Interest Rate Modeling

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Main reference :

Tomas Björk, *Arbitrage Theory in Continuous Time*, Oxford University Press, 2nd Ed. 2004. (Chapters 20-25).

Additional reference :

D. Brigo, and F. Mercurio, *Interest Rate Models - Theory and Practice*, Springer Verlag, Berlin, Heidelberg, 2nd Ed. 2006.

Basic concepts

- $T-$ bonds as main actors : *a contract guaranteeing a unit amount at a given maturity* $T > 0$

- $p(t, T)$: price, at $t \leq T$, of a $T-$bond

Assumptions :

i) $\exists\ T-$ bonds for every $T > 0$;

ii) $p(t, t) = p(T, T) = 1$;

iii) $\forall t,\ \ p(t, T)$ is differentiable in T.

To introduce interest rates starting from $T-$bonds consider the scheme :

sell a $T-$bond

$$- - - - - -| - - - - - - \quad - - | - - \quad - - | \longrightarrow$$

t $\qquad$ T $\qquad$ S

collect $p(t,T)$ and $\qquad$ pay 1 $\qquad$ collect $\frac{p(t,T)}{p(t,S)}$

buy S-bonds in no. of $\frac{p(t,T)}{p(t,S)}$

The scheme concerns a contract, in t, that refers to a unitary investment in T for the period $[T,S]$.

- $L(t;T,S)$: *simple forward rate for* $[T,S]$ *contracted at* t (LIBOR rate)

$$\longrightarrow \quad 1+(S-T)\,L(t;T,S)=\frac{p(t,T)}{p(t,S)}$$

$$\Rightarrow \quad L(t;T,S)=\frac{p(t,T)-p(t,S)}{(S-T)\,p(t,S)}=\frac{1}{S-T}\left\{\frac{p(t,T)}{p(t,S)}-1\right\}$$

- $f(t,T)$: *instantaneous forward rate with maturity T contracted at t* (rate, contracted in t for an instantaneous investment at T)

$$\rightarrow \qquad f(t,T) = \lim_{\Delta \downarrow 0} L(t;T,T+\Delta)$$

$$= \lim_{\Delta \downarrow 0} \frac{1}{\Delta} \frac{p(t,T) - p(t,T+\Delta)}{p(t,T+\Delta)} = -\frac{\partial}{\partial T} \log p(t,T)$$

$$\rightarrow \quad p(t,T) = \exp\left[-\int_t^T f(t,u)du\right]$$

- $r_t := f(t,t)$: *instantaneous short (spot) rate at t*

- *Money market account*

  (An investment that continuously matures and is reinvested at the current short rate r_t).

$$
\begin{aligned}
B_{t+\Delta} &= B_t(1+\Delta r_t) \\
&\downarrow \quad {\scriptstyle \Delta\downarrow 0} \\
dB_t &= B_t r_t dt \\
&\downarrow \\
B_T &= B_t \exp\left[\int_t^T r_s ds\right]
\end{aligned}
$$

- *Absence of arbitrage implies that*

$$\frac{p(t,T)}{B_t} = E^Q\left\{\frac{1}{B_T} \mid \mathcal{F}_t\right\}$$

$$\longrightarrow \quad p(t,T) = E^Q\left\{e^{-\int_t^T r_s ds} \mid \mathcal{F}_t\right\}$$

$\longrightarrow$ If r_t is deterministic, then

$$p(t,T) = e^{-\int_t^T r_s ds} = \frac{B_t}{B_T}$$

→ Notice that

$$D(t,T) := \exp\left[-\int_t^T r_s ds\right]$$

is the value in t of a unit amount in T *(from an amount $D(t,T)$ in t one arrives at 1 in T with a continuously compounded investment at the current short rate r_t)* and can thus be interpreted as *(stochastic) "discount factor"*

→ *On the other hand, $p(t,T)$ is the value in t of a contract that concerns a unitary amount in T.*

→ From the above one has that, if r_t is deterministic, then $D(t,T) = p(t,T)$.

- *One now has the following scheme*

$$
\begin{array}{ccc}
p(t,T) & \longleftrightarrow & f(t,T) \\
 & \Downarrow & \\
\text{dynamics for } p & \longleftrightarrow & \text{dynamics for } f \\
dp & & df
\end{array}
$$

Exemplifying the implication $df \;\longrightarrow\; dp$ let

$$df(t,T) = \alpha(t,T) + \sigma(t,T)dw_t$$

Putting

$$Y(t,T) := -\int_t^T f(t,s)ds$$

one has

$$p(t,T) = \exp\{Y(t,T)\}$$

with

$$dY(t,T) = f(t,t)dt - \int_t^T df(t,s)ds$$

Now we have

$$-\int_t^T df(t,s)ds = -\int_t^T \alpha(t,s)dtds - \int_t^T \sigma(t,s)dw_t ds$$

$$= -\left(\int_t^T \alpha(t,s)ds\right)dt - \left(\int_t^T \sigma(t,s)ds\right)dw_t$$

$$:= A(t,T)dt + S(t,T)dw_t$$

$$\rightarrow \quad dY(t,T) = r_t dt + A(t,T)dt + S(t,T)dw_t$$

and, finally,

$$dp(t,T) =$$

$$= p(t,T)\left[r_t + A(t,T) + \frac{1}{2}S^2(t,T)\right]dt + S(t,T)dw_t$$

Short rate models

$r_t = f(t,t)$ a scalar process modeled as a Markov diffusion

$$dr_t = \mu(t, r_t)dt + \sigma(t, r_t)dw_t$$

$$r_t \qquad \stackrel{?}{\Longrightarrow} \qquad \begin{cases} p(t,T) \\ f(t,T) \end{cases}$$

$$\uparrow \qquad\qquad\qquad\qquad \uparrow$$

finite dim. "factor" $\qquad\qquad$ $\infty -$ dim.

- Postulate

$$p(t,T) = F^T(t, r_t)$$

i.e. the bond is seen as a "derivative" with underlying r_t.

- How to determine $F^T(t, r_t)$?

  $\longrightarrow$ Use *Principle of absence of arbitrage (AOA)*

  $\longrightarrow$ Since the "underlying" r_t has no market, pricing is not relative to r_t but relative to other bonds.

Idea : Take two bonds

$$p(t,T) = F^T(t,r_t) \quad ; \quad p(t,S) = F^S(t,r_t)$$

and with them form a self financing portfolio that is locally riskless, i.e. such that

$$dV_t = V_t k_t dt$$

$\rightarrow$ *Since we have already the locally riskless asset*

$$dB_t = B_t r_t dt$$

absence of arbitrage implies $k_t = r_t$*.*

- How to *implement this idea* ?

- By Ito's formula

$$dF^T(t, r_t) = F^T(t, r_t)\left[\alpha^T(t, r_t)dt + \sigma^T(t, r_t)dw_t\right]$$

with

$$\begin{cases} \alpha^T(\cdot) & = & \frac{F_t^T(\cdot)+\mu(\cdot)F_r^T(\cdot)+\frac{1}{2}\sigma^2(\cdot)F_{rr}^T(\cdot)}{F^T(\cdot)} \\ \\ \sigma^T(\cdot) & = & \frac{\sigma(\cdot)F_r^T(\cdot)}{F^T(\cdot)} \end{cases}$$

- Doing the same for $p(t,S) = F^S(t,r_t)$ it is possible to form a self financing portfolio for which

$$dV_t = V_t \left[\frac{\alpha^S(\cdot)\sigma^T(\cdot) - \alpha^T(\cdot)\sigma^S(\cdot)}{\sigma^T(\cdot) - \sigma^S(\cdot)} \right] dt$$

$$\rightarrow \quad \frac{\alpha^S(\cdot)\sigma^T(\cdot) - \alpha^T(\cdot)\sigma^S(\cdot)}{\sigma^T(\cdot) - \sigma^S(\cdot)} = r_t$$

$$\rightarrow \quad \frac{\alpha^S(t,r_t) - r_t}{\sigma^S(t,r_t)} = \frac{\sigma^T(t,r_t) - r_t}{\sigma^T(t,r_t)} := \lambda(t,r_t)$$

- For any T we thus obtain for the corresponding $F^T(t,r)$ the following *Term structure equation*

$$\begin{cases} F_t^T(\cdot) + (\mu(\cdot) - \lambda(\cdot)\sigma(\cdot))F_r^T(\cdot) + \frac{1}{2}\sigma^2(\cdot)F_{rr}^T(\cdot) - rF^T(\cdot) = 0 \\ F^T(T,r) = 1 \end{cases}$$

- $\lambda(t,r)$: *market price of* (interest rate) *risk :* not *determined* endogenously by the model, but *by the market*.

- One thus has

$$\lambda(t,r) \quad \longrightarrow \quad F^T(t,r;\lambda)$$

- *Let* $\bar{p}(t,T_i)$, $i = 1,\cdots,K$ *be prices observed on the market.*

$\rightarrow$ *Calibration :* choose λ^* s.t., possibly

$$F^{T_i}(t,r;\lambda^*) \approx \bar{p}(t,T_i)\ ,\ i=1,\cdots,K$$

- *By Feynman-Kac*

$$F^T(t, r; \lambda) = E^Q \left\{ \exp\left[-\int_t^T r_s ds \right] \mid r_t = r \right\}$$

$$(Q) \qquad dr_t = (\mu(t, r_t) - \lambda(t, r_t)\sigma(t, r_t))dt + \sigma(t, r_t)dw_t^Q$$

$\longrightarrow$ *What matters is*

$$a(t, r) := \mu(t, r) - \lambda(t, r)\sigma(t, r)$$

Martingale models for the short rate

(Dynamics of r directly under the martingale measure)

$$dr_t = a(t, r_t; \theta)dt + \sigma(t, r_t; \theta)\, dw_t^Q$$

θ : an (infinite-dimensional) parameter to be *calibrated to the market*.

- *Calibration* (with respect to the "derivative" prices) then becomes

$$\theta \quad \longrightarrow \quad F^T(t, r; \theta)$$

$\longrightarrow$ Choose θ^* s.t., possibly,

$$F^{T_i}(t, r; \theta^*) \approx \bar{p}(t, T_i) \ , \quad i = 1, \cdots, K$$

$\longrightarrow$ *Interest in having models for which* $F^T(t, r; \theta)$ *can be more easily determined.*

(Exponentially) affine term structure models

These are models for which

$$p(t,T) = \exp[A(t,T) - B(t,T)r_t]$$

One has that, if

$$dr_t = a(t, r_t; \theta)dt + \sigma(t, r_t; \theta)dw_t^Q$$

$$\textit{with} \quad \begin{cases} a(t, r; \theta) & = \alpha_t r + \beta_t \\ \sigma^2(t, r_t; \theta) & = \gamma_t r + \delta_t \end{cases}$$

then the term structure (TS) is affine and

$$\theta = [\alpha_t, \beta_t, \gamma_t, \delta_t]$$

- *If the term structure is affine, then the solution of the term structure equation reduces to :*

i) the solution of a *Riccati-type equation to obtain* $B(t,T)$, i.e.

$$\begin{cases} B_t(t,T) + \alpha_t B(t,T) - \frac{1}{2}\gamma_t B^2(t,T) = -1 \\ \\ B(T,T) = 0 \end{cases}$$

ii) *an integration to obtain* $A(t,T)$, i.e.

$$\begin{cases} A_t(t,T) = \beta_t B(t,T) - \frac{1}{2}\delta_t B^2(t,T) \\ \\ A(T,T) = 0 \end{cases}$$

Example : Ho-Lee model

$$dr_t = \Phi_t dt + \sigma dw_t^Q \quad (\sigma \text{ supposed known})$$

$$\begin{cases} B_t(t,T) &=& -1 \\ B(T,T) &=& 0 \end{cases} \quad \Rightarrow \quad B(t,T) = T - t$$

$$\begin{cases} A_t(t,T) &=& \Phi_t B(t,T) - \frac{1}{2}\sigma^2 B^2(t,T) \\ A(T,T) &=& 0 \end{cases}$$

$$\Rightarrow \quad A(t,T) = \int_t^T \Phi_s \cdot (s-T)ds + \frac{\sigma^2}{2}\frac{(T-t)^3}{3}$$

Since

$$f(t,T) = -\frac{\partial}{\partial T}\log p(t,T) = -A_T(t,T) + B_T(t,T)r_t$$

$$\Rightarrow \quad f^*(0,T) = \int_0^T \Phi_s ds - \frac{\sigma^2 T^2}{2} + r_0$$

and so

$$\Phi_t = \frac{\partial f^*(0,t)}{\partial T} + \sigma^2 t$$

Calibration (Issues)

Purpose : Obtaining a possibly good fit

$$F^{T_i}(t, r; \theta^*) \approx \bar{p}(t, T_i)\ , \quad i = 1, \cdots, K$$

$\longrightarrow$ *for finite-dimensional* θ *procedure simpler, but fit poorer;*

$\longrightarrow$ *for* $\infty-$*dimensional* θ *procedure more complicated, but fit better.*

Alternative possibilities

i) Instead of a scalar "factor" r_t, a multivariate "factor" x_t. Furthermore, instead of an exponentially affine, an exponentially quadratic term structure.

ii) Take the entire forward curve as an $\infty-$dimensional "factor".

Heath-Jarrow-Morton (HJM) framework

- Take as "factor" the entire forward rate curve

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dw_t$$

$\rightarrow$ *The initially observed forward-rate curve enters directly into the model as initial condition (thus reducing the burden of model calibration), i.e.*

$$f(t,T) = f(0,T) + \int_0^t \alpha(s,T)ds + \int_0^t \sigma(s,T)dw_s$$

- One (vector) Wiener and many (infinite) assets, namely $T-$bonds for various T $\Rightarrow$ *possibility of arbitrage*

Absence of arbitrage for HJM

- Impose conditions so that $\exists$ at least one martingale measure *(under which the discounted prices of all* $T-$*bonds are martingales)*.

- Define the model directly under a martingale measure.

$\longrightarrow$ The condition becomes the *HJM "drift condition"*

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,u)du$$

- Recall in fact that from

$$df(t,T) = \alpha(t,T) + \sigma(t,T)dw_t$$

we obtained

$$dp(t,T) =$$

$$= p(t,T)\left[r_t + A(t,T) + \tfrac{1}{2}S^2(t,T)\right]dt + S(t,T)dw_t$$

with

$$\begin{cases} A(t,T) = -\displaystyle\int_t^T \alpha(t,s)ds \\ \\ S(t,T) = -\displaystyle\int_t^T \sigma(t,s)ds \end{cases}$$

- In order that the discounted value of $p(t,T)$, i.e. $B_t^{-1}p(t,T)$ is a martingale, one needs that

$$dp(t,T) = p(t,T)\left[r_t dt + S(t,T)\right] dw_t$$

i.e. that, for all T,

$$A(t,T) + \frac{1}{2}S^2(t,T) = 0$$

Differentiating with respect to T leads to the required condition (HJM drift condition).

Inputs to the HJM model

i) volatility structure $\sigma(t, T)$;

ii) initially observed forward rate.

$\longrightarrow$ *Calibration reduces to determining possible parameters in* $\sigma(t, T)$ (which is a priori simpler than in the finite dimensional factor models).

Procedure

1. Choose $\sigma(t,T)$;

2. put $\alpha(t,T) = \sigma(t,T) \int_t^T \sigma(t,u)du$;

3. from the market obtain $\{f^*(0,T)\,;\ \forall T > 0\}$ *(reconstruction and interpolation)*;

4. $f(t,T) = f^*(0,T) + \int_0^t \alpha(s,T)ds + \int_0^t \sigma(s,T)dw_s$;

5. $p(t,T) = \exp\left[-\int_t^T f(t,u)du\right]$.

Example (leading to Ho-Lee)

$$\text{Choose} \quad \sigma(t,T) \equiv \sigma$$

$$\Rightarrow \quad \alpha(t,T) = \sigma \int_t^T \sigma ds = \sigma^2(T-t)$$

It follows

$$\begin{aligned} f(t,T) &= f^*(0,T) + \int_0^t \sigma^2(T-s)ds + \int_0^t \sigma dw_s \\ &= f^*(0,T) + \sigma^2 t \left(T - \tfrac{t}{2}\right) + \sigma w_t \end{aligned}$$

and

$$r_t = f^*(0,t) + \sigma^2 \frac{t^2}{2} + \sigma w_t$$

From the last relation one obtains

$$dr_t = \left[f_T^*(0,t) + \sigma^2 t \right] dt + \sigma dw_t = \Phi_t dt + \sigma dw_t$$

with

$$\Phi_t = f_T^*(0,t) + \sigma^2 t$$

$\longrightarrow$ *One reobtains thus the Ho-Lee model. Notice the perfect "fitting" without the need of "inverting the yield curve" as in the classical model.*

Change of numeraire
(forward measures)

- By default one takes B_t as *numeraire*, i.e. for Q a martingale measure one has that

$$\frac{S_s}{B_s} = E^Q\left\{\frac{S_t}{B_t} \mid \mathcal{F}_s\right\} , \quad \forall\, s < t \text{ and all asset prices } S_t$$

- In particular, for the price Π_t of a derivative asset with claim H_T one has (recall $B_t = B_0 \exp\left[\int_0^t r_s ds\right]$)

$$\frac{\Pi_t}{B_t} = E^Q\left\{\frac{H_T}{B_T} \mid \mathcal{F}_t\right\} \Rightarrow \begin{cases} \Pi_t = B_t E^Q\left\{\frac{H_T}{B_T} \mid \mathcal{F}_t\right\} \\ \Pi_t = E^Q\left\{e^{-\int_t^T r_s ds} H_T \mid \mathcal{F}_t\right\} \end{cases}$$

$\rightarrow$ *A change of numeraire induces a change in the MM, which may result in certain cases convenient for the calculation of derivative prices.*

- Since for a generic numeraire S_t^0 with associated MM Q^0 one has

$$\frac{\Pi_t}{S_t^0} = E^{Q^0}\left\{\frac{H_T}{S_T^0} \mid \mathcal{F}_t\right\} \quad \leftrightarrow \quad \Pi_t = S_t^0 E^{Q^0}\left\{\frac{H_T}{S_T^0} \mid \mathcal{F}_t\right\}$$

the following two examples illustrate possible advantages resulting from a change of numeraire.

1. Assume a *claim can be factored* as $H_T = S_T^0 \bar{H}_T$.

- *With B_t as numeraire one would have to calculate*

$$\Pi_t = B_t E^Q \left\{ \frac{S_T^0 \bar{H}_T}{B_T} \mid \mathcal{F}_t \right\}$$

- *with S_t^0 as numeraire, and Q^0 the corresponding MM, the same Π_t can be obtained as*

$$\Pi_t = S_t^0 E^{Q^0} \left\{ \bar{H}_T \mid \mathcal{F}_t \right\}$$

2. Let the *short rate be stochastic*.

- *With B_t as numeraire one would have to calculate*

$$\Pi_t = E^Q \left\{ e^{-\int_t^T r_s ds} H_T \mid \mathcal{F}_t \right\}$$

i.e. the conditional expectation of the product of the (correlated) random variables $\exp\left[-\int_t^T r_s ds\right]$ *and* H_T.

- *With $p(t,T)$ as numeraire, and denoting by Q^T the corresponding MM (forward measure)*

$$\Pi_t = p(t,T) E^{Q^T} \left\{ \frac{H_T}{p(T,T)} \mid \mathcal{F}_t \right\} = p(t,T) E^{Q^T} \{ H_T \mid \mathcal{F}_t \}$$

Theorem

- Let Q be a MM for the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, Q)$ ($t \leq T$ and $\mathcal{F}_T = \mathcal{F}$).

- Let S_t be the price process, under AOA, of any given asset on the market, i.e. $\frac{S_t}{B_t}$ is a $(Q, \mathcal{F}_t)-$martingale.

- For any fixed price process $S_t^0 > 0$ there exists then a probability measure $Q^0 \sim Q$ such that $\frac{S_t}{S_t^0}$ is a $(Q^0, \mathcal{F}_t)-$martingale. The R.-N. derivative is

$$L = \frac{dQ^0}{dQ} = \frac{S_T^0}{B_T S_0^0} \quad \text{(assuming } B_0 = 1\text{)}$$

from which one obtains the mean-1 $(Q, \mathcal{F}_t)-$martingale

$$L_t := E^Q\{L \mid \mathcal{F}_t\} = \frac{S_t^0}{B_t S_0^0}$$

Proof :

$$E^{Q^0}\left\{\frac{S_t}{S_t^0} \mid \mathcal{F}_s\right\} = \frac{E^Q\left\{L\frac{S_t}{S_t^0}|\mathcal{F}_s\right\}}{E^Q\{L|\mathcal{F}_s\}}$$

$$= \frac{E^Q\left\{\frac{S_t}{S_t^0}E^Q\{L|\mathcal{F}_t\}|\mathcal{F}_s\right\}}{L_s} = \frac{E^Q\left\{\frac{S_t}{S_t^0}L_t|\mathcal{F}_s\right\}}{L_s}$$

$$= \frac{S_0^0 B_s}{S_s^0}E^Q\left\{\frac{S_t}{S_t^0}\,\frac{S_t^0}{B_t S_0^0} \mid \mathcal{F}_s\right\} = \frac{S_0^0 B_s}{S_s^0}\,\frac{S_s}{B_s S_0^0} = \frac{S_s}{S_s^0}$$

Caps, Floors, Swaptions

$$- \quad \underset{t}{|} \quad - - - - \quad \underset{T_0}{|} \quad - - \quad \underset{T_1}{|} \quad - - | - - | - - - - \quad \underset{T_N}{|} \quad - \longrightarrow$$

$$\alpha_i := T_i - T_{i-1}\,, \quad i = 1, \cdots, N$$ ("tenors")

T_i : *"resettlement dates"*

$$p_i(t) := p(t, T_i)$$

$$L_i(t) := L(t; T_{i-1}, T_i)$$

$$L_i(t) = \frac{1}{\alpha_i} \frac{p_{i-1}(t) - p_i(t)}{p_i(t)} = \frac{1}{\alpha_i} \left(\frac{p_{i-1}(t)}{p_i(t)} - 1 \right)$$

Caps

- Assume one has to make a series of interest rate payments on a (unitary) notional/principal at the current spot LIBOR rate. The rate is set at T_{i-1}, payment is made at T_i ; $i = 1, \cdots, N$ ("settlement in arrears")

- *Cap*=$\sum$ *Caplets* : a contract, established at $t < T_0$, that puts a "cap" on the interest rate; denote it by R.

- i*–th Caplet :* an option with expiration T_i and claim

$$X_i = \alpha_i \max[L_i(T_{i-1}) - R, 0] = \alpha_i \left(L_i(T_{i-1}) - R\right)^+$$

$\longrightarrow$ *One always has to pay at the rate* $L_i(T_{i-1})$*, the exceeding part* $(L_i(T_{i-1}) - R)^+$ *is however reimbursed.*
$\longrightarrow$ *The rate at which one actually pays is therefore*

$$L_i(T_{i-1}) - (L_i(T_{i-1}) - R)^+ = \min\{L_i(T_{i-1}), R\}$$

LIBOR market model

- Since a Caplet is of the form of an European call option, it would be convenient to price it with the Black and Scholes formula, which requires the underlying - here $L_i(t)$ - to be lognormal.

  $\longrightarrow$ *Despite of the fact that the B.&S. formula considers a deterministic r_t, while here $L_i(t)$ and thus also r_t are stochastic, this is in fact market practice and to justify it one introduces a model guaranteeing lognormality* $\longrightarrow$ *"market model"*.

  $\longrightarrow$ Market models also turn out to be convenient for calibration.

- The $i-$th Caplet can be evaluated as follows *(E^i denotes expectation w.r. to Q^i, i.e. the MM corresponding to $p(t,T_i)$ as numeraire)*

$$Capl_i(t) = \begin{cases} \alpha_i E^Q \left\{ e^{-\int_t^{T_i} r_s ds} \left(L_i(T_{i-1}) - R \right)^+ \mid \mathcal{F}_t \right\} \\ \\ \alpha_i p_i(t) E^i \left\{ \left(L_i(T_{i-1}) - R) \right)^+ \mid \mathcal{F}_t \right\} \end{cases}$$

Lemma : $L_i(t)$ is, on $[0, T_{i-1}]$, a martingale w.r. to Q^i.

Proof : Since $\alpha_i L_i(t) = \frac{p_{i-1}(t)}{p_i(t)} - 1$, it suffices to show that $\frac{p_{i-1}(t)}{p_i(t)}$ is a martingale, which follows from the fact that the price $p_{i-1}(t)$ is here expressed in units of $p_i(t)$.

- For the "LIBOR market model" one would now like to define the LIBOR dynamics such that, for each $i = 1, \cdots, N$, $L_i(t)$ is a lognormal martingale in the measure Q^i.

$\rightarrow$ *This reduces to putting*

$$dL_i(t) = L_i(t)\sigma_i(t)dw_t^i, \quad i = 1, \cdots, N$$

with $\sigma_i(t)$ deterministic and w_t^i Wiener in the measure Q^i.

$\rightarrow$ Existence of the Wiener processes w_t^i can be shown.

Under Q^i one then has

$$\log\, L_i(t) \sim \mathcal{N}\left(-\frac{1}{2}\int_0^t \sigma_i^2(s)ds, \int_0^t \sigma_i^2(s)ds\right)$$

and so the price of the $i-$th Caplet can be computed by the following Black& Scholes formula (setting $r = 0$)

$$Capl_i(t) = \alpha_i p_i(t)\left[L_i(t)\mathcal{N}(d_1) - R\mathcal{N}(d_2)\right]$$

with $$\begin{cases} d_1 = \frac{1}{\Sigma_i(t,T_{i-1})}\left[\log\left(\frac{L_i(t)}{R}\right) + \frac{1}{2}\Sigma_i^2(t,T_{i-1})\right] \\ \\ d_2 = d_1 - \Sigma_i(t,T_{i-1}) \end{cases}$$

where $\Sigma_i^2(t,T_{i-1}) = \int_t^{T_{i-1}} \sigma_i^2(s)ds$

$\longrightarrow$ *Need to determine* $\sigma_i(t)$ *(implied volatility)*

Pricing of Caps via the standard MM Q

$$
\begin{aligned}
&Capl_i(t) = \alpha_i E^Q \left\{ e^{-\int_t^{T_i} r_s ds} \left(L_i(T_{i-1}) - R\right)^+ \mid \mathcal{F}_t \right\} \\
&= \alpha_i E^Q \left\{ E^Q \left\{ e^{-\int_t^{T_i} r_s ds} \left(L_i(T_{i-1}) - R\right)^+ \mid \mathcal{F}_{T_{i-1}} \right\} \mid \mathcal{F}_t \right\} \\
&= \alpha_i E^Q \left\{ e^{-\int_t^{T_{i-1}} r_s ds} p(T_{i-1}, T_i) \left(L_i(T_{i-1}) - R\right)^+ \mid \mathcal{F}_t \right\} \\
&= E^Q \left\{ e^{-\int_t^{T_{i-1}} r_s ds} p(T_{i-1}, T_i) \left(\frac{1}{p(T_{i-1}, T_i)} - 1 - R\alpha_i \right)^+ \mid \mathcal{F}_t \right\} \\
&= E^Q \left\{ e^{-\int_t^{T_{i-1}} r_s ds} \left[1 - (1 + R\alpha_i) p(T_{i-1}, T_i)\right]^+ \mid \mathcal{F}_t \right\} \\
&= (1 + R\alpha_i) E^Q \left\{ e^{-\int_t^{T_{i-1}} r_s ds} \left(\frac{1}{1 + R\alpha_i} - p(T_{i-1}, T_i) \right)^+ \mid \mathcal{F}_t \right\}
\end{aligned}
$$

which can be computed as a European-type put option with maturity T_{i-1}, exercise price $K_i = \frac{1}{1+R\alpha_i}$ and having as underlying the bond with maturity T_i *(r_t may be stochastic)*.

Floors

- *Floor*$=\sum$ *Floorlets* : a contract that puts a "floor" on the interest rate; denote it by R.

- $i-$*th Floorlet :* an option with expiration T_i and claim

$$X_i = \alpha_i \max[R - L_i(T_{i-1}), 0] = \alpha_i \left(R - L_i(T_{i-1})\right)^+$$

$\longrightarrow$ *One always receives at the rate* $L_i(T_{i-1})$, *the missing part* $(R - L_i(T_{i-1}))^+$ *is however reimbursed.*
$\longrightarrow$ *The rate at which one actually receives payments is therefore*

$$L_i(T_{i-1}) + (R - L_i(T_{i-1}))^+ = \max\{L_i(T_{i-1}), R\}$$

$\longrightarrow$ While a Cap protects those who have to make payments against rising rates, Floors protect those who receive payments against falling rates.

$\longrightarrow$ *Floors can be priced in perfect duality with Caps. Since the former correspond to a put option and the latter to a call option, a put-call-parity-type relation can be exploited here as well.*

Interest Rate Swaps (IRS)

(forward swaps settled in arrears)

- Exchanging a sequence of payments at a *"floating rate"* *("floating leg")* against a sequence of payments at a *fixed rate* *"fixed leg"* (the sequence concerns the same resettlement dates $T_0, \cdots, T_n$ as before).

- Referring always to the fixed leg, one distinguishes :

  i) *receiver swap :* one receives at the dates $T_1, \cdots, T_N$ the fixed leg and pays the floating leg.

  ii) *payer swap :* one receives the floating and pays the fixed leg.

More precisely : *Payer IRS*

- Payments occur at dates $T_1, \cdots, T_n$;
- *for each period* $[T_i, T_{i+1}]$; $i = 0, \cdots, N-1$ *the LIBOR rate* $L_{i+1}(T_i)$ *is set at time* T_i *and the payment of* $\alpha_{i+1} L_{i+1}(T_i)$ *(for unitary nominal/principal)* is received at time T_{i+1};
- the fixed amount $\alpha_{i+1} K$ is paid at T_{i+1}.

$\longrightarrow$ *Notice that, due to the equality*

$$(L - R)^+ - (R - L)^+ = L - R$$

the difference between a Cap and a Floor is equivalent to a forward payer IRS. (For IRS we use K *for the fixed rate, the symbol* R *will be reserved for the Swap rate to be introduced below).*

Arbitrage free value of a payer IRS

1. *Value in $t < T_0$ of the floating leg ?*

- Recalling that $L_{i+1}(t)$ is, on $[0, T_i]$ a $Q^{i+1}-$martingale, the value in t of the floating payment made in T_{i+1} is

$$\Pi^{i+1}(t) = p_{i+1}(t) E^{i+1} \left\{ \alpha_{i+1} L_{i+1}(T_i) \mid \mathcal{F}_t \right\}$$

$$= p_{i+1}(t)\alpha_{i+1}L_{i+1}(t) = p_{i+1}(t)\frac{p_i(t)-p_{i+1}(t)}{p_{i+1}(t)} = p_i(t) - p_{i+1}(t)$$

- *The total value in $t < T_0$ of the floating leg is then*

$$\sum_{i=0}^{N-1} \left[p_i(t) - p_{i+1}(t)\right] = p_0(t) - p_N(t)$$

2. *Value in $t < T_0$ of the fixed leg ?*

The total value is just the sum of the discounted individual amounts, i.e.

$$\sum_{i=0}^{N-1} p_{i+1}(t)\alpha_{i+1}K = K\sum_{i=1}^{N} \alpha_i p_i(t)$$

3. *Total value $PS_0^N(t;K)$ in $t < T_0$ of the payer IRS*

It is given by the difference of the values of the two legs, i.e.

$$PS_0^N(t;K) = p_0(t) - p_N(t) - K\sum_{i=1}^{N} \alpha_i p_i(t)$$

Forward **swap rate** (payer IRS)

- The swap rate for a forward payer IRS is that value of the fixed amount K for which $PS_0^N(t;K) = 0$, i.e. it is the value $R_0^N(t)$ given by

$$R_0^N(t) := \frac{p_0(t) - p_N(t)}{\sum_{i=1}^N \alpha_i p_i(t)}$$

$\longrightarrow$ *For* $N = 1$ *the swap rate coincides with the LIBOR rate*.

$\longrightarrow$ The denominator $S_0^N(t) := \sum_{i=1}^N \alpha_i p_i(t)$ (also referred to as *accrual factor*) can be interpreted as the value of a traded asset, namely a buy-and-hold portfolio consisting, for each i, of α_i units of the T_i-bond.

$\longrightarrow$
$$R_0^N(t) := \frac{p_0(t) - p_N(t)}{S_0^N(t)}$$

is, on $[0, T_0]$, a martingale in the measure corresponding to $S_0^N(t)$ as numeraire.

Swaption (Payer IRS)

- It is a contract, established at $t < T_0$, that gives the holder the right but not the obligation to enter, at $t = T_0$, a payer IRS with given fixed rate K.

  $\rightarrow$ *A Swaption (swap-option) for a payer IRS and a fixed rate K is thus an option with maturity T_0 and claim*

$$X_0^N = \left(PS_0^N(T_0; K)\right)^+$$

$\longrightarrow$ Noticing that in terms of $R_0^N(t)$ *(swap rate)* and $S_0^N(t)$ *(accrual factor)* one may rewrite

$$PS_0^N(t;K) = \left(R_0^N(t) - K\right) S_0^N(t)$$

the Swaption claim also takes the form

$$X_0^N = \left(PS_0^N(t;K)\right)^+ = \left(R_0^N(T_0) - K\right)^+ S_0^N(T_0)$$

which reminds one of a call option with underlying the swap rate R_0^N and strike K.

Arbitrage-free value in $t < T_0$ of a Swaption

- Given the factor $S_0^N(T_0)$ in the claim, one may choose as numeraire $S_0^N(t)$ so that for the value $PSW_0^N(t)$ in $t < T_0$ of a payer IRS swaption one has

$$PSW_0^N(t) = S_0^N(t) E^{0,N} \left\{ (R_0^N(T_0) - K)^+ \mid \mathcal{F}_t \right\}$$

  with $E^{0,N}$ denoting the expectation in the martingale measure $Q^{0,N}$ that corresponds to $S_0^N(t)$ as numeraire.

  $\rightarrow$ *Convenient to have lognormality of $R_0^N(t)$ in the measure $Q^{0,N}$.*

Swap market model

- Analogously to the LIBOR market model one would now like to define the swap rate dynamics such that $R_0^N(t)$ is a lognormal martingale in the measure $Q^{0,N}$.

  $\rightarrow$ *This reduces to putting*

$$dR_0^N(t) = R_0^N(t)\,\sigma_t^{0,N}\,dw_t^{0,N}$$

  with $\sigma_t^{0,N}$ *deterministic and* $w_t^{0,N}$ *a Wiener in the measure* $Q^{0,N}$*.*

$\rightarrow$ One can then use the *B.& S. formula* obtaining

$$PSW_0^N(t) = S_0^N(t)\left[R_0^N(t)\mathcal{N}(d_1) - K\mathcal{N}(d_2)\right]$$

with
$$\begin{cases} d_1 = \frac{1}{\Sigma_{0,N}(t)}\left[\log\left(\frac{R_0^N(t)}{K}\right) + \frac{1}{2}\Sigma_{0,N}^2(t)\right] \\ \\ d_2 = d_1 - \Sigma_{0,N}(t) \end{cases}$$

where $\Sigma_{0,N}^2(t) = \int_t^{T_0} \sigma_s^{0,N} ds$

$\rightarrow$ $\sigma_t^{0,N}$ *has to be calibrated to the Swaption market analogously to what happens for the LIBOR market models.*

- *Remark 1 :* The *Swap market model is incompatible with the LIBOR market model* in the sense that lognormality in one model does not imply lognormality in the other *(this incompatibility is however more theoretical than practical)*.

- *Remark 2 :* The preceding development concerned payer IRS and corresponding swaptions. For a receiver IRS and corresponding swaption the development can be obtained analogously by inverting the floating and fixed legs *(like passing from a call to a put option)*. Notice furthermore that, always due to the equality

$$(R-K)^+ - (K-R)^+ = R-K$$

one has that *the difference between a payer swaption and the corresponding receiver swaption is equivalent to a forward payer IRS*.

A1. Option pricing when interest rate is stochastic

(an application of the change of numeraire technique)

- Consider an *European call option* on an underlying with price S_t. The claim is

$$H_T = (S_T - K)^+ = (S_T - K)\mathbf{1}_{\{S_T > K\}}$$

- Considering besides B_t also S_t and $p(t,T)$ as numeraires, one obtains the following expressions for the same arbitrage free price Π_0 at time t=0, namely (put $B_0 = 1$)

$$\Pi_0 = E^Q\left\{\frac{H_T}{B_T}\right\} = \begin{cases} S_0\, E^S\left\{\frac{H_T}{S_T}\right\} \\ p(0,T)\, E^T\{H_T\} \end{cases}$$

with E^Q, E^S, E^T denoting expectation w.r. to the martingale measures corresponding to the above three numeraires respectively.

- From the foregoing it follows that one can write

$$\begin{aligned}\Pi_0 &= E^Q\left\{B_T^{-1}S_T\mathbf{1}_{\{S_T>K\}}\right\} - KE^Q\left\{B_T^{-1}\mathbf{1}_{\{S_T>K\}}\right\} \\ &= S_0Q^S\{S_T>K\} - Kp(0,T)Q^T\{S_T>K\}\end{aligned}$$

i.e. Π_0 *can be computed by computing the probability of the event* $S_T > K$ *under the two measures* Q^S *and* Q^T.

- *For actual computation* assume that S_t is such that $Z_t := \frac{S_t}{p(t,T)}$ satisfies *(for any measure under which w_t is a Wiener process)*

$$dZ_t = Z_t \left[m_t dt + \sigma_t^Z dw_t \right]$$

with σ_t^Z a deterministic time function.

$\rightarrow$ $Z_t := \frac{S_t}{p(t,T)}$ is a Q^T-martingale so that, under Q^T,

$$dZ_t = Z_t \sigma_t^Z dw_t^T$$

with w_t^T a Q^T-Wiener and, consequently, under Q^T

$$Z_T = \frac{S_0}{p(0,T)} \exp \left[-\frac{1}{2} \int_0^T (\sigma_t^Z)^2 dt + \int_0^T \sigma_t^Z dw_t^T \right]$$

- Putting $\Sigma_T^2 := \int_0^T (\sigma_t^Z)^2 dt$ one then has

$$Q^T\{S_T > K\} = Q^T\left\{\frac{S_T}{p(T,T)} > K\right\} = Q^T\{Z_T > K\}$$

$$= P\left\{\log\frac{S_0}{p(0,T)} + \mathcal{N}\left(-\tfrac{1}{2}\Sigma_T^2, \Sigma_T^2\right) > \log K\right\}$$

$$P\left\{\mathcal{N}(0,1) > \frac{\log\frac{K\,p(0,T)}{S_0} + \frac{1}{2}\Sigma_T^2}{\Sigma_T}\right\} = \mathcal{N}(d_2)$$

where

$$d_2 = \frac{\log\frac{S_0}{K\,p(0,T)} - \frac{1}{2}\Sigma_T^2}{\Sigma_T}$$

- Next, putting
$$Y_t := \frac{1}{Z_t} = \frac{p(t,T)}{S_t}$$
which is a Q^S-martingale, under Q^S one obtains

$$dY_t = -Y_t \sigma_t^Z dw_t^S$$

with w_t^S a Q^S-Wiener and, consequently, under Q^S

$$Y_T = \frac{p(0,T)}{S_0} \exp\left[-\frac{1}{2}\int_0^T (\sigma_t^Z)^2 dt - \int_0^T \sigma_t^Z dw_t^S\right]$$

- It follows that

$$Q^S\left\{S_T > K\right\} = Q^S\left\{\frac{p(T,T)}{S_T} < \frac{1}{K}\right\} = Q^S\left\{Y_T < \frac{1}{K}\right\}$$

$$= P\left\{\log\frac{p(0,T)}{S_0} + \mathcal{N}\left(-\tfrac{1}{2}\Sigma_T^2, \Sigma_T^2\right) < -\log K\right\}$$

$$P\left\{\mathcal{N}(0,1) < \frac{\log\frac{S_0}{K\,p(0,T)} + \frac{1}{2}\Sigma_T^2}{\Sigma_T}\right\} = \mathcal{N}(d_1)$$

where

$$d_1 = d_2 + \Sigma_T$$

- *Summarizing*, one has that, if $Z_t := \frac{S_t}{p(t,T)}$ satisfies

$$dZ_t = Z_t \left[m_t dt + \sigma_t^Z dw_t \right]$$

with σ_t a deterministic time function, then the price Π_0 at time $t = 0$ of the claim $H_T = (S_T - K)^+$ is given by

$$\Pi_0 = S_0 \mathcal{N}(d_1) - K p(0,T) \mathcal{N}(d_2)$$

where $\mathcal{N}(\cdot)$ denotes the cumulative Gaussian distribution function and

$$\begin{cases} d_1 &= \frac{\log \frac{S_0}{K\, p(0,T)} + \frac{1}{2}\Sigma_T^2}{\Sigma_T} \\ \\ d_2 &= d_1 - \Sigma_T \end{cases}$$

with $\Sigma_T^2 := \int_0^T (\sigma_t^Z)^2 dt$.

Application

(Pricing a call option on a bond)

- Consider the *Hull-White model* for the spot rate, i.e.

$$dr_t = [\Phi_t - a r_t]dt + \sigma\, dw_t^Q$$

- *This leads to an affine term structure, i.e.*

$$p(t,T) = \exp[A(t,T) - B(t,T) r_t]$$

with

$$B(t,T) = \frac{1}{a}\left[1 - e^{-a(T-t)}\right]$$

- Putting $T_1 = T$ with T the maturity of the option, take $S_t = p(t, T_2)$ with $T_2 > T_1$.

- For this case one has

$$
\begin{aligned}
Z_t &= \tfrac{p(t,T_2)}{p(t,T_1)} \\
&= \exp\left[(A(t,T_2) - A(t,T_1)) - (B(t,T_2) - B(t,T_1))r_t\right]
\end{aligned}
$$

and therefore *(one has to verify only the volatility)*

$$dZ_t = Z_t[\cdots\cdots]dt + Z_t\sigma_t^Z dw_t$$

with $\sigma_t^Z = -\sigma[B(t,T_2) - B(t,T_1)] = \frac{\sigma}{a}\, e^{at}\left[e^{-aT_1} - e^{-aT_2}\right]$ which is indeed deterministic.

- *Summarizing*, for $H_{T_1} = (p(T_1, T_2) - K)^+$ one then has

$$\Pi_0 = p(0, T_2)\mathcal{N}(d_1) - Kp(0, T_1)\mathcal{N}(d_2)$$

with $d_1 = \frac{\log \frac{p(0,T_2)}{K\,p(0,T_1)} + \frac{1}{2}\Sigma_{T_1}^2}{\Sigma_{T_1}}$; $d_2 = d_1 - \Sigma_{T_1}$ and where

$$\begin{aligned}
\Sigma_{T_1}^2 &= \frac{\sigma^2}{a^2}\left[e^{-aT_1} - e^{-aT_2}\right]^2 \int_0^{T_1} e^{2at} dt \\
&= \frac{\sigma^2}{2a^3}\left(e^{2aT_1} - 1\right)\left[e^{-aT_1} - e^{-aT_2}\right]^2 \\
&= \frac{\sigma^2}{2a^3}\left(1 - e^{-2aT_1}\right) e^{2aT_1}\left[e^{-aT_1} - e^{-aT_2}\right]^2 \\
&= \frac{\sigma^2}{2a^3}\left(1 - e^{-2aT_1}\right)\left(1 - e^{-a(T_2 - T_1)}\right)^2
\end{aligned}$$

$\rightarrow$ Can be used to *calibrate the parameter* a.

A2. Solved problems/additional topics

Problem 1.

The problem : Consider the following dynamics, under a martingale measure Q, of the spot rate r_t,

$$dr_t = ar_t dt + \sigma dw_t$$

and assume that the term structure has the form

$$p(t,T) = F^T(t, r_t)$$

$$= \exp\left[A(t,T) + B(t,T)r_t + C(t,T)r_t^2 + D(t,T)r_t^3\right]$$

a) Imposing on $F^T(t,r)$ to satisfy the term structure equation, show that this implies $D(t,T) \equiv 0$. What is the resulting system of equations for $A(t,T), B(t,T), C(t,T)$?

b) *What changes if r_t satisfies lognormal dynamics, i.e.*

$$dr_t = r_t[adt + \sigma dw_t] \quad ?$$

Knowing that for a deterministic spot rate r_t one has $p(t,T) = \exp\left[-\int_t^T r_s ds\right]$, verify the correctness of the result in the case of $\sigma = 0$.

Solution :

a) The term structure equation is

$$F_t + arF_r + \frac{1}{2}\sigma^2 F_{rr} - rF = 0$$

Since

$$\begin{aligned} F_t &= F[A_t + B_t r + C_t r^2 + D_t r^3] \\ F_r &= F[B + 2Cr + 3Dr^2] \\ F_{rr} &= F\{[B + 2Cr + 3Dr^2]^2 + [2C + 6Dr]\} \end{aligned}$$

and since the term structure equation has to hold for any value of r, the coefficients of the various powers of r have to vanish and this implies the following :

$$\begin{cases} A_t + \frac{1}{2}\sigma^2[B^2 + 2C] = 0 & , \quad A(T,T) = 0 \\ B_t + aB + 2BC\sigma^2 + 3D\sigma^2 - 1 = 0 & , \quad B(T,T) = 0 \\ C_t + 2Ca + 2C^2\sigma^2 + 3BD\sigma^2 = 0 & , \quad C(T,T) = 0 \\ D_t + 3Da + 3CD\sigma^2 = 0 & , \quad D(T,T) = 0 \\ \frac{9}{2}D^2\sigma^2 = 0 & \end{cases}$$

As a consequence $D(t,T) = 0$ and one has the remaining system

$$\begin{cases} A_t + \frac{1}{2}\sigma^2[B^2 + 2C] = 0 \\ B_t + aB + 2BC\sigma^2 - 1 = 0 \\ C_t + 2Ca + 2C^2\sigma^2 = 0 \end{cases}$$

b) In this case the system becomes (with the terminal conditions always equal to zero)

$$\begin{cases} A_t = 0 \\ B_t + aB + \frac{1}{2}\sigma^2[B^2 + 2C] - 1 = 0 \\ C_t + 2Ca + 2\sigma^2 BC + 3\sigma^2 D = 0 \\ D_t + 3Da + 2\sigma^2 C^2 + 3\sigma^2 BD = 0 \\ 3CD\sigma^2 = 0 \\ \frac{9}{2}\sigma^2 D^2 = 0 \end{cases}$$

The system thus reduces to

$$\begin{cases} A_t = 0 \\ B_t + aB + \frac{1}{2}\sigma^2 B^2 - 1 = 0 \end{cases}$$

which implies $A(t,T) = 0$ and thus one finds

$$F(t,r) = \exp[B(t,T)r]$$

with

$$\begin{cases} B_t + aB + \frac{1}{2}\sigma^2 B^2 = 1 \\ B(T,T) = 0 \end{cases}$$

- In the case when $\sigma = 0$ the equation for B becomes $B_t + aB = -1$ with solution $B(t,T) = \frac{e^{a(T-t)}-1}{a}$ and thus
$$p(t,T) = \exp\left[\frac{r_t}{a}\left(e^{a(T-t)} - 1\right)\right]$$
On the other hand, for $\sigma = 0$ one has as solution of the equation for r_t the following

$$r_s = r_t \exp[a(s-t)]$$

but

$$p(t,T) = \exp\left[-\int_t^T r_s ds\right] = \exp\left[\frac{r_t}{a}\left(e^{a(T-t)} - 1\right)\right]$$

and this coincides in fact with the above expression.

Problem 2.

The problem : Assume that, under a martingale measure Q, the spot rate r_t satisfies the following model (CIR)

$$dr_t = (a - br_t)dt + \sigma\sqrt{r_t}\, dw_t$$

a) On the basis of the term structure equation show that, for $\beta, \lambda > 0$, one has

$$E_{t,r}\left\{e^{-\beta r_T} \exp\left[-\int_t^T r_s ds\right]\right\} = \exp\left[A(t,T) - B(t,T)r\right]$$

where $B(t,T)$ and $A(t,T)$ satisfy

$$\begin{cases} B_t(t,T) = b\,B(t,T) + \frac{1}{2}\sigma^2 B^2(t,T) - 1\,, & B(T,T) = \beta \\ A_t(t,T) = aB(t,T)\,, & A(T,T) = 0 \end{cases}$$

b) Taking as known that the solution of previous system is

$$\begin{cases} B(t,T) &= \frac{\beta[\gamma+b+e^{\gamma(T-t)}(\gamma-b)]+2(e^{\gamma(T-t)}-1)}{\beta\sigma^2(e^{\gamma(T-t)}-1)+\gamma-b+e^{\gamma(T-t)}(\gamma+b)} \\ \\ A(t,T) &= \frac{2a}{\sigma^2}\, log\left(\frac{2\gamma e^{\frac{(T-t)(\gamma+b)}{2}}}{\beta\sigma^2(e^{\gamma(T-t)}-1)+\gamma-b+e^{\gamma(T-t)}(\gamma+b)}\right) \end{cases}$$

with $\gamma := \sqrt{b^2+2\sigma^2}$, obtain an explicit expression for the price $p(t,T)$ of a $T-$bond.

Solution :

a) Putting

$$F(t,r) := E_{t,r}\left\{e^{-\beta r_T}\exp\left[-\int_t^T r_s ds\right]\right\}$$

the function $F(t,r)$ gives the price, at t, of a claim $\exp[-\beta\, r_T]$ with maturity T. It therefore satisfies the term structure equation

$$\begin{cases} F_t + (a-br)F_r + \frac{1}{2}\sigma^2 r F_{rr} - rF = 0 \\ \\ F(T,r) = \exp[-\beta\, r] \end{cases}$$

Since

$$\begin{aligned} F_t(t,r) &= F(t,r)[A_t(t,T) - B_t(t,T)r] \\ F_r(t,r) &= -F(t,r)B(t,T) \\ F_{rr}(t,r) &= F(t,r)B^2(t,T) \end{aligned}$$

one thus obtains

$$A_t - B_t r - aB + bBr + \frac{1}{2}\sigma^2 B^2 r - r = 0$$

which, having to hold for all values of r, leads to

$$\begin{cases} A_t = aB & A(T,T) = 0 \\ \\ B_t = bB + \frac{1}{2}\sigma^2 B^2 - 1 & B(T,T) = \beta \end{cases}$$

b) One has $p(t,T) = F(t,r)$ where $A(t,T), B(t,T)$ satisfy the system in point a) of the problem statement for $\beta = 0$. Consequently

$$p(t,T) =$$

$$= \exp\left[\frac{2a}{\sigma^2}\,log\left(\frac{2\gamma e^{\frac{(T-t)(\gamma+b)}{2}}}{\gamma-b+e^{\gamma(T-t)}(\gamma+b)}\right) - \frac{2(e^{\gamma(T-t)}-1)}{\gamma-b+e^{\gamma(T-t)}(\gamma+b)}\,r_t\right]$$

$$= \left(\frac{2\gamma e^{\frac{(T-t)(\gamma+b)}{2}}}{\gamma-b+e^{\gamma(T-t)}(\gamma+b)}\right)^{\frac{2a}{\sigma^2}} \exp\left[\frac{\gamma-b+e^{\gamma(T-t)}(\gamma+b)}{2(e^{\gamma(T-t)}-1)}\,r_t\right]$$

Problem 3.

The problem : Consider the following multifactor model for the term structure

$$p(t,T) = F^T(t,x_t) = \exp\left[A(t,T) + B(t,T)x_t\right]$$

where $x_t \in \mathbb{R}^p$ is a multivariate Gaussian process which, under a martingale measure Q, satisfies

$$dx_t = Gx_t dt + H dw_t$$

with w_t a multivariate Wiener process and G and H are deterministic matrices (to be calibrated to market data). Put $x_t^1 = r_t$ namely the spot rate and let

$$a(t,T) = -A_T(t,T) \quad , \quad b(t,T) = -B_T(t,T)$$

a) Show that

$$df(t,T) = [a_t(t,T) + b_t(t,T)x_t + b(t,T)Gx_t]\,dt + b(t,T)H\,dw_t$$

b) Imposing the *HJM drift condition* (which justifies the dynamics of x_t under a martingale measure) show that under absence of arbitrage the following has to hold

$$\begin{cases} b_t(t,T) & = & -b(t,T)G \\ a_t(t,T) & = & b(t,T)HH' \displaystyle\int_t^T b'(t,u)du \end{cases}$$

with terminal conditions

$$b(T,T) = (1,\underline{0}) \quad (\underline{0} \in \mathbb{R}^{p-1})\,; \quad a(T,T) = 0$$

(to show the latter, determine the expression for r_t in $t = T$ and recall that we had put $x_t^1 = r_t$).

c) Take as known that $F^T(t,x)$ satisfies the following PDE, *a generalization of the term structure equation*,

$$\begin{cases} F_t^T(t,x) + F_x^T(t,x)Gx + \frac{1}{2}tr(HH'F_{xx}^T(t,x)) \\ \qquad\qquad\qquad\qquad\qquad\qquad -rF^T(t,x) = 0 \\ F^T(T,x) = 1 \end{cases}$$

Show that from this PDE follows the same system of equations as in the previous point b). To this effect notice that, since B is a row vector, one has $tr(HH'B'B) = B(HH')B'$. After having obtained the conditions on $A(t,T)$ e $B(t,T)$, the system of equations of the previous point b) follows by differentiating with respect to T and using the definitions of $a(t,T)$ and $b(t,T)$.

Solution :

a) One has

$$f(t,T) = -\frac{\partial}{\partial T}\log p(t,T) = -A_T(t,T) - B_T(t,T)x_t$$

$$= a(t,T) + b(t,T)x_t$$

from which

$$df(t,T) = [a_t(t,T)+b_t(t,T)x_t+b(t,T)Gx_t]dt+b(t,T)Hdw_t$$

b) The *HJM drift condition* is

$$a_t(t,T)+(b_t(t,T)+b(t,T)G)\,x_t = b(t,T)HH'\int_t^T b'(t,u)du$$

and it has to hold for all x_t. Therefore

$$\begin{cases} b_t(t,T) &= -b(t,T)G \\ a_t(t,T) &= b(t,T)HH'\int_t^T b'(t,u)du \end{cases}$$

From the first relation in point a) above one obtains $r_t = f(t,t) = a(t,t) + b(t,t)x_t$ which, for $t = T$ *(one initial or terminal condition for the above system suffices)* and recalling that $x_t^1 = r_t$, gives the desired terminal conditions.

c) From $F^T(t,x) = \exp[A(t,T) + B(t,T)x]$ one has

$$\begin{aligned} F^T_t(t,x) &= F^T(t,x)\left(A_t(t,T) + B_t(t,T)x\right) \; ; \\ F^T_x(t,x) &= F^T(t,x)B(t,T) \\ F^T_{xx}(t,x) &= F^T(t,x)B'(t,T)B(t,T) \end{aligned}$$

so that the PDE becomes

$$\begin{aligned} 0 &= A_t(\cdot) + B_t(\cdot)x + B(\cdot)Gx + \tfrac{1}{2}tr(HH'B'(\cdot)B(\cdot)) - r \\ &= A_t(\cdot) + B_t(\cdot)x + B(\cdot)Gx + \tfrac{1}{2}B'(\cdot)(HH')B(\cdot) - x^1 \end{aligned}$$

Since the equation has to hold for all values of $x \in \mathbb{R}^p$, one obtains

$$\left\{ \begin{array}{l} B_t^1(t,T) + (B(t,T)G)^1 - 1 = 0 \\ B_t^k(t,T) + (B(t,T)G)^k = 0 \quad \text{for } k = 2, \cdots, p \\ A_t(t,T) + \frac{1}{2} B(t,T)(HH')B'(t,T) = 0 \end{array} \right.$$

These equations in turn have to hold for all $T \geq t$ and so, being $A(\cdot)$ and $B(\cdot)$ differentiable, one finally obtains

$$\left\{ \begin{array}{l} b_t(t,T) + b(t,T)G = 0 \\ a_t(t,T) + b(t,T)(HH')B'(t,T) = 0 \end{array} \right.$$

where *(see the definition of* $b(t,T)$ *in the problem statement)* $B'(t,T) = -\int_t^T b'(t,u)du$. The terminal conditions result from the same considerations as in the previous point b) using the fact that $r_t = f(t,t) =$
$= a(t,t) + b(t,t)x_t$.

Problem 4.

The problem : Following the HJM approach suppose that $\sigma(t,T) \equiv \sigma$ with σ a given parameter. Let the initially observed forward rate, namely $f^*(0,T)$, be given.

a) Determine the explicit expression of $f(t,T)$ and r_t (in terms of w_t^Q)

b) Show that this setup leads to an affine term structure. Using this fact and following the classical approach, determine $A(t,T)$ and $B(t,T)$ for which

$$p(t,T) = \exp\left[A(t,T) - B(t,T)r_t\right]$$

Determine the explicit expression of $p(t,T)$ in terms of r_t.

c) Using the two explicit expressions in terms of w_t^Q obtained under point a) for $f(t,T)$ and r_t, determine the explicit expression of $p(t,T)$ on the basis of $p(t,T) = \exp\left[-\int_t^T f(t,s)ds\right]$, first in terms of w_t^Q and then of r_t.

d) Using the expression of the moment generating function of a Gaussian, in particular

$$E\left\{\exp\left[\gamma\mathcal{N}(0,\sigma^2)\right]\right\} = \exp\left[\frac{\gamma^2\sigma^2}{2}\right],$$

on the basis of the explicit expression of $p(t,T)$ in terms of w_t^Q, obtained under point c), determine the expected value, in $t=0$, of $p(t,T)$.

Solution :

a) By the HJM drift condition, having $\sigma(t,T) \equiv \sigma$, one obtains $a(t,T) = \sigma^2(T-t)$, from which

$$\begin{aligned} f(t,T) &= f^*(0,T) + \int_0^t \sigma^2(T-s)ds + \int_0^t \sigma dw_s \\ &= f^*(0,T) + \sigma^2 t\left(T - \tfrac{t}{2}\right) + \sigma\, w_t \end{aligned}$$

and

$$r_t = f^*(0,t) + \sigma^2 \frac{t^2}{2} + \sigma\, w_t$$

b) Differentiating the last relation one finds

$$dr_t = \left[f_T^*(0,t) + \sigma^2 t \right] dt + \sigma\, dw_t$$

and so one ends up with a Ho-Lee model, i.e. we obtain an affine term structure. Therefore

$$p(t,T) = F^T(t,r_t) = \exp[A(t,T) - B(t,T) r_t]$$

with $A(t,T)$ and $B(t,T)$ satisfying the known ordinary differential equations which leads to

$$\begin{cases} B(t,T) = T - t \\ A(t,T) = \int_t^T \left(f_T^*(0,s) + \sigma^2 s \right) (s-T) ds + \frac{\sigma^2}{2} \int_t^T (T-s)^2 ds \end{cases}$$

Taking into account that $f(t,T) = -\frac{\partial}{\partial T}\log p(t,T)$, and performing an integration by parts, one obtains for $A(t,T)$ the expression

$$A(t,T) = f^*(0,t)(T-t) + \log\frac{p^*(0,T)}{p^*(0,t)} - \frac{\sigma^2 t}{2}(T-t)^2$$

and so, finally,

$$p(t,T) =$$

$$= \frac{p^*(0,T)}{p(0,t)} \exp\left[(T-t)f^*(0,t) - \frac{\sigma^2 t}{2}(T-t)^2 - (T-t)r_t\right]$$

c) One has

$$p(t,T) = \exp\left[-\int_t^T f^*(0,s)ds - \frac{\sigma^2}{2}tT(T-t) - \sigma(T-t)w_t^Q\right]$$
$$= \frac{p^*(0,T)}{p^*(0,t)}\exp\left[-\frac{1}{2}\sigma^2 tT(T-t) - \sigma(T-t)w_t^Q\right]$$

From the expression of r_t in terms of w_t^Q one obtains

$$\sigma\, w_t^Q = r_t - f^*(0,t) - \frac{\sigma^2 t^2}{2}$$

Replacing this in the first relation one ends up with

$$p(t,T) = \frac{p^*(0,T)}{p^*(0,t)}\exp\left[(T-t)f^*(0,t) - \frac{\sigma^2 t}{2}(T-t)^2 - (T-t)r_t\right]$$

d) One has

$$E\{p(t,T)\} =$$

$$= \frac{p^*(0,T)}{p^*(0,t)} \exp\left[-\tfrac{1}{2}\sigma^2 tT(T-t)\right] E\left\{e^{-\sigma(T-t)w_t^Q}\right\}$$

$$= \frac{p^*(0,T)}{p^*(0,t)} \exp\left[-\tfrac{1}{2}\sigma^2 tT(T-t)\right] \exp\left[\sigma^2(T-t)^2 t\right]$$

$$= \frac{p^*(0,T)}{p^*(0,t)} \exp\left[\sigma^2(T-t)t\left(\tfrac{T}{2}-t\right)\right]$$

Problem 5.

The problem : Assume that, under a martingale measure Q, the prices S_t, S_t^0 of two risky assets satisfy

$$dS_t = S_t\left[r_t dt + \sigma_t dw_t^Q\right] \quad , \quad dS_t^0 = S_t^0\left[r_t dt + v_t dw_t^Q\right]$$

a) Denoting by Q^0 the martingale measure equivalent to Q that corresponds to having S_t^0 as numeraire, show that

$$L_t := E^Q\left\{\frac{dQ^0}{dQ} \mid \mathcal{F}_t\right\} = \exp\left[\int_0^t v_s dw_s^Q - \frac{1}{2}\int_0^t v_s^2 ds\right]$$

b) What are the dynamics of S_t under Q^0 ?

c) What are the dynamics of $Z_t := S_t/S_t^0$ under Q^0 ?

Solution :

a) It is known that $L_t = \frac{S_t^0}{S_0^0 B_t}$ so that, under Q,

$$\begin{aligned} dL_t &= \frac{1}{S_0^0}\left[B_t^{-1} S_t^0 r_t dt + S_t^0 B_t^{-1} v_t dw_t^Q - r_t B_t^{-1} S_t^0 dt\right] \\ &= L_t v_t dw_t^Q \end{aligned}$$

The solution of this equation leads immediately to the desired formula.

b) Given the expression for L_t obtained in a), one has that the process $w_t^{Q^0}$ obtained from w_t^Q by translating it by v_t, i.e.

$$dw_t^{Q^0} = dw_t^Q - v_t dt$$

is, under Q^0, a Wiener process and this implies that, under Q^0,

$$dS_t = S_t \left[(r_t + \sigma\, v_t) dt + \sigma dw_t^{Q^0} \right]$$

c) Under Q^0 the process Z_t is a martingale so that in its dynamics there are no terms in dt. It thus follows immediately that

$$dZ_t = Z_t \sqrt{\sigma_t^2 + v_t^2}\, dw_t^{Q^0}$$

Problem 6.

The problem : Suppose that, under a martingale measure Q,

$$dr_t = ar_t dt + \sigma\, dw_t$$

It follows that one has an affine term structure, i.e.

$$p(t,T) = \exp\left[A(t,T) - B(t,T)r_t\right]$$

Assuming $T_i - T_{i-1} = 1$ it is known that

$$L(t,T_i) = \frac{p(t,T_{i-1})}{p(t,T_i)} - 1$$

a) Deriving first the expression for $\frac{p(t,T_{i-1})}{p(t,T_i)}$ as it results from the affine term structure, determine the dynamics, for $t \leq T_{i-1}$, of $L(t,T_i)$ under the martingale measure Q^i corresponding to $p(t,T_i)$ as numeraire.

b) Verify whether, under Q^i, one has lognormality of $L(t,T_i)$.

Solution : Under Q^i the process $L(t, T_i)$ is a martingale and therefore its drift vanishes. It thus suffices to consider the term in dw^i (where w^i is a Wiener under Q^i).

a) One has

$$dL(t, T_i) =$$

$$= d\left(\exp\left[(A(t, T_{i-1}) - A(t, T_i)) - (B(t, T_{i-1}) - B(t, T_i))r_t\right]\right)$$

$$= \frac{p(t,T_{i-1})}{p(t,T_i)}\left[\cdots dt - (B(t, T_{i-1}) - B(t, T_i))\sigma dw_t^i\right]$$

$$= -[L(t, T_i) + 1]\left[(B(t, T_{i-1}) - B(t, T_i))\sigma dw_t^i\right]$$

b) Due to the term $+1$ in the first factor of the rightmost expression, one cannot have lognormality for $L(t, T_i)$ under Q^i.

Problem 7.

The problem : Assume that, under a martingale measure Q, the spot rate r_t satisfies the following Ho-Lee model

$$dr_t = \Phi_t dt + \sigma dw_t$$

so that

$$p(t,T) = \exp\left[A(t,T) - B(t,T)r_t\right]$$

with $B(t,T) = T - t$. Given two time points T_1, T_2 with $t < T_1 < T_2$, consider an European call option with maturity T_1 and $p(t,T_2)$ as underlying. Its value in T_1 is thus

$$H^c_{T_1} = (p(T_1,T_2) - K)^+$$

a) Show that the arbitrage free price in $t = 0$ of this option is

$$\Pi_0^c = p(0, T_2)\Phi(d_1) - Kp(0, T_1)\Phi(d_2)$$

with

$$\begin{cases} d_1 & = & \dfrac{\log\left(\frac{p(0,T_2)}{Kp(0,T_1)}\right) + \frac{1}{2}\sigma^2(T_2 - T_1)^2 T_1}{\sigma(T_2 - T_1)\sqrt{T_1}} \\ d_2 & = & d_1 - \sigma(T_2 - T_1)\sqrt{T_1} \end{cases}$$

b) Show that for the corresponding put option

$$H^p_{T_1} = (K - p(T_1, T_2))^+$$

the price in $t = 0$ is given by

$$\Pi^p_0 = p(0, T_2)[\Phi(d_1) - 1] + Kp(0, T_1)[1 - \Phi(d_2)]$$

To this effect notice that one has

$$E^Q \left\{ e^{-\int_0^{T_1} r_s ds} p(T_1, T_2) \right\} = p(0, T_2)$$

c) Assume as known that the price in $t = 0$ of a Caplet for the generic interval $[T_{i-1}, T_i]$ and with cap rate R is, putting $\alpha_i := (T_i - T_{i-1})$,

$$Capl_i(0) =$$

$$= \alpha_i E^Q \left\{ e^{-\int_0^{T_i} r_s ds} \left(L(T_{i-1}, T_i) - R \right)^+ \right\}$$

$$= (1 - R\alpha_i) E^Q \left\{ e^{-\int_0^{T_i} r_s ds} \left(\frac{1}{1 - R\alpha_i} - p(T_{i-1}, T_i) \right)^+ \right\}$$

What does this last expression become in the context of the given problem ?

Solution :

a) Use the formula for the pricing of options when the short rate is stochastic (section A1). It can be applied whenever the process

$$Z_t := \frac{p(t,T_2)}{p(t,T_1)}$$

has a deterministic volatility.

Since

$$Z_t = \exp[(A(t,T_2) - A(t,T_1)) - (T_2 - T_1)r_t]$$

taking into account the dynamics of r_t, one obtains

$$dZ_t = Z_t \{[\cdots]dt - \sigma \cdot (T_2 - T_1)dw_t\}$$

The result now follows immediately from the formula that was derived in the subsection "Applications" of section A1. for the case of the Hull and White model. Since Ho-Lee corresponds to Hull and White for $a = 0$, it suffices to let $a \to 0$.

b) Apply the *put-call-parity* that here is based on

$$(K - p(T_1, T_2))^+ = K + (p(T_1, T_2) - K)^+ - p(T_1, T_2)$$

One then has

$$E^Q \left\{ e^{-\int_0^{T_1} r_s ds} (K - p(T_1, T_2))^+ \right\}$$

$$= Kp(0, T_1) + p(0, T_2)\Phi(d_1) - Kp(0, T_1)\Phi(d_2) - p(0, T_2)$$

$$= p(0, T_2)[\Phi(d_1) - 1] + Kp(0, T_1)[1 - \Phi(d_2)]$$

The first equality in the previous formula follows from

$$
\begin{aligned}
& E^Q\left\{e^{-\int_0^{T_1} r_s ds} p(T_1,T_2)\right\} \\
& = E^Q\left\{e^{-\int_0^{T_1} r_s ds} E^Q\left\{e^{-\int_{T_1}^{T_2} r_s ds} \mid \mathcal{F}_{T_1}\right\}\right\} \\
& = E^Q\left\{e^{-\int_0^{T_2} r_s ds}\right\} = p(0,T_2)
\end{aligned}
$$

Notice that the difference $\Pi_0^p - \Pi_0^c$ of the two prices in in $t = 0$ is $Kp(0,T_1) - p(0,T_2)$ in accordance with the fact that the value of this difference in $t = T_1$, i.e. $K - p(T_1,T_2)$ has to be discounted to $t = 0$.

c) Except for the factor $(1 - R\alpha_i)$ we are in the context of point b) with $K = \frac{1}{1-R\alpha_i}$, $T_1 = T_{i-1}$ and $T_2 = T_i$.