$$

where

$$c = S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma b\right).$$

Hence, the knock-out option $H1_A$ can be written as

$$H1_A = (S_T - K)_+ 1_{\{S_T < c\}} = (S_T - K)_+ - (S_T - c)_+ - (c - K)1_{\{S_T > c\}},$$

which is a combination of two call options and of a binary option. We then calculate the maximal probability of success set

$$\mathbb{P}[A] = N(b/\sqrt{T}),$$

with a constant b determined from the condition (5.17) written explicitly as :

$$\begin{aligned} x = \mathbb{E}^{\mathbb{Q}}[H1_A] &= \mathbb{E}^{\mathbb{Q}}[(S_T - K)_+] - \mathbb{E}^{\mathbb{Q}}[(S_T - c)_+] - (c - K)\mathbb{E}^{\mathbb{Q}}[1_{\{S_T > c\}}] \\ &= S_0 N(d_1) - K N(d_2) - S_0 N\left(\frac{-b - \frac{\mu}{\sigma}T + \sigma T}{\sqrt{T}}\right) + K N\left(\frac{-b - \frac{\mu}{\sigma}T}{\sqrt{T}}\right). \end{aligned}$$

- *Case (ii) : $\mu > \sigma^2$.*

In this case, the success set takes the form

$$A = \{S_T < c_1\} \cup \{S_T > c_2\} = \{W_T < b_1\} \cup \{W_T > b_2\},$$

for some $c_1 < c_2$. Hence, the knock-out option $H1_A$ can be written again as a combination of call options and digital options. We calculate the maximal probability of success set

$$\mathbb{P}[A] = N(b_1/\sqrt{T}) + N(-b_2/\sqrt{T}),$$

with constants b_1 and b_2 determined from the condition (5.17), which can be written explicitly (after some straightforward computations) as :

$$\begin{aligned} x &= \mathbb{E}^{\mathbb{Q}}[H1_A] \\ &= S_0N(d_1) - KN(d_2) - S_0N\left(\frac{-b_1 - \frac{\mu}{\sigma}T + \sigma T}{\sqrt{T}}\right) + KN\left(\frac{-b_1 - \frac{\mu}{\sigma}T}{\sqrt{T}}\right) \\ &\quad + S_0N\left(\frac{-b_2 - \frac{\mu}{\sigma}T + \sigma T}{\sqrt{T}}\right) - KN\left(\frac{-b_2 - \frac{\mu}{\sigma}T}{\sqrt{T}}\right). \end{aligned}$$

We illustrate the amount of initial capital that can be saved by accepting a certain shortfall probability, see Remark 5.2. Let us consider the following numerical example : $T = 0,25$ (i.e. 3 months), $\sigma = 0,3$, $\mu = 0,08$ ($< \sigma^2$), $S_0 = 100$, $K = 110$. For the values $\varepsilon = 1\%$, 5% , 10% , we compute the corresponding proportions $x/\mathbb{E}^{\mathbb{Q}}[H]$ given respectively by 89% , 59% , 34% . Thus, if we are ready to accept a shortfall probability of 5% , we can reduce the initial capital by 41% .

6 Conclusion

The portfolio problems studied in these notes were mostly formulated in a complete continuous-time market model, typically the Black-Scholes model. This simplifying assumption of market model is relaxed and refined in recent and numerous works to make models more realistic by considering various market imperfections. An important extension, both from a theoretical and practical viewpoint, of Merton's model and method was studied by Davis and Norman [2]. The martingale approach to portfolio optimization was largely developed and extended in the literature to the context of general incomplete models, in the presence of portfolio constraints or with transaction costs. We refer to Schachermayer [15] for a survey of related results.

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