Smallest asymptotic bound of solutions to positive mixed fractional-order inhomogeneous linear systems with time-varying delays

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January 30, 2021

Abstract

This paper is concerned with the asymptotic behaviour of solutions to a positive mixed fractional-order inhomogeneous linear system subject to time-varying delays and bounded disturbance. In particular, when its linear part is asymptotically stable and the disturbance is small, we have obtained the smallest bound (as the time is large enough) of the solutions with the arbitrary initial condition. In the case where the assumption that the disturbance converges to 0 as the time t tends to infinity is added, we have shown that every solution of the system tends to the origin. Numerical simulations are finally presented to illustrate the theoretical findings.

Key words: Mixed fractional-order systems, Time-varying delays, Positive inhomogeneous linear systems, Boundedness, Attractivity

1 Introduction

Delay fractional-order systems have received considerable research attention recently. They provide mathematical models of practical systems in which the fractional rate of change depends on the influence of their present and hereditary effects. Below we list some of the most important contributions to this topic.

Let $\hat{\alpha} = (\alpha_1, \dots, \alpha_d) \in (0,1] \times \cdots \times (0,1]$ and $h_k : [0,\infty) \to \mathbb{R}_{\geq 0}$ is continuous for $1 \leq k \leq m$. Consider the mixed fractional-order system having time-varying delays

$$^{C}D^{\hat{\alpha}}_{0+}x(t) = Ax(t) + \sum_{1\leq k\leq m} A_k x(t-h_k(t)) + f(t, x(t), x(t-h_1(t)), \dots, x(t-h_m(t))), t > 0 \tag{1}$$

with the initial condition $x(\cdot) = \phi(\cdot) \in C([-r,0];\mathbb{R}^d)$. Here A, A_k $(1 \leq k \leq m)$ are real $d \times d$ matrices, $^{C}D^{\hat{\alpha}}_{0+}x(t) = ({}^{C}D^{\alpha_1}_{0+}x_1(t), \dots, {}^{C}D^{\alpha_d}_{0+}x_d(t))^{\mathrm{T}}$ with $^{C}D^{\alpha_k}_{0+}x_k(t)$ is the

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Caputo derivative of the order α_k of the function $x_k(\cdot)$ which is defined later, $r := \max_{1\leq k\leq m}\sum_{t\geq 0} h_k(t)$ and $f : [0,\infty)\times\mathbb{R}^d\times\cdots\times\mathbb{R}^d\to\mathbb{R}^d$ is a continuous perturbation.

In [1], under a mild Lipschitz condition of f, by using properties of Mittag-Leffler functions, a weighted norm, and the Banach fixed point theorem, the authors showed the existence and uniqueness of global solutions to System (1) for the case $\alpha_1 = \cdots = \alpha_d$ and delays h_k $(1\leq k\leq m)$ are constant. After that, in [7], Tuan and Trinh have improved this result (they have proved the existence and uniqueness of solutions to System (1) where the fractional-orders $\alpha_1,\dots,\alpha_d$ are different and delays h_k $(1\leq k\leq m)$ vary (bounded or unbounded)).

Regarding the asymptotic behaviour of solutions to System (1), in [11], the authors studied the case $f = 0$, $A = 0$ and have obtained a necessary and sufficient condition for the stability of the system via eigenvalues of the system matrix and their location in a specific area of the complex plane. In [10], the authors have corrected and strengthened results in [11] and have given the stability of (1) with $A = 0$ while in [5], by exploring delayed Mittag-Leffler type matrix functions, the authors have studied the finite time stability of solutions to these systems. Recently, by combining the linearization method, a Lyapunov–Perron type operator and establishing a new weighted type norm associated with a Mittag-Leffler function which is compatible with the dependence on history and the hereditary property of the model, Tuan and Trinh [7] have proposed a sufficient condition for the Mittag-Leffler stability of (1). For the scalar case, in [9], the authors have solved an answer to the open question about the relationship between the stability of the trivial solution of (1) and that of its linearization. By another approach (using a generalized Halanay inequality), in [4], the authors have also obtained a result on the stability of this equation. Notice that the results mentioned above are proved for the case $\alpha_1 = \cdots = \alpha_d$ or $d = 1$. In our opinion, it is clear that the stability theory of mixed fractional-order systems with delays is far from being fully understood. However, the situation becomes simpler if we only focus on positive systems. In this direction, one of the most interesting works is the paper by Shen and Lam [6] where the author has considered the stability and performance analysis of System (1) with the assumptions that matrix A is Metzler, A_1 is nonnegative and $f = 0$. Then, in [8], Tuan, Trinh and Lam have established a necessary and sufficient condition for the positivity of mixed fractional-order linear systems with time-varying delays and a criterion to characterize the stability of these systems where the time-varying delays are bounded or unbounded. Relying on vector Lyapunov-like functions, comparison arguments, reducing the asymptotic stability problem to verify a Hurwitz property on a suitable matrix, in [3], the authors have obtained a general method to establish the asymptotic behaviour of solutions to the multi-order multiple time-varying delays nonlinear systems.

This paper is devoted to discussion on the asymptotic behaviour of solutions to mixed fractional-order inhomogeneous linear systems, that is, System (1) with the term f having the form $Dw(t)$. First, we have obtained the smallest asymptotic bound of solutions to (1) in the case its homogeneous part is asymptotically stable and w is small. Second, when the term w converges to 0 as the time t approaches to infinity, we have proved that every solution of (1) tends to the origin. We have also provided some specific examples to illustrate the proposed result.

Before concluding this section, we introduce some notations which are used throughout this paper. Let $\mathbb{N}$ be the set of natural numbers, $\mathbb{R}$ ($\mathbb{R}_+$) be the set of real numbers (nonnegative real numbers, respectively), and $\mathbb{R}^d$ be the d-dimensional Euclidean space endowed with a norm $\|\cdot\|$, $\mathbb{R}^d_{0,+} := \{x = (x_1, \dots, x_i, \dots, x_d)^{\mathrm{T}} \in \mathbb{R}^d : x_i \geq 0\}$ and $\mathbb{R}^d_+ := \{x = (x_1, \dots, x_i, \dots, x_d)^{\mathrm{T}} \in \mathbb{R}^d : x_i > 0\}$. For any $[a,b] \subset \mathbb{R}$, let $C([a,b];\mathbb{R}^d)$ be the space of continuous functions $\xi : [a,b] \to \mathbb{R}^d$. A matrix $A = (a_{ij})_{1\leq i,j\leq d} \in \mathbb{R}^{d\times d}$ is called Metzler if $a_{ij} \geq 0$ for all $1 \leq i \neq j \leq d$. A matrix $A \in \mathbb{R}^{d\times d}$ is said to be Hurwitz if its spectrum $\sigma(A)$ satisfies

$$\sigma(A) \subset \{\lambda \in \mathbb{C} : \Re\lambda < 0\}.$$

Let $n, m \in \mathbb{N}$ and $A = (a_{ij})^{1\leq j\leq m}_{1\leq i\leq n}, B = (B_{ij})^{1\leq j\leq m}_{1\leq i\leq n} \in \mathbb{R}^{n\times m}$. We write $A \succ B$ ($A \succeq B$) if $a_{ij} > b_{ij}$ ($a_{ij} \geq b_{ij}$, respectively) for all $1 \leq i \leq n,\ 1 \leq j \leq m$. The matrix A is said to be nonnegative if $a_{ij} \geq 0$ for all $1 \leq i \leq n,\ 1 \leq j \leq m$. For $\alpha \in (0,1)$ and an integrable function $x : [a,b] \to \mathbb{R}$, the Riemann–Liouville integral operator of $x(\cdot)$ with the order α is defined by

$$(I^\alpha_{a+}x)(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} x(\tau)\, d\tau, \quad t \in (a,b],$$

where $\Gamma(\cdot)$ is the Gamma function. The *Riemann–Liouville fractional derivative* ${}^{RL}D^\alpha_{a+}x$ of a integrable function $x : [0,T] \to \mathbb{R}$ is defined by

$${}^{RL}D^\alpha_{a+}x(t) = DI^{1-\alpha}_{a+}x(t) \text{ for almost } t \in (a,b],$$

$D = \frac{d}{dt}$ is the usual derivative. The Caputo fractional derivative ${}^{C}D^\alpha_{a+}x$ of a continuous function $x : [a,b] \to \mathbb{R}$ is defined

$$({}^{C}D^\alpha_{a+}x)(t) :=^{RL} D^\alpha_{a+}(x(t) - x(a)) \text{ for almost } t \in (a,b].$$

For more details on fractional calculus, we would like to introduce the reader to the monograph by K. Diethelm [2].

2 The asymptotic behaviour of solutions to mixed-order fractional systems with time-varying delays

Let $\hat{\alpha} = (\alpha_1, \dots, \alpha_d)^{\mathrm{T}} \in (0,1] \times \cdots \times (0,1] \subset \mathbb{R}^d$, $m \in \mathbb{N}$. Assume that $h_k : [0,\infty) \to \mathbb{R}_+$ ($1 \leq k \leq m$) are continuous such that

(F1) $h_k(0) > 0$;

(F2) $h_k(t) \leq h_k$ for all $t \in [0,\infty)$;

(F3) $h_k(0) \neq h_l(0)$ for any $1 \leq k \neq l \leq m$.

Consider the following system

$${}^{C}D^{\hat{\alpha}}_{0+}x(t) = Ax(t) + \sum_{1\leq k\leq m} A_k x(t - h_k(t)) + Dw(t), \quad t \in (0,\infty), \tag{2}$$

$$x(\cdot) = \phi(\cdot) \in C([-r,0];\mathbb{R}^d_{0,+}) \quad \text{on } [-r,0], \tag{3}$$

where $A = (a_{ij})_{1\le i,j\le d}$ is Metzler, $A_k = (a^k_{ij})_{1\le i,j\le d}$ $(1 \le k \le m)$, $D = (d_{ij})_{1\le i,j\le d} \in \mathbb{R}^{d\times d}$ are nonnegative, $w(\cdot) \in C([0,\infty);\mathbb{R}^d_{0,+})$ such that $w(t) \preceq \bar{w}$ on $[0,\infty)$ and $r := \max_{1\le k\le m} h_k$. As showed in [8, Proposition 1], the system (2)–(3) with the delay functions $h_i(\cdot)$ $(1 \le i \le m)$ satisfy the conditions (F1)–(F3) is positive. Denote $U_{\bar{w}} = \{w \in C([0,\infty);\mathbb{R}^d_{0,+}) : w(t) \le \bar{w},\ \forall t \ge 0\}$ and $V_\gamma = \{\phi \in C([-r,0];\mathbb{R}^d_{0,+}) : \phi(t) \le \gamma\}$. In this note we will give some estimates concerning the solution to System (2)–(3) under the further assumption that there is a positive vector $\lambda \in \mathbb{R}^d_+$ such that

(H1) $(A + \sum_{1\le i\le m} A_i)\lambda < 0$,

(H2) $(A + \sum_{1\le i\le m} A_i)\lambda + D\bar{w} \le 0$.

Remark 2.1. The assumption (H1) implies that the matrix $A + \sum_{1\le i\le m} A_i$ is inverse.

The main result is the following theorem.

Theorem 2.2 (Main result)**.** Consider the system (2)–(3). The following statements hold.

(i) For any $\phi \in C([-r,0];\mathbb{R}^d_{0,+})$, we have

$$\sup_{w\in U_{\bar{w}}} \limsup_{t\to\infty} \Phi(t;w,\phi) = -(A + \sum_{1\le i\le m} A_i)^{-1} D\bar{w},$$

where $\Phi(\cdot;w,\phi)$ denotes the solution to System (2)–(3).

(ii) If we add the assumption that $\lim_{t\to\infty} w(t) = 0$. Then,

$$\lim_{t\to\infty} \Phi(t;w,\phi) = 0$$

for every $\phi \in C([-r,0];\mathbb{R}^d_{0,+})$.

To prove Theorem 2.2, we need the preparatory results below.

Proposition 2.3. Consider the system (2)–(3). Assume that there exit $\bar{w} \in \mathbb{R}^d_{0,+}$, $\lambda \in \mathbb{R}^d_+$ such that the conditions (H1) and (H2) are satisfied. Then the solution $\Phi(\cdot;w,\lambda)$ to the system (2) with the initial condition

$$\Phi(t;w,\lambda) = \lambda, \quad \forall t \in [-r,0], \tag{4}$$

satisfies

$$\Phi(t;w,\lambda) \le \lambda, \quad \forall t \ge -r. \tag{5}$$

Proof. Let $e(t) = \lambda - \Phi(t;w,\lambda)$ with $t \ge -r$. Then $e(t) = 0$ for all $t \in [-r,0]$. On the other hand,

$$\begin{aligned} {}^{C}D^{\hat{\alpha}}_{0+} e(t) &= -A\Phi(t;w,\lambda) - \sum_{1\le i\le m} A_i \Phi(t - h_i(t);w,\lambda) - Dw(t) \\ &= Ae(t) + \sum_{1\le i\le m} A_i e(t - h_i(t)) - (A + \sum_{1\le i\le m} A_i)\lambda - Dw(t),\ \forall t > 0. \end{aligned} \tag{6}$$

Furthermore, due to (H1) and (H2), we have

$$-(A+\sum_{1\le i\le m} A_i)\lambda - Dw(t) = -(A+\sum_{1\le i\le m} A_i)\lambda - D\bar{w} + D(\bar{w}-w(t)) \ge 0$$

for all $t \ge 0$. This implies that (6) is positive and $e(t) \ge 0$ for all $t \ge 0$, that is, $\Phi(t; w, \lambda) \le \lambda$ for all $t \ge -r$. □

Remark 2.4. From Proposition 2.3, to obtain the smallest upper bound for the solution to the system (2)–(3), we need to estimate

$$\inf\left\{\|\lambda\| : \lambda \in \mathbb{R}^d_+,\ \lambda \ge \bar{\phi}, (A+\sum_{1\le i\le m} A_i)\lambda < 0,\ (A+\sum_{1\le i\le m} A_i)\lambda + D\bar{w} \le 0\right\}. \tag{7}$$

Remark 2.5. Proposition 2.3 is still true if the condition (F2) is replaced by
(F2)' $t - h_k(t) \ge -r$ for all $t \in [0,\infty)$ with $r > 0$ is fixed and given.

Next, we introduce some properties of the solution to the initial valued problem

$$^{C}D^{\hat{\alpha}}_{0+}x(t) = Ax(t) + \sum_{1\le k\le m} A_k x(t-h_k) + D\bar{w}, \quad t \in (0,\infty), \tag{8}$$

$$x(\cdot) = \lambda \quad \text{on } [-r, 0]. \tag{9}$$

Proposition 2.6. The solution $\Phi_1(\cdot; \bar{w}, \lambda)$ to the system (8)–(9) has following properties.

(i) Let $c > 0$ be arbitrary, then $\Phi_1(t+c; \bar{w}, \lambda) \le \Phi_1(t; \bar{w}, \lambda)$ for any $t \ge 0$.

(ii)
$$\lim_{t\to\infty} \Phi_1(t; \bar{w}, \lambda) = -(A+\sum_{1\le i\le m} A_i)^{-1} D\bar{w}. \tag{10}$$

Proof. (i) Fix $c > 0$ and let $e(t) = \Phi_1(t; \bar{w}, \lambda) - \Phi_1(t+c; \bar{w}, \lambda)$ for every $t \ge -r$. From Proposition 2.3, we have

$$e(t) \ge 0, \quad t \in [-r, 0].$$

Furthermore, for $t \ge 0$,

$$^{C}D^{\hat{\alpha}}_{0+}e(t) = Ae(t) + \sum_{1\le i\le m} A_i e(t-h_i).$$

Hence, $e(t) \ge 0$ and thus $\Phi_1(t+c; \bar{w}, \lambda) \le \Phi_1(t; \bar{w}, \lambda)$ for all $t \ge 0$.

(ii) From (i), we see that $\Phi_1(\cdot; \bar{w}, \lambda)$ is decreasing. Hence, the limit $\lim_{t\to\infty} \Phi_1(t; \bar{w}, \lambda)$ exists. Put $\ell = \lim_{t\to\infty} \Phi(t; \bar{w}, \lambda)$. First, we have

$$\lim_{t\to\infty} {}^{C}D^{\hat{\alpha}}_{0+}\Phi_1(t; \bar{w}, \lambda) = (A+\sum_{1\le i\le m} A_i)\ell + D\bar{w}.$$

Then, by Final Value Theorem, we obtain

$$\begin{aligned}\lim_{t\to\infty} {}^C D^{\hat{\alpha}}_{0+}\Phi_1(t;\bar{w},\lambda) &= \lim_{s\to+0} s\mathcal{L}\{{}^C D^{\hat{\alpha}}_{0+}\Phi_1(t;\bar{w},\lambda)\}\\ &= \lim_{s\to+0} s\left[s^{\alpha_1}\mathcal{L}\{\Phi_1^1(\cdot;\bar{w},\lambda)\}(s) - s^{\alpha_1-1}\lambda_1,\dots,s^{\alpha_d}\mathcal{L}\{\Phi_1^d(\cdot;\bar{w},\lambda)\}(s) - s^{\alpha_d-1}\lambda_d\right]\\ &= \lim_{s\to+0}\left[s^{\alpha_1}(s\mathcal{L}\{\Phi_1^1(\cdot;\bar{w},\lambda)\}(s) - \lambda_1),\dots,s^{\alpha_d}(s\mathcal{L}\{\Phi_1^d(\cdot;\bar{w},\lambda)\}(s) - \lambda_d)\right]\\ &= 0,\end{aligned}$$

where $\mathcal{L}$ denote the Laplace transform. Thus, $(A + \sum_{1\le i\le m} A_i)\ell + D\bar{w} = 0$ and

$$\ell = -(A + \sum_{1\le i\le m} A_i)^{-1} D\bar{w}.$$

□

Proposition 2.7. Consider the system

$${}^C D^{\hat{\alpha}}_{0+}x(t) = Ax(t) + \sum_{1\le k\le m} A_k x(t-h_k) + Dw(t), \quad t\in(0,\infty), \tag{11}$$

where $w:[0,\infty)\to\mathbb{R}^d$ is continuous and non-negative which satisfies $w(t)\preceq\bar{w}$ for any $t\in[0,\infty)$ and $\lim_{t\to\infty} w(t) = 0$. Moreover, there exists $\lambda\in\mathbb{R}^d_+$ such that the conditions (H1) and (H2) hold. Then, the solution $\Phi_1(\cdot;w,\lambda)$ of (11) with the initial condition $\Phi_1(t;w,\lambda) = \lambda$ on $[-r,0]$ has the limit

$$\lim_{t\to\infty}\Phi_1(t;w,\lambda) = 0.$$

Proof. From Proposition 2.3, we see that the solution $\Phi_1(\cdot;w,\lambda)$ is non-negative. By using the same arguments as in the proof of Proposition 2.6, we observe that this solution is decreasing. Thus, there is a $\ell_1\in\mathbb{R}^d_{0,+}$ such that

$$\lim_{t\to\infty}\Phi_1(t;w,\lambda) = \ell_1.$$

From (11), let $t\to\infty$, we see that

$$\lim_{t\to\infty} {}^C D^{\hat{\alpha}}_{0+}\Phi_1(t;w,\lambda) = (A + \sum_{1\le i\le m} A_i)\ell_1.$$

On the other hand, by using Final Value Theorem, then

$$\begin{aligned}\lim_{t\to\infty} {}^C D^{\hat{\alpha}}_{0+}\Phi_1(t;w,\lambda) &= \lim_{s\to+0} s\mathcal{L}\{{}^C D^{\hat{\alpha}}_{0+}\Phi_1(t;w,\lambda)\}\\ &= \lim_{s\to+0} s\left[s^{\alpha_1}\mathcal{L}\{\Phi_1^1(\cdot;w,\lambda)\}(s) - s^{\alpha_1-1}\lambda_1,\dots,s^{\alpha_d}\mathcal{L}\{\Phi_1^d(\cdot;w,\lambda)\}(s) - s^{\alpha_d-1}\lambda_d\right]\\ &= \lim_{s\to+0}\left[s^{\alpha_1}(s\mathcal{L}\{\Phi_1^1(\cdot;w,\lambda)\}(s) - \lambda_1),\dots,s^{\alpha_d}(s\mathcal{L}\{\Phi_1^d(\cdot;w,\lambda)\}(s) - \lambda_d)\right]\\ &= 0,\end{aligned}$$

which implies that $\ell_1 = 0$. The proof is complete. □

Denote by $\Phi(\cdot;\bar{w},\lambda)$ the solution to the system

$$
{}^{C}D^{\hat{\alpha}}_{0+}x(t) = Ax(t) + \sum_{1\le k\le m} A_k x(t-h_k(t)) + D\bar{w}, \quad t\in(0,\infty), \tag{12}
$$

$$
x(\cdot) = \lambda \quad \text{on } [-r,0]. \tag{13}
$$

Using Proposition 2.6 and the positivity of the system, we obtain the following proposition.

Proposition 2.8. Under the assumptions as above on A, A_k $(1\le k\le m)$, D, w and λ, we have

$$
\Phi_1(t;\bar{w},\lambda) \ge \Phi(t;\bar{w},\lambda)
$$

for every $t\ge 0$.

Proof. Let $e(t) = \Phi_1(t;\bar{w},\lambda) - \Phi(t;\bar{w},\lambda)$ for all $t\ge -r$. For any $t\ge 0$, we have

$$
\begin{aligned}
{}^{C}D^{\hat{\alpha}}_{0+}e(t) &= A(\Phi_1(t;\bar{w},\lambda) - \Phi(t;\bar{w},\lambda)) + \sum_{1\le k\le m} A_k(\Phi_1(t-h_k(t);\bar{w},\lambda) - \Phi(t-h_k(t);\bar{w},\lambda)) \\
&\quad + \sum_{1\le k\le m} A_k(\Phi_1(t-h_k;\bar{w},\lambda) - \Phi_1(t-h_k(t);\bar{w},\lambda)) \\
&= Ae(t) + \sum_{1\le k\le m} A_k e_k(t-h_k(t)) + \sum_{1\le k\le m} A_k(\Phi_1(t-h_k;\bar{w},\lambda) - \Phi_1(t-h_k(t);\bar{w},\lambda)).
\end{aligned}
$$

Note that from Proposition 2.6, $\Phi_1(t-h_k;\bar{w},\lambda) - \Phi_1(t-h_k(t);\bar{w},\lambda) \ge 0$ for all $t\ge 0$ and $1\le k\le m$. Hence, $e(t)\ge 0$ and $\Phi_1(t;\bar{w},\lambda)\ge\Phi(t;\bar{w},\lambda)$ for $t\ge 0$. $\square$

Let $\hat{\Phi}(\cdot;\bar{w},0)$ be the solution of the system

$$
\begin{aligned}
{}^{C}D^{\hat{\alpha}}_{0+}x(t) &= (A + \sum_{1\le i\le m} A_i)x(t) + D\bar{w}, \quad t>0, \\
x(0) &= 0.
\end{aligned}
$$

We have the following relations between $\hat{\Phi}(\cdot;\bar{w},0)$, the solution $\Phi_1(\cdot;\bar{w},0)$ of System (8) satisfying the initial condition $\Phi_1(\cdot;\bar{w},0) = 0$ on $[-r,0]$ and the solution $\Phi(\cdot;\bar{w},0)$ of System (12) with the initial condition 0.

Proposition 2.9. The following statements hold.

(i) For any $t\ge 0$, we have

$$
\Phi_1(t;\bar{w},0) \le \Phi(t;\bar{w},0) \le \hat{\Phi}(t;\bar{w},0).
$$

(ii)

$$
\lim_{t\to\infty}\Phi(t;\bar{w},0) = \lim_{t\to\infty}\Phi_1(t;\bar{w},0) = \lim_{t\to\infty}\hat{\Phi}(\cdot;\bar{w},0) = -(A + \sum_{1\le i\le m} A_i)^{-1}D\bar{w}.
$$

Proof. (i) See [6, Lemma 7].
(ii) Note that the solution $\hat{\Phi}(\cdot;\bar{w},0)$ is increasing. Hence, the limit $\lim_{t\to\infty}\hat{\Phi}(t;\bar{w},0)$ exists. By the similar arguments as in the proof of Proposition 2.6, we have

$$\lim_{t\to\infty}\hat{\Phi}(t;\bar{w},0) = -(A+\sum_{1\le i\le m}A_i)^{-1}D\bar{w}.$$

Furthermore, following [6, Lemma 8], then

$$\lim_{t\to\infty}\Phi_1(t;\bar{w},0) = -(A+\sum_{1\le i\le m}A_i)^{-1}D\bar{w},$$

which combining with (i) completes the proof. □

We are now in the position to complete the proof of Theorem 2.2.

Proof of Theorem 2.2. (i) Let $\lambda\in\mathbb{R}^d_+$ satisfying (H1) and (H2). For any $\phi\in C([-r,0];\mathbb{R}^d_{0,+})$, we can find a constant $c\ge 1$ such that $\phi\in V_{c\lambda}$. Notice that $c\lambda$ also satisfies the conditions (H1) and (H2). It is obvious that for any $w\in U_{\bar{w}}$, by using Proposition 2.8 and the same arguments as in the proof of this result, we have

$$\Phi(t;w,\phi)\preceq\Phi(t;w,c\lambda)\preceq\Phi(t;\bar{w},c\lambda)\le\Phi_1(t;\bar{w},c\lambda)$$

for all $t\ge 0$. Furthermore,

$$\lim_{t\to\infty}\Phi_1(t;\bar{w},c\lambda) = -(A+\sum_{1\le i\le m}A_i)^{-1}D\bar{w}.$$

Thus,

$$\limsup_{t\to\infty}\Phi(t;w,\phi)\preceq -(A+\sum_{1\le i\le m}A_i)^{-1}D\bar{w}. \tag{14}$$

On the other hand, from Proposition 2.9, we see that

$$\lim_{t\to\infty}\Phi(t;\bar{w},0) = -(A+\sum_{1\le i\le m}A_i)^{-1}D\bar{w},$$

which together with (14) implies that

$$\sup_{w\in U_{\bar{w}}}\limsup_{t\to\infty}\Phi(t;w,\phi) = -(A+\sum_{1\le i\le m}A_i)^{-1}D\bar{w}.$$

(ii) For any $\phi\in C([-r,0];\mathbb{R}^d_{0,+})$, as above, there exits $c\ge 1$ such that (H1) and (H2) hold for $c\lambda$. From the estimates

$$0\preceq\Phi(t;w,\phi)\preceq\Phi(t;w,c\lambda)\preceq\Phi(t;\bar{w},c\lambda)\le\Phi_1(t;\bar{w},c\lambda)$$

and the fact that

$$\lim_{t\to\infty}\Phi_1(t;\bar{w},c\lambda) = 0,$$

it implies that

$$\lim_{t\to\infty}\Phi(t;w,\phi) = 0.$$

The proof is complete. □

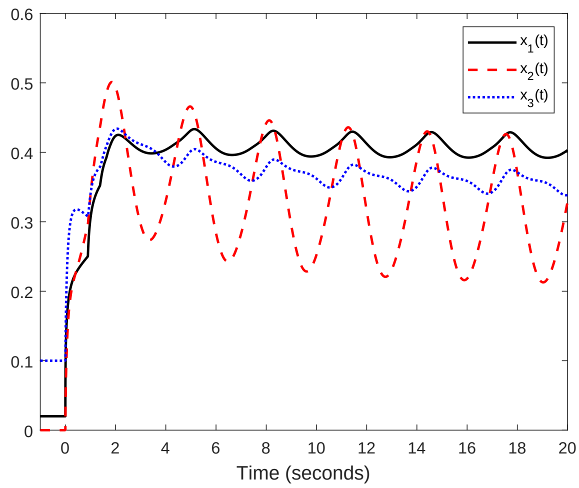

Figure 1: Trajectories of the solution $\varphi(\cdot,\phi)$ to system (15) when $\phi(t) = (0.02, 0, 0.1)^{\mathrm{T}}$ on $[-1, 0]$

3 Illustrative example

This section introduces some examples to illustrate the effectiveness of proposed results.

Example 3.1. Let $\hat{\alpha} = (\alpha_1, \alpha_2, \alpha_3) = (0.5, 0.7, 0.8)$, $h : [0,\infty) \to \mathbb{R}_{\geq 0}$ is defined by $h(t) = \sin^2(t+1)$ and $w : [0,\infty) \to \mathbb{R}^3_{0,+}$ having the form

$$w(t) = (\cos^2(t), \sin^2(t), \frac{1}{1+t})^{\mathrm{T}}.$$

Consider the fractional-order system with a time-varying delay

$$^{C}D^{\hat{\alpha}}_{0+}x(t) = Ax(t) + A_1 x(t-h(t)) + Dw(t), \quad t > 0, \tag{15}$$

where

$$A = \begin{pmatrix} -5 & 1 & 0 \\ 0.5 & -4 & 0.5 \\ 1 & 0 & -6 \end{pmatrix}, \ A_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \text{ and } D = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

By choosing $\lambda = \bar{w} = (1,1,1)^{\mathrm{T}}$ then the conditions (H1) and (H2) are satisfied. Thus, from Theorem 2.2, for any $\phi \in C([-1,0];\mathbb{R}^3_{0,+})$, the solution $\Phi(\cdot; w, \phi)$ of (15) with the initial condition ϕ is bounded by $b = (1.002, 1.001, 0.999)^{\mathrm{T}}$ in $\mathbb{R}^3_{0,+}$ as the time t is large enough. In Figure 1, we simulate trajectories of the solution $\Phi(\cdot; w, \phi)$ to (15) with the initial condition $\phi(t) = (0.02, 0, 0.1)^{\mathrm{T}}$ on $[-1, 0]$.

Example 3.2. Let $\hat{\alpha} = (\alpha_1, \alpha_2, \alpha_3) = (0.5, 0.7, 0.8)$, $h : [0,\infty) \to \mathbb{R}_{\geq 0}$ is defined by $h(t) = \sin^2(t+1)$ and $w : [0,\infty) \to \mathbb{R}^3_{0,+}$ having the form

$$w(t) = (0, \frac{\sin^2(t)}{1+t^2}, \frac{1}{1+t^2})^{\mathrm{T}}.$$

Consider the fractional-order system with a time-varying delay

$$^{C}D^{\hat{\alpha}}_{0+}x(t) = Ax(t) + A_1 x(t-h(t)) + Dw(t), \quad t > 0, \tag{16}$$

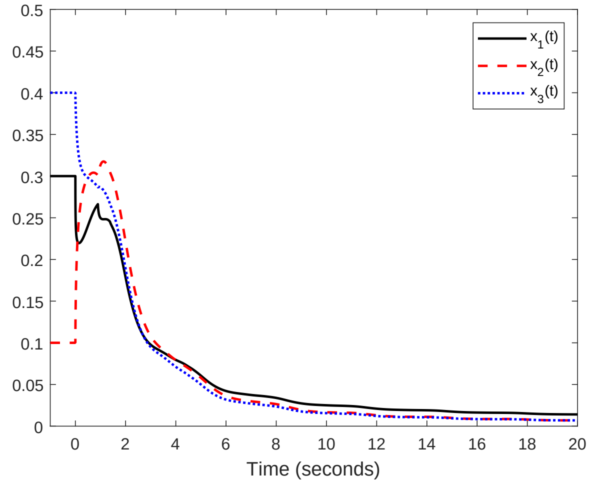

Figure 2: Trajectories of the solution $\varphi(\cdot, \phi)$ to system (16) when $\phi(t) = (0.3, 0.1, 0.4)^{\mathrm{T}}$ on $[-1, 0]$

where

$$A = \begin{pmatrix} -5 & 1 & 0 \\ 0.5 & -4 & 0.5 \\ 1 & 0 & -6 \end{pmatrix}, \ A_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \text{ and } D = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

By Theorem 2.2, for any $\phi \in C([-1, 0]; \mathbb{R}^3_{0,+})$, the solution $\Phi(\cdot; w, \phi)$ of (16) with the initial condition ϕ converges to the origin as the time $t \to \infty$. In Figure 2, we simulate trajectories of the solution $\Phi(\cdot; w, \phi)$ to (16) with the initial condition $\phi(t) = (0.3, 0.1, 0.4)^{\mathrm{T}}$ on $[-1, 0]$.

4 Conclusion

This paper is devoted to discussing the asymptotic behaviour of solutions to a positive mixed fractional-order inhomogeneous linear system subject to time-varying delays and a bounded disturbance. First, we have obtained the smallest bound of solutions (as the time is large enough) with the arbitrary initial conditions in the case its homogeneous part is asymptotically stable and the disturbance is small. When the assumption that the disturbance converges to 0 is added, we have shown that every solution of the system tends to the origin. To do these, our approach is using the positivity of the system and a new comparison method that is suitable for fractional-order systems.

Acknowledgement

This research is supported by The International Center for Research and Postgraduate Training in Mathematics–Institute of Mathematics–Vietnam Academy of Science and Technology under the Grant ICRTM01-2020.09. A part of this paper was completed while the first author was a postdoc at the Vietnam Institute for Advanced Study in Mathematics (VIASM). He would like to thank VIASM for support and hospitality.

References

[1] N. D. Cong and H. T. Tuan. Existence, uniqueness, and exponential boundedness of global solutions to delay fractional differential equations. *Mediterr. J. Math.*, 14(5):Paper No. 193, 12, 2017.

[2] Kai Diethelm. *The analysis of fractional differential equations*, volume 2004 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2010. An application-oriented exposition using differential operators of Caputo type.

[3] Javier A. Gallegos, Norelys Aguila-Camacho, and Manuel Duarte-Mermoud. Vector Lyapunov-like functions for multi-order fractional systems with multiple time-varying delays. *Commun. Nonlinear Sci. Numer. Simul.*, 83:105089, 12, 2020.

[4] Bin-Bin He, Hua-Cheng Zhou, YangQuan Chen, and Chun-Hai Kou. Asymptotical stability of fractional order systems with time delay via an integral inequality. *IET Control Theory Appl.*, 12(12):1748–1754, 2018.

[5] Mengmeng Li and JinRong Wang. Exploring delayed Mittag-Leffler type matrix functions to study finite time stability of fractional delay differential equations. *Appl. Math. Comput.*, 324:254–265, 2018.

[6] Jun Shen and James Lam. Stability and performance analysis for positive fractional-order systems with time-varying delays. *IEEE Trans. Automat. Control*, 61(9):2676–2681, 2016.

[7] H. T. Tuan and H. Trinh. A qualitative theory of time delay nonlinear fractional-order systems. *SIAM J. Control Optim.*, 58(3):1491–1518, 2020.

[8] Hoang The Tuan, Trinh Hieu, and James Lam. Positivity and stability of mixed fractional-order systems with unbounded delays: Necessary and sufficient conditions. *International Journal of Robust and Nonlinear Control*, 31(1):37–50, 2021.

[9] Hoang The Tuan and Stefan Siegmund. Stability of scalar nonlinear fractional differential equations with linearly dominated delay. *Fract. Calc. Appl. Anal.*, 23(1):250–267, 2020.

[10] Hoang The Tuan and Hieu Trinh. A linearized stability theorem for nonlinear delay fractional differential equations. *IEEE Trans. Automat. Control*, 63(9):3180–3186, 2018.

[11] Jan Čermák, Jan Horníček, and Tomáš Kisela. Stability regions for fractional differential systems with a time delay. *Commun. Nonlinear Sci. Numer. Simul.*, 31(1-3):108–123, 2016.