TANNAKIAN DUALITY AND GAUSS-MANIN CONNECTIONS FOR A FAMILY OF CURVES

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ABSTRACT. Let R be a Dedekind ring that contains a field k of characteristic 0. Let X/R be a smooth projective scheme with connected fibers. We relate the absolute differential fundamental group of X over k with the fundamental group of $\operatorname{Spec}(R)$ over k and the geometric relative fundamental group in a short exact sequence. There are natural maps from the group cohomology of the geometric relative fundamental group to the Gauss-Manin connections. We prove that these maps are isomorphisms when R is a complete DVR and X has relative dimension 1 over R.

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1. Introduction

For algebraic schemes, Grothendieck's étale fundamental group serves as a replacement for the topological fundamental group. Let X/k be a separated scheme, and let $x \in X \times_k \bar{k}$ be a $\bar{k}$-point of $X \times_k \bar{k}$, where $\bar{k}$ is an algebraic closure of k. Grothendieck defines in [SGA1] the étale fundamental group

$\pi^{\text{et}}(X,x)$. This group is a profinite group which classifies Galois coverings of X (with a distinguished point above x). Then Grothendieck deduces the following exact sequences.

- The fundamental exact sequence which relates the étale fundamental groups of X, with the geometric fundamental group $\pi^{\text{et}}(X\times_k \bar{k},x)$ and the Galois group $\text{Gal}(\bar{k}/k)$:

$$1\longrightarrow \pi^{\text{et}}(X\times_k \bar{k},x)\longrightarrow \pi^{\text{et}}(X,x)\longrightarrow \text{Gal}(\bar{k}/k)\longrightarrow 1. \tag{1}$$

- The homotopy exact sequence which relates the fundamental group of X, of S and of the fiber $X_s := f^{-1}(s)$:

$$\pi^{\text{et}}(X_s,x)\longrightarrow \pi^{\text{et}}(X,x)\longrightarrow \pi^{\text{et}}(S,s)\longrightarrow 1, \tag{2}$$

provided that f is proper with connected fibers.

There is yet another algebraic replacement of the topological fundamental group motivated by the Riemann-Hilbert correspondence between systems of analytic linear differential equations, i.e., $\mathcal{O}_X$-coherent modules equipped with a flat connection, and local systems on X, i.e., finite dimensional complex linear representations of the topological fundamental group of X. For a smooth scheme X over a field k of characteristic 0, consider the category of flat connections on X/k. A k-rational point x of X yields a fiber functor, i.e., it associates to each connection its fiber at x. Thus, we have a Tannakian category and Tannakian duality yields a pro-algebraic affine group scheme. This group scheme is called the *differential fundamental group of X at x* and is denoted by $\pi(X,x)$.

Let k be an algebraically closed field of characteristic zero and R be a k-algebra which is a Dedekind domain. Let $f: X\longrightarrow S := \text{Spec}(R)$ be a smooth, projective morphism with connected fibers. Fix a point $x\in X(k)$ and its image $s = f(x)\in S(k)$. This yields an exact sequence similar to the one mentioned above. The homotopy sequence

$$\pi(X_s/k,x)\longrightarrow \pi(X/k,x)\longrightarrow \pi(R/k)\longrightarrow 1, \tag{3}$$

which was shown to be exact in characteristics zero by L. Zhang [Zha14] and in any characteristic by J. P. dos Santos [dS15] (see Theorem 3.1).

Assume that S is affine and has dimension one, that is, $S = \text{Spec}(R)$, where R is Dedekind ring of equal characteristic 0. Then one also has an analog of the fundamental exact sequence as follows.

Let $\eta: S\longrightarrow X$ be a section to f, or, in other words, an R-point of X as an R-schemes (by means of f). Let s be a k-point of $\text{Spec}(R)$ and $x=\eta(s)$. Tannakian duality applied to the category of flat connections on X/k, R/k, and X/R, equipped with natural fiber functors, yields the fundamental groupoids $\Pi(X/k)$, $\Pi(R/k)$, and the *relative differential fundamental group* $\pi(X/R)$ (see Appendix A.2). An absolute connection on X/k can be *inflated* to a relative connection on X/R (cf. 2.2.3). This gives a functor from $\text{MIC}(X/k)$ to $\text{MIC}(X/R)$, and, by Tannakian duality as homomorphism $\pi(X/R)\longrightarrow \Pi(X/k)$. This maps fits in to a sequence

$$\pi(X/R)\longrightarrow \Pi(X/k)\longrightarrow \Pi(R/k)\longrightarrow 1, \tag{4}$$

which is shown to be exact (see 3.2), and we call it the *fundamental exact sequence.* More precisely, we introduce the category of relative connections of geometric origin, which determines the *geometric relative fundamental group* $\pi^{\text{geom}}(X/R)$. Then the above sequence splits into the following exact sequences:

$$\pi(X/R)\longrightarrow \pi^{\text{geom}}(X/R)\longrightarrow 1$$

and

$$1\longrightarrow \pi^{\text{geom}}(X/R)\longrightarrow \Pi(X/k)\longrightarrow \Pi(R/k)\longrightarrow 1. \tag{5}$$

We note that this result generalizes Theorem 5.11 of [EH06] and was implicitly mentioned in [DHdS18, Theorem 9.2]. It was formulated and proved in the PhD thesis of Hugo Bay-Rousson [BR19]; however, some parts of the argument in the proof there are missing. For the sake of completeness, we provide here a detailed proof.

Given the exact sequences, our aim is to compare the de Rham cohomology and the group cohomology as well as to express the Gauss-Manin connection in terms of group cohomology. This is motivated by a remark in [EH06, Section 6] that the Gauss-Manin connection can be interpreted as the action of the fundamental group of the base S on the group cohomology of the geometric relative fundamental group.

More precisely, the exact sequence (5) yields a map

$$\mathrm{H}^0(\pi^{\mathrm{geom}}(X/R), V) \longrightarrow \mathrm{H}^0_{\mathrm{dR}}(X/R, \mathrm{inf}(\mathcal{V}))$$

which is bijective, hence the maps

$$\mathrm{R}^i_{\mathrm{Rep}(R:\Pi(X/k))}\mathrm{H}^0(\pi^{\mathrm{geom}}(X/R), V) \longrightarrow \mathrm{H}^{\mathrm{i}}_{\mathrm{dR}}(X/R, \mathrm{inf}(\mathcal{V})).$$

Our first main result is:

Theorem 1.1 (Theorem 5.1)**.** *Let X be a projective smooth geometrically connected fiber over Dedekind domain R. Let V be an object in* $\mathrm{Rep}^{\mathrm{f}}(R:\Pi(X/k))$. *For $i \geq 0$, the canonical homomorphism*

$$\mathrm{R}^i_{\mathrm{Rep}(R:\Pi(X/k))}\mathrm{H}^0(\pi^{\mathrm{geom}}(X/R), V) \longrightarrow \mathrm{H}^i(\pi^{\mathrm{geom}}(X/R), V)$$

is an isomorphism. Consequently, it induces a representation of $\Pi(R/k)$ *on* $\mathrm{H}^i(\pi^{\mathrm{geom}}(X/R), V)$ *which has the property that the canonical homomorphism*

$$\delta^i : \mathrm{H}^i(\pi^{geom}(X/R), V) \longrightarrow \mathrm{H}^i_{\mathrm{dR}}(X/R, \mathcal{V}). \tag{6}$$

is $\Pi(R/k)$*-equivariant for $i \geq 0$.*

In general, the maps δ^i are far from being isomorphisms. However, for the case $f: X \longrightarrow \mathrm{Spec}(A)$ is a smooth family of curves of genus $g \geq 1$, where A is a complete discrete valuation ring, we show they are isomorphisms for all i (cf. Corollary 7.9). In summary, the main goal of this paper is to prove the following result.

Theorem 1.2 (Theorem 6.9, Corollary 7.9)**.** *Let X be a projective curve with genus $g \geq 1$ over complete discrete valuation ring A. Let $\mathcal{V}$ be an object in* $\mathrm{MIC}^{\mathrm{geom}}(X/A)$. *Denote V as the representation corresponding to $\mathcal{V}$. Then the map*

$$\delta^i : \mathrm{H}^i(\pi(X/A), V) \longrightarrow \mathrm{H}^i_{\mathrm{dR}}(X/A, \mathcal{V})$$

is bijective for all $i \geq 0$. Moreover, if $\mathcal{V}$ be an object in $\mathrm{MIC}^{\circ}(X/k)$, *then*

$$\delta^i : \mathrm{H}^i(\pi^{\mathrm{geom}}(X/A), V) \longrightarrow \mathrm{H}^i(\pi(X/A), V) \longrightarrow \mathrm{H}^i_{\mathrm{dR}}(X/A, \mathcal{V})$$

is bijective for all $i \geq 0$.

1.1. **Outline of the paper.**

(1) Section 2 is dedicated to presenting the main objects and the categories we work with throughout the paper. We also discuss a range of differential fundamental groups and differential fundamental groupoids, in which the geometric relative fundamental group $\pi^{\mathrm{geom}}(X/R)$ (see 2.2.4) plays a very important role.

(2) In Section 3, we prove the exactness of the fundamental exact sequence (4), see Theorem 3.2. We prove this by showing that the kernel L of the following map

$$\Pi(X/k) \longrightarrow \Pi(R/k)$$

coincides with the "image" of the following map

$$\pi(X/R) \longrightarrow \Pi(X/k).$$

This "image" is actually the relative geometric fundamental group $\pi^{\mathrm{geom}}(X/R)$, this fact is a consequence of the homotopy exact sequence of J. P. dos Santos (see Theorem 3.1). Thus, we not only obtain the fundamental exact sequence but also show that the kernel L is the Tannakian fundamental group, which is very important for study its cohomology in Section 7.

(3) In Section 4, we review the concept of relative de Rham cohomology and the Gauss-Manin connection (see 4.1 and 4.2). We also show that the de Rham cohomology of inflated connection is a finite projective R-module provided that the map $X \longrightarrow S = \mathrm{Spec}(R)$ is projective and smooth (see Lemma 4.2). However, in the case of interest, where $X \longrightarrow \mathrm{Spec}(A)$ with A is a cDVR, the structure morphism is not of finite type. Therefore, in this section, we introduce the notion of de Rham cohomology for relative formal schemes with coefficients is a formal connection $\mathrm{H}^{\bullet}_{\mathrm{dR}}(\mathfrak{X}/\mathfrak{S},(\mathcal{V},\widehat{\nabla}))$, see 4.3.3.

By means of Grothendieck's existence theorem (also known as GFGA principle), we have

$$\mathrm{H}^{\bullet}_{\mathrm{dR}}(X/A,(\mathcal{V},\nabla_{/A})) \cong \mathrm{H}^{\bullet}_{\mathrm{dR}}(\mathfrak{X}/\mathfrak{S},(\widehat{\mathcal{V}},\widehat{\nabla}_{/A})),$$

as shown in Theorem 4.10. Thus, we can equip de Rham cohomology sheaf $\mathrm{H}^{\bullet}_{\mathrm{dR}}(X/A,(\mathcal{V},\nabla_{/A}))$ with the Gauss-Manin connection of $\mathrm{H}^{\bullet}_{\mathrm{dR}}(\mathfrak{X}/\mathfrak{S},(\widehat{\mathcal{V}},\widehat{\nabla}_{/A}))$. This implies that $\mathrm{H}^{\bullet}_{\mathrm{dR}}(X/A,(\mathcal{V},\nabla_{/A}))$ is locally free. Working with the formal setting also allows us to use the Leray spectral sequence (see Theorem 4.12), which is truly essential in proving the following map

$$\mathrm{H}^{n}(\pi(\mathfrak{X}/k),V) \longrightarrow \mathrm{H}^{n}_{\mathrm{dR}}(\mathfrak{X}/k,(\mathcal{V},\widehat{\nabla}))$$

is bijective for $n \geq 0$ (see Lemma 7.7). This comparison is acctually key to studying the cohomology of geometric relative fundamental group in Section 7.

(4) In Section 5, we prove Theorem 5.1. From the short exact sequence (5):

$$1 \longrightarrow \pi^{\mathrm{geom}}(X/R) \longrightarrow \Pi(X/k) \longrightarrow \Pi(R/k) \longrightarrow 1,$$

we prove that $\mathrm{H}^0(\pi^{\mathrm{geom}}(X/R),V)$ is a $\Pi(R/k)$-representation for any V, where V is a $\Pi(X/k)$-representation. Consider the following diagram of functors:

$$\begin{array}{ccc} \mathrm{Rep}(R:\Pi(X/k)) & \xrightarrow{\mathrm{H}^0(\pi^{\mathrm{geom}}(X/R),-)} & \mathrm{Rep}(R:\Pi(R/k)) \\ \downarrow{\scriptstyle \mathrm{Res}^{\Pi(X/k)}_{\pi^{\mathrm{geom}}(X/R)}} & & \downarrow{\scriptstyle \mathrm{Res}^{\Pi(R/k)}_{1}} \\ \mathrm{Rep}(\pi^{\mathrm{geom}}(X/R)) & \xrightarrow{\mathrm{H}^0(\pi^{\mathrm{geom}}(X/R),-)} & \mathrm{Mod}_R. \end{array}$$

(5) In Section 6, we present a comparison between the group cohomology of the relative fundamental group $\mathrm{H}^{\bullet}(\pi(X/A),V)$ and the relative de Rham cohomology $\mathrm{H}^{\bullet}_{\mathrm{dR}}(X/R,\mathcal{V})$, where X/A is a smooth families of curves of genus $g \geq 1$ over a complete discrete valuation ring A (see Theorem 6.9). We establish this comparison by reducing it to comparisons at a generic fiber and at a closed fiber, requiring an understanding of the fiber of Tannakian groups and the scalar extension of Tannakian categories. In the case of closed fibers, everything is nearly completed in [DH18]. However, in the generic fiber case, it poses a real challenge to us. In case of X has genus $g \geq 2$, the proof of the comparison in the generic fiber case (see Proposition 6.5) relies on the following results:
- Poincaré duality for de Rham cohomology $\mathrm{H}^{\bullet}_{\mathrm{dR}}(X/K,\mathcal{V}_K)$.
- There are infinitely many non-isomorphism simple connections in $\mathrm{MIC}^{\circ}(X_K/K)$.
- The first de Rham cohomology $\mathrm{H}^{1}_{\mathrm{dR}}(X/K,\mathcal{V}_K)$ is non-vanishing.

Last but not least, the case $g = 1$ is separately proven using the fact that the fundamental group $\pi(X_K/K)$ is commutative.

(6) For $i \geq 0$, the natural map δ^i (6) can be factored as follows:

$$\delta^i : \mathrm{H}^i(\pi^{\mathrm{geom}}(X/R), V) \longrightarrow \mathrm{H}^i(\pi(X/R), V) \longrightarrow \mathrm{H}^i_{\mathrm{dR}}(X/R, (\mathcal{V}, \nabla_{/R})).$$

In section 7, we aim to prove the main corollary (see Corollary 7.9), which states that the cohomology of geometric relative fundamental group $\mathrm{H}^n(\pi^{\mathrm{geom}}(X/A), V)$ is the same as the cohomology of relative differential group $\mathrm{H}^n(\pi(X/A), V)$, where X/A is a projective smooth curves of genus $g \geq 1$ over complete discrete valuation ring A.

We prove that

$$\mathrm{H}^1(\pi^{\mathrm{geom}}(X/R), V) \cong \mathrm{H}^1(\pi(X/R), V)$$

through the universal extension theorem (see Theorem 7.1). To compare at the higher cohomologies, we move the problem to the formal setting. We construct the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{H}^i(\pi^{\mathrm{geom}}(X/A), V) & \longrightarrow & \mathrm{H}^i(\pi(X/A), V) \\
\text{Corollary7.9} \uparrow \cong & & \text{Theorem 6.9} \downarrow \cong \\
\mathrm{H}^i(\Pi(\mathfrak{X}/k)^{\Delta}, V) & & \mathrm{H}^i_{\mathrm{dR}}(X/A, (\mathcal{V}, \nabla_{/A})) \\
 & & \text{Theorem 4.10} \downarrow \cong \\
\text{Lemma7.8} \uparrow \cong & & \mathrm{H}^i_{\mathrm{dR}}(\mathfrak{X}/\mathfrak{S}, (\widehat{\mathcal{V}}, \widehat{\nabla}_{/A})) \\
 & & \text{Lemma7.3} \uparrow \cong \\
\mathrm{H}^i(\pi(\mathfrak{X}/k), V_s) \otimes A & \xrightarrow[\cong]{\text{Lemma7.7}} & \mathrm{H}^i_{\mathrm{dR}}(\mathfrak{X}/k, (\widehat{\mathcal{V}}, \widehat{\nabla})) \otimes_k A,
\end{array}$$

and then we prove that all arrows in the above diagram are bijective.

In case V is the restriction to $\mathrm{Rep}(\pi^{\mathrm{geom}}(X/R))$ of a representation of $\Pi(X/k)$, there exists a canonical homomorphism

$$\mathrm{R}^i_{\mathrm{Rep}(R:\Pi(X/k))}\mathrm{H}^0(\pi^{\mathrm{geom}}(X/R), V) \longrightarrow \mathrm{H}^i(\pi^{\mathrm{geom}}(X/R), V).$$

Theorem 5.1 asserts that this morphism is an isomorphism.

(7) In Appendix 1, we discuss affine group schemes over a Dedekind domain. We also recall the Tannakian description of group scheme homomorphisms (Theorem A.1) and the Tannakian description of exact sequences (Theorem A.2). In Appendix A2, we remark the concept of affine k-groupoid scheme acting on S in [De90, Section 3].

(8) In Appendix A3, we define the induction functor for groupoid schemes (see A.3.2), which helps us prove that any $\Pi(X/k)^{\Delta}$-module is a quotient of a $\Pi(X/k)$-module (see Corollary A.7). In Appendix A4, we define the cohomology of groupoid schemes, which help us to prove Theorem 5.1.

(9) In Appendices A5 and A6, we review the Tannakian duality theories used in this paper:
- Tannakian duality over a field includes two duality theories. One is called the neutral Tannakian category, and the other is the general Tannakian category.
- Tannakian duality over a Dedekind ring.

(10) In Appendix B, we prove Theorem 3.5, which generalizes the results in [DHdS18], where the ring of interest is a discrete valuation ring. The proof is similar, but we need to extend the necessary results to Dedekind rings. These results are given in Proposition B.5 and Proposition B.8.

1.2. **Notations and Conventions.**

(1) The ring R (resp. A) is a dedekind domain (resp. complete discrete valuation ring) with quotient field K and residue field k.
(2) The spectrum of R is denoted by S, and the formal spectrum of A is denoted by $\mathfrak{S}$.
(3) There are several types of connections. For convenience of the reader, we will denote absolute connections by $(\mathscr{V},\nabla)$ and relative connections by $(\mathscr{M},\nabla)$. Sometimes, if there is no risk of confusion, we will use $\mathscr{V}$ or $\mathscr{M}$ for short.
(4) $\mathrm{MIC}(X/R)$ denotes the category of quasi-coherent $\mathscr{O}_X$-sheaves equipped with R-linear flat connections.
(5) $\mathrm{MIC}^{\mathrm{coh}}(X/R)$ denotes the full subcategory of $\mathrm{MIC}(X/R)$ consisting of coherent $\mathscr{O}_X$-sheaves equipped with R-linear flat connections.
(6) $\mathrm{MIC}^{\circ}(X/R)$ denotes the full subcategory of $\mathrm{MIC}(X/R)$ consisting of $\mathscr{O}_X$-locally free coherent sheaves.
(7) $\mathrm{MIC}^{\mathrm{se}}(X/R)$ denotes the full subcategory of $\mathrm{MIC}(X/R)$ consisting of objects which can be presented as quotients of objects from $\mathrm{MIC}^{\circ}(X/R)$.
(8) $\mathrm{MIC}^{\mathrm{geom}}(X/R)$ denotes the full subcategory of $\mathrm{MIC}^{\mathrm{se}}(X/R)$ of objects which can be presented as subquotients of objects of the form $\inf(\mathscr{V})$, where $\mathscr{V}\in\mathrm{Obj}(\mathrm{MIC}^{\circ}(X/k))$.
(9) For an absolute connection $(\mathscr{V},\nabla)$ in $\mathrm{Obj}(\mathrm{MIC}^{\circ}(X/k)$, the inflated connection to X/R is denoted by $\nabla_{/R}$. The relative de Rham cohomology $\mathrm{H}^{\bullet}_{\mathrm{dR}}(X/R,(\mathscr{V},\nabla_{/R}))$ is equipped with a natural connection – the Gauss-Manin connection.
(10) When we compare the group cohomology of the fundamental group and the de Rham cohomology, we always assume V is a representation corresponding to the flat connection $(\mathscr{V},\nabla)$.

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2. FLAT CONNECTIONS AND THE DIFFERENTIAL FUNDAMENTAL GROUPOID

Let k be an algebraically closed field of characteristic 0, and let R be a Dedekind domain over k. Let $f: X \longrightarrow \mathrm{Spec}(R)$ be a smooth map with geometrically connected fibers.

2.1. **Connections.**

Let $\Omega^1_{X/R}$ denote the sheaf of relative Kähler differentials on X/R. Since X is smooth over R, $\Omega^1_{X/R}$ is a locally free sheaf.

2.1.1. *Flat connections.* A *connection* on a sheaf of $\mathscr{O}_X$-modules $\mathscr{M}$ on X is an R-linear map

$$\nabla:\mathscr{M}\longrightarrow\Omega^1_{X/R}\otimes\mathscr{M},$$

satisfying the Leibniz rule and is *flat* in the sense that the composed map $\nabla_1\circ\nabla=0$, where

$$\nabla_1:\Omega^1_{X/R}\otimes\mathscr{M}\longrightarrow\Omega^2_{X/R}\otimes\mathscr{M};\quad \omega\otimes e\longmapsto d\omega\otimes e-\omega\otimes\nabla(e)$$

(cf. [Ka70, (1.0)]). When no confusion may arise we shall address a sheaf with a flat connection simply as a connection. The notation is $(\mathscr{M},\nabla)$ and is usually abbreviated to $\mathscr{M}$.

2.1.2. *Categories of consideration.* We consider the following categories:

- $\mathrm{MIC}(X/R)$ – the category of *quasi-coherent* $\mathscr{O}_X$-sheaves equipped with R-linear flat connections;
- $\mathrm{MIC}^{\mathrm{coh}}(X/R)$ – the full subcategory of *coherent* $\mathscr{O}_X$-sheaves equipped with R-linear flat connections;

- $\mathrm{MIC}^{\circ}(X/R)$ – the full subcategory of locally free $\mathscr{O}_X$-sheaves equipped with R-linear flat connections;
- $\mathrm{MIC}^{\mathrm{se}}(X/R)$ – the full subcategory of $\mathrm{MIC}^{\mathrm{coh}}(X/R)$ consisting of objects which can be presented as quotients of objects from $\mathrm{MIC}^{\circ}(X/R)$.

$\mathrm{MIC}^{\mathrm{se}}(X/R)$ satisfies Serre's condition (whence the supscription "se"). Hence, it is a Tannakian category over R in the sense of Saavedra (cf. [Sa72] or [DH18]).

According to [DH18], an object in $\mathrm{MIC}^{\mathrm{coh}}(X/R)$ is locally free if (and only if) it is torsion free over R. We recall the following concept for easy referencing.

Definition 2.1 (Special sub-quotients)**.** Let $\mathscr{M}$ be an object in $\mathrm{MIC}^{\circ}(X/R)$. We write $T^{a,b}(\mathscr{M})$ for $\mathscr{M}^{\otimes a}\otimes \mathscr{M}^{\vee\otimes b}$.

(i) The full subcategory in $\mathrm{MIC}^{\mathrm{coh}}(X/R)$ of all sub-quotients of objects of the form $T^{a_1,b_1}(\mathscr{M})\oplus\cdots\oplus T^{a_m,b_m}(\mathscr{M})$ is denoted by $\langle\mathscr{M}\rangle_{\otimes}$.

(ii) We recall the notion of a special subquotient of a connection whose sheaf is locally free as an $\mathscr{O}_X$-module. An monomorphism (i.e. injective map) $\alpha:\mathscr{M}'\to\mathscr{M}$. is said to be special if $\mathrm{Coker}(\alpha)$ is locally free. Then $\mathscr{M}'$ is called a special subobject of $\mathscr{M}$.

(iii) Call an object $\mathscr{M}''$ in $\mathrm{Obj}(\mathrm{MIC}^{\circ}(X/R))$ a *special sub-quotient* of $\mathscr{M}$ if there exists a special monomorphism $\mathscr{M}'\to\mathscr{M}$ and an epimorphism $\mathscr{M}'\to\mathscr{M}''$.

(iv) The category of all special sub-quotients of various $T^{a_1,b_1}(\mathscr{M})\oplus\cdots\oplus T^{a_m,b_m}(\mathscr{M})$ is denoted by $\langle\mathscr{M}\rangle_{\otimes}^{s}$.

2.1.3. *Absolute connections.* If R is of finite type (resp. essentially of finite type) over k, then X is smooth (resp. essentially of finite type) as a scheme over k. We denote by $\mathrm{MIC}(X/k)$ the category of $\mathscr{O}_X$-quasi-coherent sheaves with k-linear flat connections. It is known that each $\mathscr{O}_X$-coherent sheaf with flat connection over k is locally free, and the dual sheaf is equipped with a (dual) connection in a canonical way. Hence, $\mathrm{MIC}^{\mathrm{coh}}(X/k)=\mathrm{MIC}^{\circ}(X/k)$ is a rigid tensor k-linear abelian category. Similarly, we have the categories $\mathrm{MIC}^{\circ}(R/k)$ and $\mathrm{MIC}^{\mathrm{se}}(X/R)$. The pull-back by f yields the functor (denoted by the same notation) $f^*:\mathrm{MIC}^{\circ}(R/k)\longrightarrow\mathrm{MIC}^{\circ}(X/k)$.

2.1.4. *Formal connections.* If R is a complete discrete valuation ring over k, let $\mathfrak{X}$ be the completion of X with respect to the adic topology on R. Denote $\mathfrak{S}:=\mathrm{Spf}(R)$. Then by $\mathrm{MIC}(\mathfrak{X}/\mathfrak{S})$ and $\mathrm{MIC}(\mathfrak{X}/k)$ we understand the category of continuous connections (see 4.3.2). According to Lemma 7.5, there is an equivalence between $\mathrm{MIC}^{\circ}(\mathfrak{X}/k)$ and $\mathrm{MIC}^{\circ}(X_0/k)$, where X_0 denotes the fiber of X at the closed point of $\mathrm{Spec}(R)$. That is, there is an equivalence between vector bundles with flat connection on $\mathfrak{X}/k$ and vector bundles with flat connection on X_0/k.

Furthermore, if X is proper over R, then by the GFGA principle [FGIKNV05, Theorem 8.4.2],

$$\mathrm{MIC}^{\mathrm{coh}}(\mathfrak{X}/\mathfrak{S})\cong\mathrm{MIC}^{\mathrm{coh}}(X/R),$$

as shown in Lemma 4.9. Consequently, we have

$$\mathrm{MIC}^{\mathrm{se}}(X/A)\cong\mathrm{MIC}^{\mathrm{se}}(\mathfrak{X}/\mathfrak{S}).$$

2.2. Fundamental groups and groupoids. Let $\eta : \mathrm{Spec}(R) \longrightarrow X$ be an R-point of X (as an R-scheme). This section also yields a k-point x in the fiber X_s of X:

$$\begin{array}{ccc} \mathrm{Spec}(k) & \xrightarrow{s} & \mathrm{Spec}(R) \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle \eta} \\ X_s & \longrightarrow & X. \end{array}$$

These points yield various Tannakian group and groupoid schemes which we shall refer to as groups and groupoids for short.

2.2.1. *The fundamental groups.* The assumption that $f : X \longrightarrow \mathrm{Spec}(R)$ has connected fibers allows us to define the fundamental group $\pi(X/k) = \pi(X/k, x)$ is the Tannakian dual of $\mathrm{MIC}^\circ(X/k)$ with respect to the fiber functor x^* (cf. Appendix A.17):

$$\mathrm{Rep}^{\mathrm{f}}(\pi(X/k)) \cong \mathrm{MIC}^\circ(X/k).$$

The fundamental group $\pi(R/k)$ is the Tannakian dual of $\mathrm{MIC}^\circ(X/k)$ with respect to the fiber functor s^*:

$$\mathrm{Rep}^{\mathrm{f}}(\pi(R/k)) \cong \mathrm{MIC}^\circ(R/k).$$

The map $f : X \longrightarrow S$ induces a group homomorphism

$$f_* : \pi(X/k) \longrightarrow \pi(R/k).$$

This map is surjective as it admits a section induced from the section $\eta : S \longrightarrow X$.

2.2.2. *The fundamental groupoid.* Assume that $\mathrm{End}_{\mathrm{MIC}^\circ(X/k)}(R, d_{R/k}) = k$, then by [De90, Théorème 1.12] we have the absolute fundamental groupoid (scheme) $\Pi(X/k) = \Pi(X/k, \eta)$ which is the Tannakian dual of $\mathrm{MIC}^\circ(X/k)$ with respect to the fiber functor η^* (cf. Appendix A.19 for more details):

$$\mathrm{Rep}^{\mathrm{f}}(R : \Pi(X/k)) \cong \mathrm{MIC}^\circ(X/k).$$

The absolute fundamental groupoid $\Pi(R/k) = \Pi(R/k, \mathrm{id})$ is the Tannakian dual of $\mathrm{MIC}^\circ(R/k)$ with respect to the forgetful functor $\mathrm{id} : (V, \nabla) \longmapsto V$:

$$\mathrm{Rep}^{\mathrm{f}}(R : \Pi(R/k)) \cong \mathrm{MIC}^\circ(R/k).$$

The map $f : X \longrightarrow S$ induces a groupoid homomorphism

$$f^* : \Pi(X/k) \longrightarrow \Pi(R/k).$$

This map is surjective as it admits a section induced from the section $\eta : S \longrightarrow X$.

We notice that $\pi(X/k)$ is the base change of $\Pi(X/k)$ with respect to the map $(s, s) : \mathrm{Spec}(k) \longrightarrow \mathrm{Spec}(R) \times \mathrm{Spec}(R)$.

2.2.3. *The relative fundamental group scheme.* The *relative fundamental group scheme* $\pi(X/R) = \pi(X/R, \eta)$ is the Tannakian dual of $\mathrm{MIC}^{\mathrm{se}}(X/R)$ with respect to the fiber functor η^* (cf. Appendix A.21):

$$\mathrm{Rep}^{f}(\pi(X/R)) \cong \mathrm{MIC}^{\mathrm{se}}(X/R).$$

We have the *inflation functor*

$$\inf : \mathrm{MIC}^\circ(X/k) \longrightarrow \mathrm{MIC}^{\mathrm{se}}(X/R)$$

which assigns to each connection $(\mathcal{V}, \nabla)$ in $\mathrm{MIC}^\circ(X/k)$ the R-linear connection $(\mathcal{V}, \nabla_{/R})$ in $\mathrm{MIC}^{\mathrm{se}}(X/R)$:

$$\nabla_{/R} : \mathcal{V} \xrightarrow{\nabla} \Omega^1_{X/k} \otimes \mathcal{V} \longrightarrow \Omega^1_{X/R} \otimes \mathcal{V}.$$

This relative connection is called an *inflated connection*, and is denoted by $\inf(\mathcal{V})$ or $(\mathcal{V}, \nabla_{/R})$.

The functor inf induces a homomorphism of group-groupoid (schemes): $\pi(X/R) \longrightarrow \Pi(X/k)$, which factors through the diagonal subgroup scheme $\Pi(X/k)^{\Delta}$ of $\Pi(X/k)$ (see Appendix A.2):

$$\begin{array}{ccc} \pi(X/R) & \longrightarrow & \Pi(X/k)^{\Delta} \\ & \searrow & \downarrow \\ & & \Pi(X/k). \end{array}$$

2.2.4. *The geometric relative fundamental group.* One of our key observation is that the group scheme which is the image of the map

$$\mathrm{inf} : \pi(X/R) \longrightarrow \Pi(X/k)^{\Delta}$$

turns out to be the Tannakian dual to the category of *relative connections of geometric origin*, i.e., subquotients of inflated connections.

Definition 2.2. Let $\mathrm{MIC}^{\mathrm{geom}}(X/R)$ be the full subcategory of $\mathrm{MIC}^{\mathrm{se}}(X/R)$ of objects which can be presented as subquotients of objects of the form $\mathrm{inf}(\mathcal{V})$, where $\mathcal{V} \in \mathrm{Obj}(\mathrm{MIC}^{\circ}(X/k))$.

Lemma 2.3. *The category* $\mathrm{MIC}^{\mathrm{geom}}(X/R)$ *defined above, together with the fiber functor* η^**, is a Tannakian category. The canonical map* $\pi(X/R) \longrightarrow \pi^{\mathrm{geom}}(X/R)$ *is surjective (faithfully flat).*

2.2.5. *The Ind-categories.* The Tannakian dualities mentioned above extend to the ind-categories. Recall that the ind-category of a (notherian) category is the category of inductive directed systems of object from the original category. We shall use the notation $\mathrm{MIC}^{\mathrm{ind}}(X/R)$ to denote the ind-category of $\mathrm{MIC}^{\mathrm{se}}(X/R)$ and $\mathrm{MIC}^{\mathrm{ind}}(X/k)$ to denote the ind-category of $\mathrm{MIC}^{\circ}(X/k)$. Since the category of all connections is cocomplete these ind-categories can be identified with the categoy of connections which can be presented as the union of their coherent subconnections (or equivalently, as direct limits of coherent connections).

The have the following equivalences:

$$\begin{aligned} \mathrm{Rep}(\pi(X/k)) \cong \mathrm{Rep}(R : \Pi(X/k)) \quad &\cong \quad \mathrm{MIC}^{\mathrm{ind}}(X/k) \subsetneq \mathrm{MIC}(X/k); \\ \mathrm{Rep}(\pi(X/R)) \quad &\cong \quad \mathrm{MIC}^{\mathrm{ind}}(X/R) \subsetneq \mathrm{MIC}(X/R). \end{aligned}$$

We notice that there are quasi-coherent connections which cannot be presented as a union of coherent subconnections; for instance, the sheaf of algebras of differential operators. Therefore, the rightmost inclusions above are proper.

3. The fundamental exact sequence

3.1. **The homotopy exact sequence.** The following sequence is shown to be exact by L. Zhang [Zha14] in equal characteristics zero and J. P. dos Santos [dS15] in the general case and is called the homotopy exact sequence, it resembles the topological homotopy exact sequence. In this section, we assume more that $f : X \to S$ is a projective morphism.

Theorem 3.1 (dos Santos, Zhang)**.** *The following sequence is exact*

$$\pi(X_s/k, x) \longrightarrow \pi(X/k, x) \longrightarrow \pi(R/k) \longrightarrow 1.$$

This exact sequence generalizes Grothendieck's homotopy exact sequence for étale fundamental groups [SGA1, Théorème IX.6.1].

3.2. **The fundamental exact sequence.** The functor f^* and the inflation functor induce the following sequence of homomorphisms of group and groupoid schemes:

$$\pi(X/R) \xrightarrow{\text{inf}} \Pi(X/k) \xrightarrow{f_*} \Pi(R/k) \longrightarrow 1.$$

We first notice that the composition $f_* \circ \text{inf}$ corresponds to the functor that takes the pull-back of connections on R/k along f and then inflates to relative connections on X/R, thereby sending any connection on R/k to a trivial relative connection on X/R. Therefore, it is a trivial homomorphism.

Further, we notice that the map inf factors into the composition of

$$\pi(X/R) \longrightarrow \pi^{\text{geom}}(X/R) \longrightarrow \Pi(X/k).$$

The aim of this section is to show the following theorem.

Theorem 3.2. *Let $f : X \longrightarrow \text{Spec}(R)$ be a smooth projective map with geometrically connected fibers. Then the following sequences*

$$\pi(X/R) \longrightarrow \pi^{\text{geom}}(X/R) \longrightarrow 1$$

and

$$1 \longrightarrow \pi^{\text{geom}}(X/R) \longrightarrow \Pi(X/k) \xrightarrow{f^*} \Pi(R/k) \longrightarrow 1$$

of flat affine group schemes and groupoids schemes over R are exact.

We notice that the exactness of the second sequence amounts to the exactness of the following sequence of R-group schemes (see Appendix A.2.4):

$$1 \longrightarrow \pi^{\text{geom}}(X/R) \longrightarrow \Pi(X/k)^{\Delta} \longrightarrow \Pi(R/k)^{\Delta} \longrightarrow 1.$$

The name "fundamental exact sequence" is motivated by Grothendieck's fundamental exact sequences of the etale fundamental groups:

$$1 \longrightarrow \pi^{\text{et}}(X \otimes_k \bar{k}, \bar{x}) \longrightarrow \pi^{\text{et}}(X, \bar{x}) \longrightarrow \text{Gal}(\bar{k}/k) \longrightarrow 1.$$

Indeed, the fundamental groupoid $\Pi(R/k)$ can be seen as a generalization of the Galois group $\text{Gal}(\bar{k}/k)$, while the relative fundamental group plays the role of the geometric étale fundamental group.

For the étale fundamental groups, Grothendieck used the fundamental exact sequence to deduce the homotopy exact sequence. For the differential fundamental groups, dos Santos provided a direct proof using his criterion for the exact sequence of group schemes. In what follows, we shall use Theorem 3.1 to deduce Theorem 3.2. The proof will be given in Section 3.2.2. First, we shall need some lemmas.

3.2.1. *Some lemmas.* Combining Theorem 3.1 with the criterion for exact sequences of affine group schemes [EHS08, Theorem A.1], we deduce the following lemma (cf. [DHdS18, Theorem 9.1]).

Lemma 3.3. *Let $\mathcal{V}$ be an object of* $\text{MIC}^\circ(X/k)$.

(1) *The maximal trivial subobject of $\mathcal{V}|_{X_s}$ is the restriction of a subobject $\mathcal{T} \longrightarrow \mathcal{V}$. Moreover, $\mathcal{T}$ is the pull-back to* $\text{MIC}^\circ(X/k)$ *of an object of* $\text{MIC}^\circ(R/k)$*;*
(2) *If $\mathcal{N}$ belongs to $\langle \mathcal{V}|_{X_s}\rangle_\otimes$, then there exists $\tilde{\mathcal{N}}$ in $\langle \mathcal{V}\rangle_\otimes$ and a monic $\mathcal{N} \longrightarrow \tilde{\mathcal{N}}|_{X_s}$.*

Proof. According to Theorem 3.1, the following sequence

$$\pi(X_s/k) \longrightarrow \pi(X/k) \longrightarrow \pi(R/k) \longrightarrow 1.$$

is exact. Using the characterization of exactness presented in [EHS08, Theorem A.1], we immediately arrive at the desired conclusion. Moreover, the proof in loc. cit. shows that $\tilde{\mathcal{N}}$ can be chosen in $\langle\mathcal{V}\rangle_\otimes$. □

The following lemma is analogous to [DHdS18, Theorem 9.2], which generalizes a result of Deligne ([EH06, Theorem 5.10]).

Lemma 3.4. *Let $\mathcal{V}$ be an object of $\mathrm{MIC}^\circ(X/k)$ and let $\mathcal{M} \longrightarrow \inf(\mathcal{V})$ be a special subobject in $\mathrm{MIC}^{\mathrm{se}}(X/R)$. Then there exists $\mathcal{N}$ in $\mathrm{Obj}(\mathrm{MIC}^\circ(X/k))$ and an epimorphism $\inf(\mathcal{N}) \longrightarrow \mathcal{M}$. Moreover, $\mathcal{N}$ can be chosen in $\langle\mathcal{V}\rangle_\otimes$.*

Combining Lemmas 3.4 and 3.3, we deduce the following property of the category $\mathrm{MIC}^{\mathrm{geom}}(X/R)$. The proof of the following lemma requires techniques of flat affine group scheme and will be given in the Appendix B, see Theorem B.9.

Lemma 3.5 (cf. [DHdS18, Theorem 8.2])**.** *Let $(\mathcal{V},\nabla)$ be an absolute connection on X/k. Then each relative connection that is locally free as an $\mathcal{O}_X$-module in $\langle\inf(\mathcal{V},\nabla)\rangle_\otimes$ is indeed a* special *subobject of a tensor generated object from $\inf(\mathcal{V},\nabla)$.*

Corollary 3.6. *The map $\pi^{\mathrm{geom}}(X/R) \longrightarrow \Pi(X/k)^\Delta$ is a closed immersion.*

Proof. Since every object of $\mathrm{MIC}^{\mathrm{geom}}(X/R)^\circ$ is isomorphic to a special subquotient of an object of the form $(\mathcal{V},\nabla_{/R})$ via Lemma 3.5, Theorem A.1 implies that $\pi^{\mathrm{geom}}(X/R) \longrightarrow \Pi(X/k)^\Delta$ is a closed immersion. □

Corollary 3.7. *Any R-projective finite representation of $\pi^{\mathrm{geom}}(X/R)$ is also a special subobject of a finite representation of $\Pi(X/k)$ considered as a representation of $\pi^{\mathrm{geom}}(X/R)$. Consequently, any finite representation of $\pi^{\mathrm{geom}}(X/R)$ is a quotient of a finite representation of $\Pi(X/k)$ considered as a $\pi^{\mathrm{geom}}(X/R)$-representation.*

Proof. According to Lemma 3.5, every object of $\mathrm{MIC}^{\mathrm{geom}}(X/R)^\circ$ is a special subobject of the form $(\mathcal{V},\nabla_{/R})$. Hence, by Lemma 3.4, any object of $\mathrm{MIC}^{\mathrm{geom}}(X/R)^\circ$ is a quotient of some inflated connection. □

Let L be a kernel of the map $f : \Pi(X/k) \longrightarrow \Pi(R/k)$. Notice that L is also equal to the kernel of $\Pi(X/k)^\Delta \longrightarrow \Pi(R/k)^\Delta$ (cf. Appendix A.2.4).

Lemma 3.8. *Every finite representation of L can be embedded into the restriction to L of a finite representation of $\Pi(X/k)$. Consequently, it can be represented as a quotient of the restriction to L of a finite representation of $\Pi(X/k)$.*

Proof. Before giving the proof, we need the following claim.

Claim. Any finite representation of L is a quotient of a finite representation of $\Pi(X/k)^\Delta$ considered as a representation of L. Consequently, any finite representation of L can be embedded into a finite representation of $\Pi(X/k)^\Delta$ considered as a representation of L.

Verification. We prove this claim by considering the induction functor $\mathrm{Ind}_L^{\Pi(X/k)^\Delta}$, which is defined in [Ja87, 3.3, p. 44]. Since L is a normal subgroup of $\Pi(X/k)^\Delta$, the functor $\mathrm{Ind}_L^{\Pi(X/k)^\Delta}$ is faithfully exact by the same argument as in the proof of [DH18, Theorem 4.2.2]. Using the same argument as in the proof of Theorem A.6, we obtain the following map

$$\mathrm{Ind}_L^{\Pi(X/k)^\Delta}(W) \longrightarrow (W)$$

which is surjective for any L-finite representation W. Hence, any finite representation of L is a quotient of a finite representation of $\Pi(X/k)^{\Delta}$. The rest of the claim follows by using the same argument as in the proof of Corollary A.7. $\triangleright$

We are now ready to prove the lemma. According to the claim and Corollary A.7, we obtain that every finite representation of L can be embedded into the restriction to L of a finite representation of $\Pi(X/k)$. The rest of the lemma follows by applying the same argument as in the proof of Corollary A.7 again. $\square$

3.2.2. *Proof of 3.2.* Our strategy is to show that the group $\mathrm{MIC}^{\mathrm{geom}}(X/R)$ equals the kernel L of the map $f : \Pi(X/k) \longrightarrow \Pi(R/k)$.

By construction, we have the closed immersion $\pi^{\mathrm{geom}}(X/R) \subset L$, which induces a functor $\mathscr{F} : \mathrm{Rep}_R(L) \longrightarrow \mathrm{Rep}^{\mathrm{f}}(\pi^{\mathrm{geom}}(X/R))$. This functor is faithful by construction. We will show it is full and essentially surjective.

Step 1. We show that $\mathscr{F}^\circ : \mathrm{Rep}^\circ_R(L) \longrightarrow \mathrm{MIC}^{\mathrm{geom}}(X/R)^\circ$ is full.

Let U_0, U_1 be objects in $\mathrm{Rep}^\circ_R(L)$ and $\phi : \mathscr{F}^\circ(U_0) \to \mathscr{F}^\circ(U_1)$ a $\mathrm{MIC}^{\mathrm{geom}}(X/R)$-morphism, meaning ϕ is a R-linear map $U_0 \to U_1$ that is $\pi^{\mathrm{geom}}(X/R)$-linear. We show that ϕ is indeed L-linear. By Lemma 3.8 there are L-linear morphism $\pi : V_0 \twoheadrightarrow U_0$ and $i : U_1 \hookrightarrow V_1$, where V_0 and V_1 are object in $\mathrm{Rep}^{\mathrm{f}}(R : \Pi(X/k))$. We set $\psi = i\phi\pi$, and then we have the following diagram

$$\begin{array}{ccc} U_0 & \xrightarrow{\phi} & U_1 \\ {\scriptstyle\pi}\uparrow & & \downarrow{\scriptstyle i} \\ V_0 & \xrightarrow[\psi]{} & V_1. \end{array}$$

Thus, ϕ is L-linear if and only ψ is. This leads us to show that

$$\mathrm{Hom}_L(V_0, V_1) = \mathrm{Hom}_{\pi^{\mathrm{geom}}(X/R)}(V_0, V_1),$$

for any $V_0, V_1 \in \mathrm{Rep}_f(R : \Pi(X/k))$, which amounts to showing $V^L = V^{\pi^{\mathrm{geom}}(X/R)}$ for $V = V_0^\vee \otimes V_1$, where V^L is the submodule of V consisting of elements stable under the action of L.

Now, through Tannakian duality, Lemma 4.3 tells us that (see Remark 4.4), for $V \in \mathrm{Obj}(\mathrm{Rep}^{\mathrm{f}}(R : \Pi(X/k)))$ that correspond to the connection $(\mathscr{V}, \nabla) \in \mathrm{Obj}(\mathrm{MIC}^\circ(X/k))$,

$$V^{\pi^{\mathrm{geom}}(X/R)} = V^{\pi(X/R)} = \mathrm{H}^0_{\mathrm{dR}}(X/R, (\mathscr{V}, \nabla_{/R})) \subset V^L.$$

On the other hand, we obviously have $V^L \subset V^{\pi^{\mathrm{geom}}(X/R)}$. Thus, we have an equality.

Step 2. We show that $\mathscr{F}$ is essentially surjective. Let $\mathscr{V}$ be an object in $\mathrm{MIC}^{\mathrm{geom}}(X/R)^\circ$. Then, according to Lemma 3.5, there exists an object $(\mathscr{W}, \nabla)$ in $\mathrm{MIC}^\circ(X/k)$ such that $\mathscr{V}$ is a special subobject of $\inf(\mathscr{W}, \nabla)$. According to Lemma 3.4, $\mathscr{V}$ can be realized as the image of a morphism φ between inflated objects. By the above discussion, φ is also in the image of $\mathscr{F}$, hence so is $\mathrm{Im}(\varphi)$. It means that $\mathscr{V}$ corresponds to a representation of L and we finish the proof. $\square$

4. de Rham cohomology and Gauss-Manin connection

4.1. **de Rham cohomology.** Let $f : X \longrightarrow S = \mathrm{Spec}(R)$ be a smooth scheme over a noetherian ring R. For a sheaf with flat connection $(\mathcal{M}, \nabla)$ on X/R, the *sheaf of horizontal sections* is defined to be

$$\mathcal{M}^{\nabla} := \mathrm{Ker}(\nabla : \mathcal{M} \to \Omega^1_{X/R} \otimes \mathcal{M}). \tag{7}$$

This is a sheaf of R-module. The 0-*th de Rham cohomology* of $\mathcal{M}$ is defined to be

$$\mathrm{H}^0_{\mathrm{dR}}(X/R, (\mathcal{M}, \nabla)) := f_*(\mathcal{M}^{\nabla}).$$

The module $\mathrm{H}^0_{\mathrm{dR}}(X/R, (\mathcal{M}, \nabla))$ can be identified with the hom-set of connections in $\mathrm{MIC}(X/R)$

$$\{\varphi : (\mathcal{O}_X, d) \longrightarrow (\mathcal{M}, \nabla)\}.$$

Since $\mathrm{MIC}(X/R)$ is equivalent to the category of left modules on the sheaf of differential operators $\mathcal{D}_{X/R}$, it has enough injectives (cf. [Ka70]). Thus, we can define the *higher de Rham cohomologies* $\mathrm{H}^i_{\mathrm{dR}}(X/R, -)$ to be the derived functors of the functor

$$\mathrm{H}^0_{\mathrm{dR}}(X/R, -) : \mathrm{MIC}(X/R) \longrightarrow \mathrm{Mod}_R.$$

Convention. When it is clear which connection on a sheaf $\mathcal{M}$ is taken, we shall omit it in the notation of the de Rham cohomology and simply write:

$$\mathrm{H}^i_{\mathrm{dR}}(X/R, \mathcal{M}).$$

This cohomology groups can also be computed as the groups of extensions in $\mathrm{MIC}(X/R)$, $\mathrm{Ext}^i_{\mathrm{MIC}(X/R)}(\mathcal{M}, \mathcal{N})$, which counts i-extensions in $\mathrm{MIC}(X/R)$, that is, exact sequences

$$0 \longrightarrow \mathcal{N} \longrightarrow \mathcal{N}_1 \longrightarrow \ldots \longrightarrow \mathcal{N}_i \longrightarrow \mathcal{M} \longrightarrow 0,$$

upto an equivalence define as follows: two sequence are equivalent if there is a map of sequence between them induced by the identity maps on $\mathcal{M}$ and $\mathcal{N}$. Since $\mathrm{MIC}(X/R)$ has enough injectives, $\mathrm{Ext}^i_{\mathrm{MIC}(X/R)}(\mathcal{M}, -)$ are the right derived functors of the hom-functor

$$\mathrm{Hom}_{\mathrm{MIC}(X/R)}(\mathcal{M}, -) : \mathrm{MIC}(X/R) \longrightarrow \mathrm{Mod}_R.$$

That is, we have the following:

Lemma 4.1. *Let $(\mathcal{M}, \nabla)$ be a flat connection in $\mathrm{MIC}(X/R)$. Then we have:*

$$\mathrm{Ext}^i_{\mathrm{MIC}(X/R)}(\mathcal{O}_X, \mathcal{M}) \cong \mathrm{H}^i_{\mathrm{dR}}(X/R, \mathcal{M}).$$

4.2. **The Gauss-Manin connection.** We briefly describe the construction of Gauss-Manin connection which is based on [ABC20] and [Ka70].

Let $\mathcal{V}$ be an absolute connection, that is, an object of $\mathrm{MIC}^{\circ}(X/k)$. Consider the de Rham cohomology of the inflated connection $\inf(\mathcal{V}) = (\mathcal{V}, \nabla_{/R})$: $\mathrm{H}^0_{\mathrm{dR}}(X/R, (\mathcal{V}, \nabla_{/R}))$. In what follows we shall adopt the abbreviation:

$$\mathrm{H}^0_{\mathrm{dR}}(X/R, \mathcal{V})) := \mathrm{H}^0_{\mathrm{dR}}(X/R, (\mathcal{V}, \nabla_{/R}))$$

when it is clear that $\mathcal{V}$ is equipped with an absolute connection and the cohomology is taken over X/R for the inflated connection on it.

$\mathrm{H}^0_{\mathrm{dR}}(X/R,\mathcal{V})$), as an R-module, is equipped with a connection over R/k, which is known as the *0-th Gauss-Manin connection.* The explicit construction is as follows. The smoothness of f implies the following exact sequence of Kähler differentials:

$$0 \longrightarrow f^*\Omega^1_{R/k} \longrightarrow \Omega^1_{X/k} \longrightarrow \Omega^1_{X/R} \longrightarrow 0. \tag{8}$$

This filtration of $\Omega^1_{X/k}$ induces a filtration on $\Omega^2_{X/k}$ which is compatible with the connection, and we obtain the following commutative diagram (cf. [Ka70, (3.2)]) with exact columns:

$$\begin{array}{ccccccc}
 & & & & f^*\Omega^1_{R/k}\otimes\mathcal{V} & \xrightarrow[\nabla_0]{} & f^*\Omega^1_{R/k}\otimes\Omega^1_{X/k}\otimes V \\
 & \overset{\delta}{\nearrow} & & & \Big\downarrow i & & \Big\downarrow j \\
 & & \mathcal{V} & \xrightarrow{\nabla} & \Omega^1_{X/k}\otimes\mathcal{V} & \xrightarrow{\nabla_1} & \Omega^2_{X/k}\otimes\mathcal{V} \\
 & & \Vert & & \Big\downarrow & & \Big\downarrow \\
\mathcal{V}^{\nabla_{/R}} & \longrightarrow & \mathcal{V} & \xrightarrow[\nabla_{/R}]{} & \Omega^1_{X/R}\otimes\mathcal{V} & \longrightarrow & \Omega^2_{X/R}\otimes\mathcal{V},
\end{array}$$

here $\nabla_0 : f^*\Omega^1_{R/k}\otimes\mathcal{V} \longrightarrow f^*\Omega^1_{R/k}\otimes\Omega^1_{X/k}\otimes\mathcal{V}$ is given by (for $r\in R$ and $v\in\mathcal{V}$):

$$dr\otimes v \longmapsto -dr\otimes\nabla(v),$$

and $\nabla_1 : \Omega^1_{X/k}\otimes\mathcal{V} \longrightarrow \Omega^2_{X/k}\otimes\mathcal{V}$ is given by (for $a\in\mathcal{O}_X$ and $v\in\mathcal{V}$)

$$da\otimes v \longmapsto -da\wedge\nabla(v).$$

Recall that $\mathcal{V}^{\nabla_{/R}}$ is the kernel of the R-linear map $\nabla_{/R}$. Diagram chasing yields a map

$$\delta : \mathcal{V}^{\nabla_{/R}} \longrightarrow f^*\Omega^1_{R/k}\otimes\mathcal{V},$$

with the property that $i\circ\delta = \nabla$. Hence $j\circ\nabla_0\circ\delta = \nabla_1\circ\nabla = 0$. Therefore δ factors as:

$$\delta : \mathcal{V}^{\nabla_{/R}} \longrightarrow \Omega^1_{R/k}\otimes_R\mathcal{V}^{\nabla_{/R}}.$$

Applying f_*, we obtain a flat connection δ on $\mathrm{H}^0_{\mathrm{dR}}(X/R,\mathcal{V})$ over R/k. The resulting connection is denoted by $\mathrm{R}^0_{\mathrm{dR}}f_*(\mathcal{V},\nabla_{/R})$ or $\mathrm{R}^0_{\mathrm{dR}}f_*\mathcal{V}$ for short. Thus, we have a left exact functor

$$\mathrm{R}^0_{\mathrm{dR}}f_* : \mathrm{MIC}(X/k) \longrightarrow \mathrm{MIC}(R/k).$$

The i-th derived functor of this functor is called the *i-th Gauss-Manin connection* of $(\mathcal{M},\nabla)$:

$$\mathrm{R}^i_{\mathrm{dR}}f_* : \mathrm{MIC}(X/k) \longrightarrow \mathrm{MIC}(R/k).$$

This functor can be computed by de Rham cohomology as follows. The sequence in (8) yields an exact sequences of complexes:

$$0 \longrightarrow f^*\Omega^1_{R/k}\otimes(\Omega^{\bullet-1}_{X/R}\otimes\mathcal{V}) \longrightarrow \Omega^\bullet_{X/k}\otimes\mathcal{V} \longrightarrow \Omega^\bullet_{X/R}\otimes\mathcal{V} \longrightarrow 0. \tag{9}$$

As argued in [ABC20, 23.2.5], the module $\mathrm{H}^i_{\mathrm{dR}}(X/R,\mathcal{V})$ is canonically isomorphic to the hypercohohomology $\mathbf{R}^i f_*(\Omega^\bullet_{X/R}\otimes\mathcal{V})$.

Applying the hyper-derived functor $\mathbf{R}^i f_*$ to the exact sequence above, we obtain the long exact sequence:

$$0 \longrightarrow \cdots \longrightarrow \mathbf{R}^i f_*(\Omega^\bullet_{X/R}\otimes\mathcal{V}) \xrightarrow{\delta_i} \mathbf{R}^{i+1} f_*(f^*\Omega^1_{R/k}\otimes(\Omega^{\bullet-1}_{X/R}\otimes\mathcal{V})) \longrightarrow \cdots.$$

The connecting map

$$\delta_i : \mathbf{R}^i f_*(\Omega^\bullet_{X/R}\otimes\mathcal{V}) \longrightarrow \mathbf{R}^{i+1} f_*(f^*\Omega^1_{R/k}\otimes(\Omega^{\bullet-1}_{X/R}\otimes\mathcal{V}))$$

can be written as:

$$\delta_i : \mathbf{R}^i f_*(\Omega^\bullet_{X/R}\otimes\mathcal{V}) \longrightarrow \Omega^1_{R/k}\otimes\mathbf{R}^i f_*(\Omega^\bullet_{X/R}\otimes\mathcal{V})$$

by projection formula. The map δ_i defines a flat connection on the R-module $\mathbf{R}^i f_*(\Omega^\bullet_{X/R}\otimes\mathcal{V}) = \mathrm{H}^i_{\mathrm{dR}}(X/R,\mathcal{V})$. Moreover, we get a long exact sequence:

$$\cdots \to \mathrm{H}^i_{\mathrm{dR}}(X/k,(\mathcal{V},\nabla)) \to \mathrm{H}^i_{\mathrm{dR}}(X/R,(\mathcal{V},\nabla_{/R})) \xrightarrow{\delta_i} \Omega^1_{R/k}\otimes \mathrm{H}^i_{\mathrm{dR}}(X/R,(\mathcal{V},\nabla_{/R})) \to \cdots \tag{10}$$

4.2.1. *Finiteness of de Rham cohomology.*

Lemma 4.2. *Let $f : X \longrightarrow \mathrm{Spec}(R)$ be a proper smooth morphism of smooth k-schemes. Let $\mathcal{V}$ be an object in $\mathrm{MIC}^\circ(X/k)$. Then $\mathrm{H}^i_{\mathrm{dR}}(X/R,\mathcal{V})$ is a finite projective R-module.*

Proof. We have the Hodge-to-de Rham spectral sequence (see [Ka70, (3.5.2.0)]):

$$E_1^{p,q} = \mathrm{R}^q f_*(\Omega^p_{X/R}\otimes\mathcal{V}) \Longrightarrow \mathbf{R}^{p+q} f_*(\Omega^\bullet_{X/R}\otimes\mathcal{V}).$$

Since X is proper, every term of

$$E_1^{p,q} = \mathrm{R}^q f_*(\Omega^p_{X/R}\otimes\mathcal{V})$$

is coherent, this follows that $\mathbf{R}^{p+q} f_*(\Omega^\bullet_{X/R}\otimes\mathcal{V})$ is also coherent. Moreover, because the de Rham cohomology sheaf $\mathrm{H}^i_{\mathrm{dR}}(X/R,\mathcal{V})$ can be equipped with Gauss-Manin connection, the underlying R-module must be projective as an R-module. $\square$

Consequently, the functors $\mathrm{R}^i_{\mathrm{dR}} f_*$ restrict to functors $\mathrm{MIC}^\circ(X/k) \longrightarrow \mathrm{MIC}^\circ(R/k)$ and we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{MIC}^\circ(X/k) & \xrightarrow{\mathrm{inf}} & \mathrm{MIC}^{\mathrm{se}}(X/R) \\ {\scriptstyle \mathrm{R}^i_{\mathrm{dR}} f_*(-)}\downarrow & & \downarrow{\scriptstyle \mathrm{H}^i_{\mathrm{dR}}(X/R,-)} \\ \mathrm{MIC}^\circ(R/k) & \xrightarrow{\Gamma(S,-)} & \mathrm{Mod}^\circ(R), \end{array}$$

where $\mathrm{Mod}^\circ(R)$ is the category of finite projective modules over R.

Recall that a connection on X/R is called *trivial* if it has the form $M\otimes_R(\mathcal{O}_X,d)$ where M is an R-module. More precisely, the underlying sheaf has the form $M\otimes_R\mathcal{O}_X$ and the connection is induced from the differential on $\mathcal{O}_X$.

Lemma 4.3. *Let $(\mathcal{V},\nabla)$ be an object of $\mathrm{MIC}^\circ(X/k)$. Then the connection $f^*(\mathrm{R}^0_{\mathrm{dR}} f_*(\mathcal{V},\nabla))$ is the maximal subobject of $(\mathcal{V},\nabla)$ in $\mathrm{MIC}^\circ(X/k)$ with the property: its inflation to $\mathrm{MIC}^{\mathrm{se}}(X/R)$ is a trivial connection.*

Proof. The connection on the pull-back

$$f^*\mathrm{R}^0_{\mathrm{dR}} f_*\mathcal{V} = f^*\mathrm{H}^0_{\mathrm{dR}}(X/R,\mathcal{V}) = \mathcal{O}_X\otimes\mathrm{H}^0_{\mathrm{dR}}(X/R,\mathcal{V})$$

is given by $\nabla(a\otimes e) = da\otimes e + a\otimes\delta e$. Hence on the inflated connection

$$\nabla_{/R}(a\otimes e) = da\otimes e.$$

This means $(\mathcal{O}_X\otimes H^0_{\mathrm{dR}}(X/R,\mathcal{V}),\nabla)$ is a subconnection of $\mathcal{V}$ with the property that its inflation is a trivial relative connection.

Actually, it has to be the maximal such, as any other $\mathcal{W}\subset\mathcal{V}$ would have the property that $(\nabla_{/R})|_{\mathcal{W}}$ is generated by horizontal sections. $\square$

Remark 4.4. The Tannakian interpretation of Lemma 4.3 is as follows. Let $V = \eta^*(\mathcal{V})$ be the representation of $\Pi(X/k)$ that corresponds to $(\mathcal{V}, \nabla)$. Then $f_*\mathrm{inf}(\mathcal{V})$ corresponds through η^* with a representation of $\Pi(R/k)$ and $f^*f_*\mathrm{inf}(\mathcal{V})$ corresponds with a subrepresentation of V on which the action of $\Pi(X/k)$ factors through the action of $\Pi(R/k)$, or equivalently, the group L acts trivially, where L is the kernel of $f : \Pi(X/k) \longrightarrow \Pi(R/k)$, see Appendix A.2.4.

4.3. **De Rham cohomology for relative formal schemes.** We follow the conventions and notations of formal schemes in [EGA I, Section 10]. A formal scheme $\mathfrak{X} = \bigcup \mathrm{Spf}(R_i)$ is *locally noetherian* if the rings R_i are noetherian and adic. $\mathfrak{X}$ is *noetherian* if it is locally noetherian and quasi-compact. For further details on formal schemes and morphisms between them, the reader is referred to Appendix C.1. From now on, we always assume that the formal scheme is locally noetherian.

4.3.1. *De Rham complex on formal schemes.* For the sake of readers, we review the Kahler differential for formal schemes $\widehat{\Omega}_{\mathfrak{X}/\mathfrak{S}}$ in Appendix C.2. Let f be a morphism of pseudo finite type. There exists a unique graded morphism of degree 1 (see [TJR21, 1.8])

$$\widehat{d}^\bullet : \widehat{\Omega}^\bullet_{\mathfrak{X}/\mathfrak{S}} \longrightarrow \widehat{\Omega}^\bullet_{\mathfrak{X}/\mathfrak{S}}$$

such that

i) $\widehat{d}^0 = \widehat{d}$,
ii) $\widehat{d}^{i+1} \circ \widehat{d}^i = 0$, for all $i \in \mathbb{N}$ and
iii) Given $\mathfrak{U} \subset \mathfrak{X}$ an open set, $w_i \in \Gamma(\mathfrak{U}, \widehat{\Omega}^i_{\mathfrak{X}/\mathfrak{S}})$ and $w_j \in \Gamma(\mathfrak{U}, \widehat{\Omega}^j_{\mathfrak{X}/\mathfrak{S}})$,

$$\widehat{d}^{i+j}(w_i \wedge w_j) = \widehat{d}^i(w_i) \wedge w_j + (-1)^i w_i \wedge \widehat{d}^j(w_j)$$

for any $i, j \in \mathbb{N}$.

Then

$$(\widehat{\Omega}^\bullet_{\mathfrak{X}/\mathfrak{S}}, \widehat{d}^\bullet) : 0 \longrightarrow \mathcal{O}_{\mathfrak{X}} \xrightarrow{\widehat{d}} \widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{S}} \xrightarrow{\widehat{d}^1} \dots \xrightarrow{\widehat{d}^{k-1}} \widehat{\Omega}^k_{\mathfrak{X}/\mathfrak{S}} \xrightarrow{\widehat{d}^k} \dots$$

is a complex of $\mathcal{O}_{\mathfrak{X}}$-modules which is called *de Rham complex* of $\mathfrak{X}$ relative to $\mathfrak{S}$.

4.3.2. *Category of modules with integrable connections* $\mathrm{MIC}(\mathfrak{X}/\mathfrak{S})$.

Definition 4.5. Let $f : \mathfrak{X} \to \mathfrak{S}$ be a smooth morphism of locally noetherian formal schemes, and $\mathcal{M}$ a quasi-coherent sheaf of $\mathcal{O}_{\mathfrak{X}}$-modules. A *connection* on $\mathcal{M}$ is a homomorphism $\widehat{\nabla}$ of abelian sheaves

$$\widehat{\nabla} : \mathcal{M} \longrightarrow \widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{S}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{M} \tag{11}$$

such that

$$\widehat{\nabla}(gm) = g\widehat{\nabla}(m) + \widehat{d}g \otimes m$$

where g and m are sections of $\mathcal{O}_{\mathfrak{X}}$ and $\mathcal{M}$ respectively over an open subset of $\mathfrak{X}$, and $\widehat{d}g$ denotes the image of g under the canonical exterior differentiation $\widehat{d} : \mathcal{O}_{\mathfrak{X}} \longrightarrow \widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{S}}$.

Analogously to the construction of the category of modules with integrable connections $\mathrm{MIC}(X/S)$ (see [Ka70, Section 1]), we define the category of modules with integrable connections $\mathrm{MIC}(\mathfrak{X}/\mathfrak{S})$.

4.3.3. *de Rham cohomology of relative formal schemes.* To every module with connection $(\mathscr{M}, \widehat{\nabla})$ on $\mathfrak{X}$, we can define unique morphisms of abelian sheaves

$$\widehat{\nabla}^i : \widehat{\Omega}^i_{\mathfrak{X}/\mathfrak{S}} \otimes \mathscr{M} \longrightarrow \widehat{\Omega}^{i+1}_{\mathfrak{X}/\mathfrak{S}} \otimes \mathscr{M}$$

such that

$$\widehat{\nabla}^i (w \otimes m) = \widehat{d} w \otimes m + w \wedge (-1)^i \widehat{\nabla}(m),$$

where m and w are local sections of $\mathscr{M}$ and $\mathscr{O}_{\mathfrak{X}}$ respectively. Since the connection $\nabla = \nabla^0$ is integrable, we have

$$\widehat{\Omega}^\bullet_{\mathfrak{X}/\mathfrak{S}} \otimes \mathscr{M} : \ 0 \to \mathscr{M} \xrightarrow{\widehat{\nabla}^0} \widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{S}} \otimes \mathscr{M} \xrightarrow{\widehat{\nabla}^1} \widehat{\Omega}^2_{\mathfrak{X}/\mathfrak{S}} \otimes \mathscr{M} \to \dots$$

is a complex.

Definition 4.6. Let $f : \mathfrak{X} \to \mathfrak{S}$ be a pseudo-finite type map of noetherian formal schemes and $(\mathscr{M}, \widehat{\nabla}) \in \mathrm{Obj}(\mathrm{MIC}(\mathfrak{X}/\mathfrak{S}))$. For every $i \in \mathbb{N}$, the i-th *de Rham cohomology sheaves* on $\mathfrak{S}$ of an object $(\mathscr{M}, \widehat{\nabla})$ is defined by

$$\mathrm{H}^i_{\mathrm{dR}}(\mathfrak{X}/\mathfrak{S}, (\mathscr{M}, \widehat{\nabla})) := \mathbf{R}^i f_* (\widehat{\Omega}^\bullet_{\mathfrak{X}/\mathfrak{S}} \otimes \mathscr{M})$$

where $\mathbf{R}^i f$ is the hyperderived functor of f_*.

Remark 4.7. When $\mathfrak{S} = \mathrm{Spf}(k)$, the de Rham cohomology $\mathrm{H}^i_{\mathrm{dR}}(\mathfrak{X}/\mathfrak{S}, (\mathscr{M}, \widehat{\nabla}))$ coincides with de Rham cohomology defined in [Ha75].

Lemma 4.8. *Let $f : \mathfrak{X} \to \mathfrak{S}$ be a smooth morphism of noetherian formal schemes and $(\mathscr{M}, \widehat{\nabla})$ be an object in $\mathrm{MIC}(\mathfrak{X}/\mathfrak{S})$. The functors $\mathrm{H}^i_{\mathrm{dR}}(\mathfrak{X}/\mathfrak{S}, -)$ are the right derived functors of the left exact functor*

$$\begin{aligned} \mathrm{R}^0_{\mathrm{dR}} f_*(-) : \mathrm{MIC}(\mathfrak{X}/\mathfrak{S}) &\longrightarrow \mathrm{MIC}(\mathfrak{S}/\mathfrak{S}) = \textit{(quasicoherent sheaves on } \mathfrak{S}) \\ (\mathscr{M}, \widehat{\nabla}) &\mapsto f_*(\mathscr{M}^{\widehat{\nabla}}). \end{aligned}$$

Proof. See Lemma C.1 in Appendix C. ☐

4.3.4. *Application of Grothendieck's existence theorem.* The i-principal part of relative formal schemes $\mathscr{P}^i_{\mathfrak{X}/\mathfrak{S}}$ is presented in [EGA IV, 20.7], and [Cr25, Section 2]. Using the same argument as in [ABC20, 4.2], we can show that a connection $(\mathscr{M}, \widehat{\nabla})$, as defined in Definition 4.5, is equivalent to an isomorphism of $\mathscr{P}^1_{\mathfrak{X}/\mathfrak{S}}$-modules

$$\mathscr{M} \otimes_{\mathscr{O}_{\mathfrak{X}}} \mathscr{P}^1_{\mathfrak{X}/\mathfrak{S}} \longrightarrow \mathscr{P}^1_{\mathfrak{X}/\mathfrak{S}} \otimes_{\mathscr{O}_{\mathfrak{X}}} \mathscr{M},$$

which is the identity modulo $\widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{S}}$.

Lemma 4.9. *Let X be a proper smooth scheme over complete discrete valuation ring A. Let $\mathfrak{X}$ be the completion of X with respect to the adic topology on A. Denote $\mathfrak{S} := \mathrm{Spf}(A)$. The functor*

$$\begin{aligned} (-)^\wedge : \mathrm{MIC}^{\mathrm{coh}}(X/A) &\longrightarrow \mathrm{MIC}^{\mathrm{coh}}(\mathfrak{X}/\mathfrak{S}) \\ (\mathscr{M}, \nabla) &\mapsto (\widehat{\mathscr{M}}, \widehat{\nabla}) \end{aligned} \tag{12}$$

yields an equivalence of category. Moreover, the functor $(-)^\wedge$ from $\mathrm{MIC}^{\mathrm{se}}(X/A)$ to $\mathrm{MIC}^{\mathrm{se}}(\mathfrak{X}/\mathfrak{S})$ is an equivalence.

Proof. See Lemma C.2 in Appendix C. ☐

The above lemma give us the morphism between two de Rham complexes:

$$\phi : \wedge(\Omega^{\bullet}_{X/A} \otimes \mathscr{M}) \to \widehat{\Omega}^{\bullet}_{\mathfrak{X}/\mathfrak{S}} \widehat{\otimes} \widehat{\mathscr{M}}.$$

Thus, we have the canonical morphism:

$$\mathrm{H}^i_{\mathrm{dR}}(X/A, \mathcal{V}) \to \mathrm{H}^i_{\mathrm{dR}}(\mathfrak{X}/\mathfrak{S}, \widehat{\mathcal{V}}).$$

Theorem 4.10. *Let X be a projective smooth variety over complete discrete valuation ring A. Let $\mathfrak{X}$ be the completion of X with respect to the adic topology on A. Denote $\mathfrak{S} := \mathrm{Spf}(A)$. Let $\mathcal{V}$ be an object in $\mathrm{MIC}^{\mathrm{coh}}(X/A)$. Then*

$$\mathrm{H}^i_{\mathrm{dR}}(X/A, \mathcal{V}) \cong \mathrm{H}^i_{\mathrm{dR}}(\mathfrak{X}/\mathfrak{S}, \widehat{\mathcal{V}})$$

for $i \geq 0$.

Proof. See Theorem C.3 in Appendix C. □

4.4. **Gauss-Manin connection on de Rham cohomology of relative formal schemes.**

Proposition 4.11. *Let $f : \mathfrak{X} \to \mathfrak{S}$ be a smooth morphism of smooth $\mathfrak{Z}$-formal scheme in the category of locally noetherian formal schemes. Let $(\widehat{\mathcal{V}}, \widehat{\nabla})$ be a flat connection on $\mathfrak{X} \to \mathfrak{Z}$. There exist a canonical flat connection $\widehat{\nabla}^q_{GM}$ on the relative de Rham cohomology sheaf $\mathrm{H}^q_{\mathrm{dR}}(\mathfrak{X}/\mathfrak{S}, (\widehat{\mathcal{V}}, \widehat{\nabla}))$.*

Proof. See Proposition C.4 in Appendix C. □

Theorem 4.12 (Leray spectral sequence)**.** *Let $g : \mathfrak{X} \longrightarrow \mathfrak{S}$ be a smooth morphism of smooth k-formal schemes and $f : \mathfrak{S} \longrightarrow \mathfrak{Z}$ be a smooth morphism of smooth k-formal schemes. Let $(\mathcal{V}, \widehat{\nabla})$ be an object in $\mathrm{MIC}(\mathfrak{X}/k)$. Then there exists a spectral sequence of de Rham sheaf cohomology of relative formal schemes:*

$$E_2^{p,q} = R^p_{\mathrm{dR}} f_* \left(R^q_{\mathrm{dR}} g_* \left(\mathcal{V}, \widehat{\nabla} \right), \nabla^q_{\mathfrak{S}} \right) \Longrightarrow R^{p+q}_{\mathrm{dR}} (f \circ g)_* (\mathcal{V}, \widehat{\nabla}),$$

where ∇^q_X indicates the Gauss-Manin connection on $R^q_{\mathrm{dR}} g_* \left(\mathcal{V}, \widehat{\nabla} \right)$.

Proof. See Theorem C.5 in Appendix C. □

5. THE GAUSS-MANIN CONNECTION FROM THE TANNAKIAN VIEWPOINT

According to Theorem 3.2, we have

$$1 \longrightarrow \pi^{\mathrm{geom}}(X/R) \longrightarrow \Pi(X/k) \xrightarrow{f} \Pi(R/k) \longrightarrow 1.$$

Let $\mathcal{V}$ be an object in $\mathrm{MIC}^{\circ}(X/k)$, together with its fiber $V = \eta^*(\mathcal{V})$ which is an R-module. According to Lemma A.4, the finite R-module $\mathrm{H}^0(\pi^{\mathrm{geom}}(X/R), V)$ is a $\Pi(R/k)$-representation in a natural way. According to Lemma 4.3, the canonical morphism of objects in $\mathrm{MIC}(R/k)$

$$\mathrm{H}^0(\pi^{\mathrm{geom}}(X/R), V) \longrightarrow \mathrm{H}^0_{\mathrm{dR}}(X/R, \mathrm{inf}(\mathcal{V}))$$

is an isomorphism. Thus, the following diagram of functors is commutative:

(13)
$$\begin{array}{ccc} \mathrm{Rep}(R : \Pi(X/k)) & \xrightarrow{\mathrm{H}^0(\pi^{\mathrm{geom}}(X/R), -)} & \mathrm{Rep}(R : \Pi(R/k)) \\ \downarrow & & \downarrow \\ \mathrm{MIC}(X/k) & \xrightarrow{\mathrm{H}^0_{\mathrm{dR}}(X/R, \mathrm{inf}(-))} & \mathrm{MIC}(R/k). \end{array}$$

Consequently we obtain canonical morphisms

$$\mathrm{R}^i_{\mathrm{Rep}(R:\Pi(X/k))}\mathrm{H}^0(\pi^{\mathrm{geom}}(X/R),V) \longrightarrow \mathrm{R}^i_{\mathrm{MIC}(X/k)}\mathrm{H}^0_{\mathrm{dR}}(X/R,\inf(\mathcal{V}))$$

where, on the left-hand side, the derived functor is taken in $\mathrm{Rep}(R:\Pi(X/k))$ and on the right-hand side, the derived functor is taken in $\mathrm{MIC}(X/k)$. We know that the right-hand side is the nth relative de Rham cohomology of the inflated connection (cf. [Ka70, Remark 3.1], [ABC20, 23.2.5]):

$$\mathrm{R}^{\mathrm{i}}_{\mathrm{MIC}(X/k)}\mathrm{H}^0_{\mathrm{dR}}(X/R,\mathcal{V}) = \mathrm{H}^{\mathrm{i}}_{\mathrm{dR}}(X/R,\mathcal{V}).$$

On the other hand, consider the following diagram of functors:

$$\begin{array}{ccc} \mathrm{Rep}(R:\Pi(X/k)) & \xrightarrow{\mathrm{H}^0(\pi^{\mathrm{geom}}(X/R),-)} & \mathrm{Rep}(R:\Pi(R/k)) \\ \downarrow{\scriptstyle \mathrm{Res}^{\Pi(X/k)}_{\pi^{\mathrm{geom}}(X/R)}} & & \downarrow{\scriptstyle \mathrm{Res}^{\Pi(R/k)}_{1}} \\ \mathrm{Rep}(\pi^{\mathrm{geom}}(X/R)) & \xrightarrow{\mathrm{H}^0(\pi^{\mathrm{geom}}(X/R),-)} & \mathrm{Mod}_R. \end{array} \tag{14}$$

There exists a canonical homomorphism of the two derived functors

$$\mathrm{R}^i_{\mathrm{Rep}(R:\Pi(X/k))}\mathrm{H}^0(\pi^{\mathrm{geom}}(X/R),V) \longrightarrow \mathrm{H}^i(\pi^{\mathrm{geom}}(X/R),V)$$

(the former derived functor is taken in $\mathrm{Rep}(R:\Pi(X/k))$ and the latter one is taken in $\mathrm{Rep}(\pi^{\mathrm{geom}}(X/R))$.

Theorem 5.1. *Let X be a projective smooth geometrically connected fiber over Dedekind domain R. Let V be an object in $\mathrm{Rep}^{\mathrm{f}}(R:\Pi(X/k))$. For $i \geq 0$, the canonical homomorphism*

$$\mathrm{R}^i_{\mathrm{Rep}(R:\Pi(X/k))}\mathrm{H}^0(\pi^{\mathrm{geom}}(X/R),V) \longrightarrow \mathrm{H}^i(\pi^{\mathrm{geom}}(X/R),V)$$

is an isomorphism. Consequently, it induces a representation of $\Pi(R/k)$ on $\mathrm{H}^n(\pi^{\mathrm{geom}}(X/R),V)$ which has the property that the canonical homomorphism

$$\mathrm{H}^i(\pi^{\mathrm{geom}}(X/R),V) \longrightarrow \mathrm{H}^i_{\mathrm{dR}}(X/R,\inf(\mathcal{V}))$$

is $\Pi(R/k)$-equivariant for $i \geq 0$.

Proof. As an initial step, we describe the injective objects in the category $\mathrm{Rep}(R:\Pi(X/k))$. Indeed, each injective object F_1 in $\mathrm{Rep}(R:\Pi(X/k))$ is a direct summand of $I\otimes_t\mathcal{O}(\Pi(X/k))$, where I is regarded as a trivial $\Pi(X/k)$-module (see Lemma A.8). Next, we prove that the restriction functor maps an injective $\Pi(X/k)$-module to one that is acyclic for the functor $(-)^{\pi^{\mathrm{geom}(X/R)}}$. Indeed, thanks to Proposition A.15, we obtain

$$\mathrm{R}^i\mathrm{Ind}^{\Pi(X/k)}_{\pi^{\mathrm{geom}(X/R)}}(I) \simeq \mathrm{H}^i(\pi^{\mathrm{geom}(X/R)}, I\otimes_t\mathcal{O}(G)) = \mathrm{H}^i(\pi^{\mathrm{geom}(X/R)},F_1)\oplus\mathrm{H}^i(\pi^{\mathrm{geom}(X/R)},F_2),$$

where F_2 is a $\Pi(X/k)$-module such that $F_1\oplus F_2 = I\otimes_t\mathcal{O}(\Pi(X/k))$. Since the induction functor $\mathrm{Ind}^{\Pi(X/k)}_{\pi^{\mathrm{geom}(X/R)}}$ is exact (see Lemma 5.2), the $\Pi(X/k)$-module F_1 is acyclic for the functor $(-)^{\pi^{\mathrm{geom}}(X/R)}$. Therefore, the Remark A.9-(1),(2) imply that the

$$V \mapsto \mathrm{H}^i(\pi^{\mathrm{geom}(X/R)},V)$$

can be regarded as the derived functors of

$$V \mapsto V^{\pi^{\mathrm{geom}}(X/R)}$$

from $\mathrm{Rep}(R:\Pi(X/k))$ to $\mathrm{Rep}(R:\Pi(R/k))$. □

Lemma 5.2. *The functor $\mathrm{Ind}^{\Pi(X/k)}_{\pi^{\mathrm{geom}}(X/R)}$ is exact.*

Proof. According to Theorem 3.2, the geometric relative differential fundamental group $\pi^{\text{geom}}(X/R)$ is normal subgroup of $\Pi(X/k)^{\Delta}$. Then, the functor $\mathrm{Ind}_{\pi^{\text{geom}}(X/R)}^{\Pi(X/k)^{\Delta}}$ is exact by the same argument as in the proof of [DH18, Theorem 4.2.2]. Using Theorem A.6 combined with the following equality:

$$\mathrm{Ind}_{\Pi(X/k)^{\Delta}}^{\Pi(X/k)} \circ \mathrm{Ind}_{\pi^{\text{geom}}(X/R)}^{\Pi(X/k)^{\Delta}} \cong \mathrm{Ind}_{\pi^{\text{geom}}(X/R)}^{\Pi(X/k)},$$

we conclude that $\mathrm{Ind}_{\pi^{\text{geom}}(X/R)}^{\Pi(X/k)}$ is exact. □

6. Comparison between de Rham cohomology and group cohomology of $\pi(X/R)$

Given Theorem 5.1, our next question is whether the map

$$\mathrm{H}^i(\pi^{\text{geom}}(X/R), V) \longrightarrow \mathrm{H}^i_{\mathrm{dR}}(X/R, \inf(\mathcal{V}))$$

is bijective. Our strategy consists naturally of two steps to be carried out in this section and the next section. In this section we aim to compare, through Tannakian duality, the de Rham cohomology on X/R and the group cohomology of $\pi(X/R)$. In the next section we try to compare the cohomologies of $\pi(X/R)$ and $\pi^{\text{geom}}(X/R)$.

In general de Rham cohomology and group cohomology of $\pi(X/R)$ are not equal due to the fact that representations of the fundamental group constitute only a part of connections (cf. subsection 2.2.5). But in the case of a smooth projective curve of genus at least 1, the two cohomologies are equal [BHT25]. We expect this to hold for smooth, projective relative curves over a DVR of equal characteristic 0. Unfortunately, we can only prove this for the case when the DVR is complete, cf. Theorem 6.9.

Assuming that $R = A$ – a complete discrete valuation ring. We aim show the natural maps of A modules:

$$\delta^i : \mathrm{H}^i(\pi(X/A), V)) \to \mathrm{H}^i_{\mathrm{dR}}(X/A, \mathcal{V})$$

are isomoprhisms for all $i \geq 0$, provided that the genus of X is ≥ 1. Here, the genus of X is the rank of the first de Rham cohomology group $\mathrm{H}^1_{\mathrm{dR}}(X/A, \mathcal{O}_{\mathcal{X}})$. Since this A-module is equipped with a connection on A/k (the Gauss-Manin connection), it is a free A-module.

Our strategy is to investigate the isomorphy the maps on the generic and the closed fibers of $\mathrm{Spec}(A)$.

6.1. The comparison maps between de Rham cohomology and group cohomology.

Let k be a field of characteristic 0. Let R be a k-algebra, which is a dedekind ring. Let $f : X \longrightarrow S = \mathrm{Spec}(R)$ be a smooth map with fiber geometrically connected. We shall assume moreover that the fibers are of dimension 1. We recall from Subsection 2.2.3 that the fibre functor η^* induces

$$\mathrm{Rep}_{\mathrm{f}}(\pi(X/R)) \cong \mathrm{MIC}^{\mathrm{se}}(X/R).$$

By taking the ind-category of these categories, we have

$$\mathrm{Rep}(\pi(X/R)) \cong \mathrm{MIC}^{\mathrm{ind}}(X/R). \tag{15}$$

Let $(\mathcal{M}, \nabla)$ be an object in $\mathrm{MIC}^{\mathrm{ind}}(X/R)$, and let $M := \mathcal{M}|_x$ be its fiber at x. An element e of $\mathrm{H}^i(\pi(X), M)$ is given in terms of an i-extension of k by M [Ja87, 4.2, p. 57]. Under the equivalence (15) it corresponds to an i-extension of $(\mathcal{O}, d)$ by $(\mathcal{M}, \nabla)$ in $\mathrm{MIC}^{\mathrm{ind}}(X/R)$ and thus can be considered as an element in $\mathrm{Ext}^i_{\mathrm{MIC}(X/R)}(\mathscr{O},\mathscr{M})$, hence it determines an element $\delta^i(e)$ of $\mathrm{H}^i_{\mathrm{dR}}(X/R,\mathscr{M})$ (see Lemma 4.1). In summary, we have the following map

$$\delta^i : \mathrm{H}^i(\pi(X/R), M) \longrightarrow \mathrm{H}^i_{\mathrm{dR}}(X/R,\mathscr{M}). \tag{16}$$

It follows from the Tannakian duality mentioned above and the description of de Rham cohomology as ext-group (cf. subsection 4.1) that for any $\mathscr{M} \in \mathrm{Obj}(\mathrm{MIC}^{\mathrm{ind}}(X/R))$ and $M = \eta^*(\mathscr{M})$ we have isomorphisms

$$\mathrm{H}^0_{\mathrm{dR}}(X/R,\mathscr{M}) \cong \mathrm{H}^0(\pi(X/R), M). \tag{17}$$

With R as a Dedekind ring, we do not yet know whether

$$\delta^1 : \mathrm{H}^1(\pi(X/R), M) \longrightarrow \mathrm{H}^1_{\mathrm{dR}}(X/R,\mathscr{M})$$

is an isomorphism. In other words, we don't know if $\mathrm{MIC}^{\mathrm{se}}(X/R) = \mathrm{MIC}^{\mathrm{coh}}(X/R)$.

However, when $R = A$ is a cDVR (over k), the following lemma allows us to establish this equality.

Lemma 6.1. *Let X be a proper smooth separated noetherian scheme over complete discrete valuation ring A. Then $\mathrm{MIC}^{\mathrm{se}}(X/A) = \mathrm{MIC}^{\mathrm{coh}}(X/A)$. Consequently we have*

$$\mathrm{H}^1_{\mathrm{dR}}(X/A,\mathscr{M}) \cong \mathrm{H}^1(\pi(X/A), M) \tag{18}$$

Proof. Let $\mathfrak{X}$ be the completion of X with respect to the adic topology on A. Denote $\mathfrak{S} := \mathrm{Spf}(A)$. To prove this lemma, we adapt the proof in [DH18, Proposition 5.2.2]. Let $\mathscr{V} \in \mathrm{Obj}(\mathrm{MIC}^{\mathrm{se}}(\mathfrak{X}/\mathfrak{S}))$. For each open affine formal subschemes $\mathfrak{U}$, the connection $\mathscr{V}|_{\mathfrak{U}}$ is a quotient of an connection in $\mathrm{MIC}^{\circ}(\mathfrak{U}/\mathfrak{S})$. Therefore, $\mathscr{V}$ can be seen as the sheaftification of a quotient sheaf, that is, the connection $\mathscr{V}$ is a quotient of an object in $\mathrm{MIC}^{\circ}(\mathfrak{X}/\mathfrak{S})$. Finally, using Lemma 4.9, we see that each object in $\mathrm{MIC}^{\mathrm{coh}}(X/A)$ is a quotient of an object in $\mathrm{MIC}^{\circ}(X/A)$.

The last equality is then obvious as for any exact sequence

$$0 \longrightarrow \mathscr{M} \longrightarrow \mathscr{M}' \longrightarrow \mathscr{O}_X \longrightarrow 0,$$

in $\mathrm{MIC}^{\mathrm{se}}(X/A)$, if $\mathscr{M}$ is in $\mathrm{MIC}^{\mathrm{se}}(X/A)$, then so is $\mathscr{M}'$. □

Lemma 6.2. *Let $\mathscr{M}$ be an object in $\mathrm{MIC}^{\mathrm{se}}(X/A)$. Denote $M := \eta^*(\mathscr{M})$. Then the map*

$$\delta^2 : \mathrm{H}^2(\pi(X/A), M) \to \mathrm{H}^2_{\mathrm{dR}}(X/A,\mathscr{M})$$

is injective.

Proof. Let J be the injective envelope of M in $\mathrm{Rep}(\pi(X/A))$ and let $(\mathscr{J},\nabla_{\mathscr{J}})$ be the corresponding connection. Since J is injective in $\mathrm{Rep}(\pi(X/A))$, and since the de Rham cohomology commutes with directed limits, one has

$$\mathrm{H}^1_{\mathrm{dR}}(X/A,(\mathscr{J},\nabla_{\mathscr{J}})) \cong \mathrm{H}^1(\pi(X/A), J) = 0.$$

Hence, the long exact sequences associated with the exact sequences $0 \to M \to J \to J/M \to 0$ and $0 \to (\mathscr{M},\nabla_{\mathscr{M}}) \to (\mathscr{J},\nabla) \to (\mathscr{J}/\mathscr{V},\nabla) \to 0$ yield

$$\mathrm{H}^2(\pi(X/A), M) \cong \mathrm{H}^1(\pi(X/A), J/M) \cong \mathrm{H}^1_{\mathrm{dR}}(X/A,\mathscr{J}/\mathscr{M}) \hookrightarrow \mathrm{H}^2_{\mathrm{dR}}(X/A,\mathscr{M}). \tag{19}$$

□

For $i \geq 2$, we establish the comparison by reducing the problem to comparisons at both the generic and closed fibers. This approach requires an understanding of the fibers of Tannakian groups and the scalar extension of Tannakian categories.

6.2. **Comparison at the generic fiber.** In this subsection we show that the maps $\delta^i \otimes_K$ are isomorphisms. Recall that the isomorphy of δ^i for smooth projective curves over fields of characteristic zero is known [BHT25]. How ever we cannot apply this result straightforwardly as the two groups $\pi(X/R)_K$ and $\pi(X_K/K)$ are different. Fortunately, their cohomology groups with coefficients in representations of the former are isomorphic, cf. Proposition 6.5.

Before proceeding let us remind the reader of the result just mentioned, cf. [BHT25, Theorem 4.1]:

Lemma 6.3. *The map*

$$\delta^i(X_K/K) : \mathrm{H}^i(\pi(X_K/K), M_K) \to \mathrm{H}^i_{\mathrm{dR}}(X_K/K, \mathscr{M}_K)$$

is bijective for $i \geq 0$. Further, both side are non-vanishing for any non-zero connection $\mathscr{M}$, provided that X_K has genus ≥ 2. The same holds for the closed fiber of the map.

To define the generic fiber of $\pi(X/R)$, we recall the category $\mathrm{MIC}^{\mathrm{se}}(X/R)_K$ introduced in [DH18, 1.3.1, 1.3.2]. The objects of $\mathrm{MIC}^{\mathrm{se}}(X/R)_K$ are the same as those of $\mathrm{MIC}^{\mathrm{se}}(X/R)$ and for two objects $\mathscr{M}$ and $\mathscr{N}$ in $\mathrm{MIC}^{\mathrm{se}}(X/R)_K$ their hom-set is

$$\mathrm{Hom}_{\mathrm{MIC}^{\mathrm{se}}(X/R)_K}(\mathscr{M},\mathscr{N}) := \mathrm{Hom}_{\mathrm{MIC}^{\mathrm{se}}(X/R)}(\mathscr{M},\mathscr{N}) \otimes_R K.$$

The category $\mathrm{MIC}^{\mathrm{se}}(X/R)_K \cong \mathrm{MIC}^{\circ}(X/R)_K$ is abelian category [DH18, Lemma 5.1.2]. Moreover, this category is Tannakian category with the fiber functor

$$\eta^* \otimes K : \mathrm{MIC}^{\mathrm{se}}(X/R)_K \to \mathrm{Vec}_K,$$

this category is controlled by the generic fiber of $\pi(X/R)$, that is,

$$\mathrm{MIC}^{\mathrm{se}}(X/R)_K \cong \mathrm{Rep}^{\mathrm{f}}(\pi(X/R)_K).$$

Lemma 6.4. *The category $\mathrm{MIC}^{\mathrm{se}}(X/R)_K$ is full subcategory of category $\mathrm{MIC}^{\circ}(X_K/K)$. Moreover,*

$$\pi(X_K/K) \twoheadrightarrow \pi(X/R)_K.$$

Proof. The category $\mathrm{MIC}^{\mathrm{se}}(X/R)_K$ can be considered as full subcategory of $\mathrm{MIC}^{\circ}(X_K/K)$ through the following restriction functor

$$\begin{aligned} \mathrm{MIC}^{\mathrm{se}}(X/R) &\to \mathrm{MIC}^{\circ}(X_K/K) \\ (\mathscr{M},\nabla) &\mapsto (\mathscr{M}_K,\nabla_K). \end{aligned}$$

The restriction functor compatible with the fiber functors of these two categories. Therefore, we have a map

$$\pi(X_K/K) \to \pi(X/R)_K.$$

To prove the surjectivity of the above map, we use the criterion of Deligne-Milne [DM82, Theorem 2.21]. We show that for each object $\mathscr{M}_0 \in \mathrm{MIC}^{\mathrm{se}}(X/R)_K$, when considered as object in $\mathrm{MIC}^{\circ}(X_K/K)$ all its subobjects will be objects in $\mathrm{MIC}^{\mathrm{se}}(X/R)_K$. There exists by assumption an $\mathscr{M} \in \mathrm{Obj}(\mathrm{MIC}^{\circ}(X/R))$ such that $i^*\mathscr{M} = \mathscr{M}_0$, where $i = i_{U/R} \circ i_{K/U}$ (see the maps $i_{K/U}$ and $i_{U/R}$ in the diagram below). Let $\mathscr{N}_0$ be a subobject of $\mathscr{M}_0$. Then there exists an open subscheme U of $\mathrm{Spec}(R)$ and $\mathscr{N}_U$ is a subconnection of $\mathscr{M}_{|X_U}$ such that $i^*_{K/U}\mathscr{N}_U = \mathscr{N}_0$, where $i_{K/U}$ is the map in the following cartesian diagram:

$$\begin{array}{ccccc} X_K & \xrightarrow{i_{K/U}} & X_U & \xrightarrow{i_{U/R}} & X \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}(K) & \longrightarrow & U & \longrightarrow & \mathrm{Spec}(R). \end{array}$$

We set $\mathcal{N} := i_{U/R*}\mathcal{N}_U \cap \mathcal{M}$. The sheaf $\mathcal{N}$ is defined as sheaftification of subpresheaf of $i_* i^* \mathcal{M}$. Since $\mathcal{N}$ can be consider as subconnection of $\mathcal{M}$, the sheaf $\mathcal{N}$ is locally free thanks to Proposition 5.1.1 in [DH18].

We now ready to prove $i^*\mathcal{N} = \mathcal{N}_0$, i.e., $\mathcal{N}_0 \in \mathrm{Obj}(\mathrm{MIC}^{\mathrm{se}}(X/R)_K)$. Then

$$i^*\mathcal{N} = i^* i_{U/R*}\mathcal{N}_U \cap i^*\mathcal{M} = i^* i_{U/R*}\mathcal{N}_U \cap \mathcal{M}_0,$$

so we will prove that

$$i^* i_{U/R*}\mathcal{N}_U = \mathcal{N}_0$$

to finish the proof. Indeed, since $i^*_{U/R} i_{U/R*}\mathcal{N} = \mathcal{N}$ (as $i_{U/R}$ is an open immersion) and $i^* = i^*_{K/U} \circ i^*_{U/R}$, the result follows. □

We have $(-)^{\pi(X/R)_K} = (-)^{\pi(X_K/K)}$, hence from the "universal δ-functor" formalism we get for every $\pi(X/R)_K$-module M_K a system of maps

$$\lambda^i_K : \mathrm{H}^i(\pi(X/R)_K, M_K) \to \mathrm{H}^i(\pi(X_K/K), M_K).$$

The main result of this subsection is the following:

Proposition 6.5. *Let A be a complete discrete valuation ring. Let X be a projective curve over A with genus $g \geq 1$. Let M_K be an object in $\mathrm{Rep}^{\mathrm{f}}(\pi(X/A)_K)$. Then the maps*

$$\lambda^i_K : \mathrm{H}^i(\pi(X/A)_K, M_K) \to \mathrm{H}^i(\pi(X_K/K), M_K) \tag{20}$$

for all $i \geq 0$ are isomorphism.

Proof. Bijectivity of λ^1_K. Let M be an object in $\mathrm{Rep}^{\mathrm{f}}(\pi(X/A))$. According to (18), we have

$$\delta^1 : \mathrm{H}^1(\pi(X/A), M) \xrightarrow{\cong} \mathrm{H}^1_{\mathrm{dR}}(X/A, \mathcal{M}). \tag{21}$$

Applying the flat base change theorem for de Rham cohomology [ABC20, 23.5] and group cohomology (Proposition 4.13 in [Ja87]), we obtain the following diagram:

$$\begin{array}{ccc}
\mathrm{H}^1(\pi(X/A)_K, M_K) & \xrightarrow{\lambda^1_K} & \mathrm{H}^1(\pi(X_K/K), M_K) \\
\uparrow \cong & & \\
\mathrm{H}^1(\pi(X/A), M) \otimes_A K & & \downarrow \delta^1(X_K/K) \\
\downarrow \delta^1 \otimes K & & \\
\mathrm{H}^1_{\mathrm{dR}}(X/A, \mathcal{M}) \otimes_A K & \xrightarrow{\cong} & \mathrm{H}^1_{\mathrm{dR}}(X_K/K, \mathcal{M}_K)
\end{array}$$

From (21), we can see that the map $\delta^1 \otimes K$ is bijective. Thanks to Lemma 6.3, the morphism $\delta^1(X_K/K)$ is also bijective. Thus, the morphism λ^1_K is bijective for every representation M_K in $\mathrm{Rep}^{\mathrm{f}}(\pi(X/A)_K)$.

Injectivity of λ^2_K. Using arguments similar to those in Lemma 6.2, the map

$$\lambda^2_K : \mathrm{H}^2(\pi(X/A)_K, M_K) \longrightarrow \mathrm{H}^2(\pi(X_K/K), M_K)$$

is injective.

Surjectivity of λ^2_K. There are two cases: the genus is 1 or larger than or equal to 2.

Case $g \geq 2$. We need a version of Lemma 4.4 in [BHT25].

Claim. Let M be an object in $\mathrm{Rep}^{\mathrm{f}}(\pi(X/A)_K)$. Then there exists an object M' in $\mathrm{Rep}^{\mathrm{f}}\pi(X/A)_K$ and an injective map $j: M \hookrightarrow M'$ such that

$$\mathrm{H}^2(\pi(X_K/K), M') = 0.$$

Verification. Thanks to the bijectivity of $\delta^\bullet(X_K/K): \mathrm{H}^i(\pi(X_K/K), M_K) \to \mathrm{H}^i_{\mathrm{dR}}(X_K/K, \mathscr{M}_K)$ (Lemma 6.3), and the Poincaré duality for de Rham cohomology on X_K/K, the claim is equivalent to showing that

$$\textit{there exists a surjective morphism } N \twoheadrightarrow M^\vee \textit{ in } \mathrm{Rep}^{\mathrm{f}}\pi(X/R)_K \textit{ such that } \mathrm{H}^0(\pi(X_K/K), N) = 0.$$

By induction on the dimension of $\mathrm{H}^0(\pi(X_K/K), N)$, it suffices to show there exists N surjecting on $M^\vee$ and satisfying the strict inequality:

$$\mathrm{h}^0(\pi(X_K/K), N) < \mathrm{h}^0(\pi(X_K/K), M^\vee),$$

as long as $\mathrm{h}^0(\pi(X_K/K), M^\vee) \neq 0$.

Consider K as the trivial representation of $\pi(X/A)_K$, assume there exists an inclusion $K \hookrightarrow M^\vee$ and let $F := M^\vee/K$ be the quotient representation. Consider an arbitrary, non-trivial, irreducible representation P and apply the functor $\mathrm{Ext}^i_{\mathrm{Rep}(\pi(X_K/K))}(-, P)$ to the short exact sequence in (the full, thick subcategory) $\mathrm{Rep}^{\mathrm{f}}(\pi(X/A)_K)$:

$$0 \longrightarrow K \xrightarrow{e} M^\vee \longrightarrow F \longrightarrow 0,$$

we obtain a long exact sequence:

$$\mathrm{Ext}^1_{\mathrm{Rep}(\pi(X_K/K))}(F, P) \longrightarrow \mathrm{Ext}^1_{\mathrm{Rep}(\pi(X_K/K))}(M^\vee, P) \xrightarrow{\mathrm{ev}_e} \mathrm{Ext}^1_{\mathrm{Rep}(\pi(X_K/K))}(K, P) \longrightarrow \mathrm{Ext}^2_{\mathrm{Rep}(\pi(X_K/K))}(F, P).$$

By Poincaré duality we have

$$\begin{aligned}\mathrm{Ext}^2_{\mathrm{Rep}(\pi(X_K/K))}(F, P) &\cong \mathrm{H}^2(\pi(X_K/K), F^\vee \otimes P)\\ &\cong \mathrm{H}^0(\pi(X_K/K), F \otimes P^\vee)^\vee\\ &\cong \mathrm{Hom}_{\mathrm{Rep}^{\mathrm{f}}(\pi(X/A)_K)}(P, F)^\vee.\end{aligned}$$

We observe that $\mathrm{MIC}^\circ(X/A)_K$ has infinitely many non-trivial connections of rank 1. Indeed, consider the structure sheaf $\mathscr{O}_X$ equipped with a connection

$$\nabla: \mathscr{O}_X \longrightarrow \Omega_{X/A}, \quad 1 \longmapsto \omega,$$

for any non-zero form ω. Then, by base change, we have a structure sheaf $\mathscr{O}_{X_K}$ with a connection

$$\nabla_K: \mathscr{O}_{X_K} \longrightarrow \Omega_{X_K/K}, \quad 1 \longmapsto \omega_K.$$

Since X has genus $g \geq 1$, there are infinitely many choices for ω, and since a map $A \longrightarrow K$ is flat, we obtain infinitely ω_K. Since $\mathrm{MIC}^\circ(X/A)_K$ is equivalent to $\mathrm{Rep}^{\mathrm{f}}(\pi(X/A)_K)$, we can choose a non-trivial irreducible representation P such that

$$\mathrm{Hom}_{\mathrm{Rep}^{\mathrm{f}}(\pi(X/A)_K)}(P, F)^\vee = 0,$$

consequently, the map

$$\mathrm{ev}_e: \mathrm{Ext}^1_{\mathrm{Rep}(\pi(X_K/K))}(M^\vee, P) \longrightarrow \mathrm{Ext}^1_{\mathrm{Rep}(\pi(X_K/K))}(K, P)$$

is surjective.

Since $\delta^1(X_K/K)$ is bijective and the first group cohomology $\mathrm{H}^1(\pi(X_K/K), M_K)$ is non-vanishing (Lemma 6.3), there exists a non-split extension of connections:

$$\varepsilon: P \longrightarrow Q \longrightarrow K.$$

With the assumption that P is non-trivial and simple, we have that $\mathrm{H}^0(\pi(X_K/K), Q) = 0$. Let ϵ be a preimage of ε along the surjective map ev_e. In terms of extensions, these are related by the following diagrams in $\mathrm{Rep}^{\mathrm{f}}(\pi(X/A)_K)$

$$\begin{array}{ccccccccc} \epsilon : 0 & \longrightarrow & P & \longrightarrow & N & \longrightarrow & M^\vee & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow e & & \\ \varepsilon = \mathrm{ev}_e(\epsilon) : 0 & \longrightarrow & P & \longrightarrow & Q & \longrightarrow & K & \longrightarrow & 0. \end{array}$$

Take the associated long exact sequence of group cohomology we have

$$\begin{array}{ccccccc} 0 = \mathrm{H}^0(\pi(X_K/K), P) & \longrightarrow & \mathrm{H}^0(\pi(X_K/K), N) & \longrightarrow & \mathrm{H}^0(\pi(X_K/K), M^\vee) & \longrightarrow & \mathrm{H}^1(\pi(X_K/K), P) \\ \| & & \uparrow & & \uparrow & & \| \\ 0 = \mathrm{H}^0(\pi(X_K/K), P) & \longrightarrow & \mathrm{H}^0(\pi(X_K/K), Q) = 0 & \longrightarrow & \mathrm{H}^0(\pi(X_K/K), K) & \hookrightarrow & \mathrm{H}^1(\pi(X_K/K), P). \end{array}$$

Since $\mathrm{H}^0(\pi(X_K/K), K) = K \neq 0$, the rightmost upper horizontal map

$$\mathrm{H}^0(\pi(X_K/K), M^\vee) \longrightarrow \mathrm{H}^1(\pi(X_K/K), P)$$

should be non-zero, that is $\mathrm{H}^0(\pi(X_K/K), N) \neq \mathrm{H}^0(\pi(X_K/K), M^\vee)$, whence the desired inequality. $\triangleright$

Let M be an object in $\mathrm{Rep}^{\mathrm{f}}(\pi(X/A)_K)$ and let $M \hookrightarrow M'$ as in Claim, that is, $\mathrm{H}^2(\pi(X_K/K), M') = 0$. The short exact sequence

$$0 \to M \to M' \to M'/M \to 0,$$

in $\mathrm{Rep}^{\mathrm{f}}(\pi(X/A)_K$ can be considered as in $\mathrm{Rep}_{\mathrm{f}}(\pi(X_K/K))$. Since the map (20) is the morphism of δ-functors, we have the following diagram:

$$\begin{array}{ccccc} \mathrm{H}^1(\pi(X/A)_K, M'/M) & \longrightarrow & \mathrm{H}^2(\pi(X/A)_K, M) & \longrightarrow & \mathrm{H}^2(\pi(X/A)_K, M') \\ \cong \downarrow \lambda^1_K & & \downarrow \lambda^2_K & & \downarrow \lambda^2_K \\ \mathrm{H}^1(\pi(X_K/K), M'/M) & \twoheadrightarrow & \mathrm{H}^2(\pi(X_K/K), M) & \longrightarrow & \mathrm{H}^2(\pi(X_K/K), M') = 0. \end{array}$$

We conclude that the middle arrow is surjective.

Case $g = 1$. Since X is an elliptic curve, it has a group structure. By Künneth theorem for $\pi(X_K/K)$ [De89, Corollary 10.47], it possesses another multiplication induced from the group structure on X. Using the Eckmann-Hilton argument, we conclude that $\pi(X_K/K)$ is abelian, leading to the decomposition

$$\pi(X_K/K) = \pi(X_K/K)^{\mathrm{uni}} \times \pi(X_K/K)^{\mathrm{red}}.$$

Furthermore, $\pi^{\mathrm{uni}}(X_K/K) \cong \mathbb{G}_a \times \mathbb{G}_a$, by means of [LM82, (1.2), (1.16)].

On the other hand, Lemma 6.4 implies that the group $\pi(X/A)_K$ is also abelian, and we have the surjection

$$\pi^{\mathrm{uni}}(X_K/K) \twoheadrightarrow \pi(X/A)_K^{\mathrm{uni}}$$

Since $\mathrm{H}^1(\pi^{\mathrm{uni}}(X_K/K), \mathscr{O}_{X_K}) = \mathrm{H}^1(\pi(X/A)_K^{\mathrm{uni}}, \mathscr{O}_{X_K})$, it follows that $\pi(X/A)_K \cong \mathbb{G}_a \times \mathbb{G}_a$. Now, using Lyndon-Hochschild-Serre spectral sequence, we rewrite the map λ^i_K for $i \geq 2$ as

$$\mathrm{H}^i(\pi(X/A)_K^{\mathrm{uni}}, M_K) \longrightarrow \mathrm{H}^i(\pi^{\mathrm{uni}}(X_K/K), M_K).$$

Since $\pi^{\mathrm{uni}}(X_K/K) \cong \pi(X/A)_K^{\mathrm{uni}}$ the map λ^i_K is an isomorphism for $i \geq 2$.

Bijectivity of λ^i_K *(*$i \geq 3$*)*. Since the de Rham cohomology vanishes in degrees $i \geq 3$, it suffices to show that λ^i_K is injective. We use the same argument as in the proof of Lemma 6.2, and the result follows. $\square$

Corollary 6.6. *Let X be a projective curve with genus $g \geq 1$. Let $\mathscr{M}$ be an object in* $\mathrm{MIC}^{\mathrm{se}}(X/A)$*. Denote* $M = \eta^*(\mathscr{M})$*. Then the map*

$$\delta^i \otimes K : \mathrm{H}^i(\pi(X/A), M)) \otimes K \longrightarrow \mathrm{H}^i_{\mathrm{dR}}(X/A, \mathscr{M}) \otimes K$$

is an isomorphism for $i \geq 0$.

Proof. Using flat base change for de Rham cohomology and group cohomology, we have the following diagram:

$$\begin{array}{ccc}
\mathrm{H}^i(\pi(X/A), M) \otimes K & \xrightarrow{\cong} & \mathrm{H}^i(\pi(X/A)_K, M_K) \\
 & & \downarrow{\scriptstyle \lambda^i_K} \\
\downarrow{\scriptstyle \delta^i \otimes K} & & \mathrm{H}^i(\pi(X_K/K), M_K) \\
 & & \downarrow{\scriptstyle \delta^i(X_K/K)} \\
\mathrm{H}^i_{\mathrm{dR}}(X/A, \mathscr{M}) \otimes K & \xrightarrow{\cong} & \mathrm{H}^i_{\mathrm{dR}}(X_K/K, \mathscr{M}_K).
\end{array}$$

As λ^i_K is an isomorphism via Proposition 6.5 and $\delta^i(X_K/K)$ is also an isomorphism, so is $\delta^i \otimes K$. □

6.3. **Comparison at the closed fiber.** Our aim now is to check that the maps

$$\delta^i \otimes k_s : \pi(X/R, \mathcal{V}) \otimes k_s \longrightarrow \mathrm{H}^i_{\mathrm{dR}}(X_s/k, \mathcal{V} \otimes k_s)$$

are isomorphisms, for any point of s of $S = \mathrm{Spec}(R)$. To define the special fiber of $\pi(X/R)$ at the closed point s of $S = \mathrm{Spec}(R)$, we recall the category $\mathrm{MIC}^{\mathrm{se}}(X/R)_s$. Let $\mathfrak{m}_s$ be the maximal ideal which determines s.. $\mathrm{MIC}^{\mathrm{se}}(X/R)_s$ is the full subcategory in $\mathrm{MIC}^{\mathrm{se}}(X/R)$ of objects annihilated by m_s. Then we have (cf. [Ja87, Chap. 10]):

$$\mathrm{MIC}^{\mathrm{se}}(X/R) \cong \mathrm{Rep}^{\mathrm{f}}(\pi(X/R)_s).$$

Lemma 6.7. *Let A be a complete discrete valuation ring. Let X be a proper smooth scheme over A. The restriction functor*

$$i : \mathrm{MIC}^{\mathrm{se}}(X/A)_s \to \mathrm{MIC}^{\circ}(X_s/k_s)$$

is an equivalence. Consequently, $\pi(X/R)_s \cong \pi(X_s/k_s)$.

Proof. According to Lemma 4.9, each object of $\mathrm{MIC}^{\mathrm{se}}(X/A)$ is a quotient of an object of $\mathrm{MIC}^{\circ}(X/A)$. This implies that the restriction functor

$$i : \mathrm{MIC}^{\mathrm{se}}(X/A)_s \to \mathrm{MIC}^{\circ}(X_s/k_s)$$

yields an equivalence between two categories. □

Thanks to Lemma 6.7, the map $\lambda^i_{k_s}$

$$\lambda^i_{k_s} : \mathrm{H}^i(\pi(X/A)_s, E) \to \mathrm{H}^i(\pi(X_s/k_s), E)$$

is an isomorphism for all $i \geq 0$.

Corollary 6.8. *Let X be a projective curve with genus $g \geq 1$. Let $\mathscr{M}$ be an object in* $\mathrm{MIC}^{\mathrm{se}}(X/A)$*. Denote* $M := \eta^*(\mathscr{M})$*. Then the map*

$$\delta^i \otimes k_s : \mathrm{H}^i(\pi(X/A), M) \otimes k_s \longrightarrow \mathrm{H}^i_{\mathrm{dR}}(X/A, \mathscr{M}) \otimes k_s \tag{22}$$

is injective for all $i \geq 0$.

Proof. According to [Ja87, 4.18 - p.64], the map

$$\mathrm{H}^i(\pi(X/A),M)\otimes k_s \hookrightarrow \mathrm{H}^i(\pi(X/A)_s,M_s),$$

for $i\geq 0$, is injective. Consider the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{H}^i(\pi(X/A),M)\otimes k_s & \hookrightarrow & \mathrm{H}^i(\pi(X/A)_s,M_s) \\
\Big\downarrow{\scriptstyle \delta^i\otimes k_s} & & \Big\downarrow{\scriptstyle \lambda^i_s} \\
 & & \mathrm{H}^i(\pi(X_s/k_s),M_s) \\
 & & \Big\downarrow{\scriptstyle \delta^i(X_s/k_s)} \\
\mathrm{H}^i_{\mathrm{dR}}(X/A,\mathscr{M})\otimes k_s & \longrightarrow & \mathrm{H}^i_{\mathrm{dR}}(X_s/k_s,\mathscr{M}_s).
\end{array}$$

Since λ^i_s is bijective and so is $\delta^i(X_s/k_s)$, according to Lemma 6.3, we conclude that the map $\delta^i\otimes k_s$ is injective. □

6.4. **Comparison theorem.** We are now ready to prove:

Theorem 6.9. *Let X be a projective curve with genus $g\geq 1$ over complete discrete valuation ring A. Let $\mathcal{V}$ be an object in $\mathrm{MIC}^{\mathrm{geom}}(X/A)$. Denote V as the representation corresponding to $\mathcal{V}$. Then the map*

$$\delta^i:\mathrm{H}^i(\pi(X/A),V))\to \mathrm{H}^i_{\mathrm{dR}}(X/A,\mathcal{V})$$

is bijective for all $i\geq 0$.

Proof. The case of $i\geq 3$. Thanks to Corollary 6.6 and the flat base change theorem for de Rham cohomology [ABC20, 23.5], we obtain

$$\mathrm{H}^i(\pi(X/A),V)\otimes K\cong \mathrm{H}^i_{\mathrm{dR}}(X/A,\mathcal{V})\otimes K\cong \mathrm{H}^i_{\mathrm{dR}}(X_K/K,\mathcal{V}_K)=0.$$

For any closed point $s\in\mathrm{Spec}(A)$, using Corollary 6.8, we obtain

$$\mathrm{H}^i(\pi(X/A),V)\otimes k_s\hookrightarrow \mathrm{H}^i_{\mathrm{dR}}(X_s/k_s,\mathcal{V}_s)=0.$$

Therefore,

$$\mathrm{H}^i(\pi(X/A),V)\cong \mathrm{H}^i_{\mathrm{dR}}(X/A,\mathcal{V})=0.$$

The case of $i=2$. The proof is carried out in several steps.

1. Assume that V corresponds to an inflated connection $\mathrm{inf}(\mathcal{V})$. It then follows from Lemma 6.2, Corollary 6.6, and Corollary 6.8 that

(1) δ^2 is injective
(2) $\delta^2\otimes K$ is bijective
(3) $\delta^2\otimes k_s$ is injective for the closed point s of $\mathrm{Spec}(A)$.

Hence δ^2 is an isomorphism.

2. Assume that U corresponds to a subconnection $\mathscr{U}$ of $\mathrm{inf}(\mathcal{V})$. We have the short exact sequence:

$$0\to U\to V\to Q\to 0 \tag{23}$$

where $Q:=M/V$ is quotient representation corresponding to the connection $\mathscr{Q}:=\mathrm{inf}(\mathcal{V})/\mathscr{U}$. Thus we also have an exact sequence

$$0\to\mathscr{U}\to\mathrm{inf}(\mathcal{V})\to\mathscr{Q}\to 0.$$

The long exact sequences yield the following commutative diagram:

$$\begin{array}{ccccccccc}
\mathrm{H}^1(\pi(X/A),Q) & \longrightarrow & \mathrm{H}^2(\pi(X/A),U) & \longrightarrow & \mathrm{H}^2(\pi(X/A),V) & \longrightarrow & \mathrm{H}^2(\pi(X/A),Q) & \longrightarrow & 0 \\
\downarrow \cong & & \downarrow \delta^2(U) & & \downarrow \cong & & \downarrow & & \\
\mathrm{H}^1_{\mathrm{dR}}(X/A,\mathscr{Q}) & \longrightarrow & \mathrm{H}^2_{\mathrm{dR}}(X/A,\mathscr{U}) & \longrightarrow & \mathrm{H}^2_{\mathrm{dR}}(X/A,\inf(\mathscr{V})) & \longrightarrow & \mathrm{H}^2_{\mathrm{dR}}(X/A,\mathscr{Q}) & \longrightarrow & 0.
\end{array}$$

The Four-Lemma implies that $\delta^2(U)$ is bijective.

3. Finally, we prove the claim for the quotient U/T, where T,U correspond to the sub-connections $\mathscr{T}\subset\mathscr{U}\subset\inf(\mathscr{V})$. Similarly as above, we have a commutative diagram with exact lines:

$$\begin{array}{ccccccc}
\mathrm{H}^2(\pi(X/A),T) & \longrightarrow & \mathrm{H}^2(\pi(X/A),U) & \longrightarrow & \mathrm{H}^2(\pi(X/A),U/T) & \longrightarrow & 0 \\
\downarrow \cong & & \downarrow \cong & & \downarrow \delta^2(U/T) & & \\
\mathrm{H}^2_{\mathrm{dR}}(X/A,(\mathscr{T})) & \longrightarrow & \mathrm{H}^2_{\mathrm{dR}}(X/A,\mathscr{U}) & \longrightarrow & \mathrm{H}^2_{\mathrm{dR}}(X/A,\mathscr{U}/\mathscr{T}) & \longrightarrow & 0.
\end{array}$$

This diagram implies that $\delta^2(U/T)$ is bijective. □

7. Comparison between cohomologies of $\pi^{\mathrm{geom}}(X/R)$ and $\pi(X/R)$

Our next aim is to compare the cohomology of $\pi^{\mathrm{geom}}(X/R)$ with that of $\pi(X/R)$ (Corollary 7.9). Our key ingredient is the universal extension theorem.

7.1. **The universal extension theorem.**

By the nature of $\pi^{\mathrm{geom}}(X/R)$, we will need the following result, which is an adaption of [EH06, Thm. 4.2].

Theorem 7.1 (*Universal extension*)**.** *Let* $(\mathscr{V},\nabla)$ *be an object in* $\mathrm{MIC}^\circ(X/k)$*, then there exists an extension in* $\mathrm{MIC}^\circ(X/k)$*:*

$$0\longrightarrow\mathscr{V}\longrightarrow\mathscr{W}\longrightarrow f^*(\mathrm{R}^1_{\mathrm{dR}}f_*(\mathscr{V},\nabla))\longrightarrow 0$$

with the property that the connecting morphism in the long exact sequence of de Rham cohomology on X/R*:*

$$\mathrm{H}^0_{\mathrm{dR}}(X/R,\inf(f^*(\mathrm{R}^1_{\mathrm{dR}}f_*(\mathscr{V},\nabla)))=\mathrm{H}^1_{\mathrm{dR}}(X,(\mathscr{V},\nabla_{/R}))\xrightarrow{\text{connecting}}\mathrm{H}^1_{\mathrm{dR}}(X/R,(\mathscr{V},\nabla_{/R}))$$

is the identity map.

Proof. Let $\mathscr{Z}=f^*(\mathrm{R}^1f_*(\mathscr{V},\nabla))$ (see Subsection 4.2). Then

$$\inf(\mathscr{Z})=\mathscr{O}_X\otimes\mathrm{H}^1_{\mathrm{dR}}(X/R,(\mathscr{V},\nabla_{/R})).$$

Let

$$\mathscr{W}=\mathscr{V}\otimes\mathscr{Z}^\vee$$

as objects in $\mathrm{MIC}^{\circ}(X/k)$. Then we have isomorphisms in $\mathrm{MIC}^{\mathrm{se}}(X/R)$ of the cohomologies of Gauss-Manin connections:

$$\begin{aligned}
\mathrm{H}^1_{\mathrm{dR}}(X/R,\inf(\mathscr{W})) &\cong \mathrm{Ext}^1_{\mathrm{MIC}(X/R)}(\mathscr{O}_X,(\mathscr{V},\nabla_{/R})\otimes \inf\mathscr{Z}^{\vee})\\
&\cong \mathrm{Ext}^1_{\mathrm{MIC}(X/R)}(\inf(\mathscr{Z}),(\mathscr{V},\nabla_{/R}))\\
&\cong \mathrm{H}^1_{\mathrm{dR}}(X/R,(\mathscr{V},\nabla_{/R}))^{\vee}\otimes \mathrm{Ext}^1_{\mathrm{MIC}(X/R)}(\mathscr{O}_X,(\mathscr{V},\nabla_{/R}))\\
&\cong \mathrm{H}^1_{\mathrm{dR}}(X/R,(\mathscr{V},\nabla_{/R}))^{\vee}\otimes \mathrm{H}^1_{\mathrm{dR}}(X/R,(\mathscr{V},\nabla_{/R}))\\
&\cong \mathrm{End}_R(\mathrm{H}^1_{\mathrm{dR}}(X/R,(\mathscr{V},\nabla_{/R}))).
\end{aligned}$$

Let ε be the element in $\mathrm{Ext}^1_{\mathrm{MIC}(X/R)}(\inf(\mathscr{Z}),(\mathscr{V},\nabla_{/R}))$, the image of which the identity map in $\mathrm{End}_R(\mathrm{H}^1_{\mathrm{dR}}(X/R,(\mathscr{V},\nabla_{/R})))$. As the identity map is killed by the Gauss-Manin connections, so is ε.

Now consider the exact sequence of complexes:

$$0\longrightarrow f^*\Omega^1_{R/k}\otimes_R(\Omega^{\bullet-1}_{X/R}\otimes\mathscr{W})\longrightarrow \Omega^{\bullet}_{X/k}\otimes\mathscr{W}\longrightarrow \Omega^{\bullet}_{X/R}\otimes\mathscr{W}\longrightarrow 0.$$

In our situation, our base scheme is an affine scheme, so we get a the long exact sequence (10):

$$\cdots\longrightarrow \mathrm{H}^1_{\mathrm{dR}}(X/k,\mathscr{W})\longrightarrow \mathrm{H}^1_{\mathrm{dR}}(X/R,\inf(\mathscr{W}))\longrightarrow \Omega^1_{R/k}\otimes_R \mathrm{H}^1_{\mathrm{dR}}(X/R,\inf(\mathscr{W}))\longrightarrow\cdots$$

Now the homomorphism:

$$\delta_1:\mathrm{H}^1_{\mathrm{dR}}(X/R,\inf(\mathscr{W}))\longrightarrow \Omega^1_{R/k}\otimes_R \mathrm{H}^1_{\mathrm{dR}}(X/R,\inf(\mathscr{W}))$$

is the Gauss-Manin connection. Hence, $\varepsilon\in\mathrm{Ker}\delta_1$ and thus it is lifted to $\tilde{\varepsilon}\in\mathrm{H}^1_{\mathrm{dR}}(X/k,(\mathscr{W},\nabla_{\mathscr{W}}))$ by the long exact sequence. Consequently, there exists an extension of connections in $\mathrm{MIC}^{\circ}(X/k)$:

$$\tilde{\varepsilon}:\quad 0\longrightarrow(\mathscr{W},\nabla_{\mathscr{W}})\longrightarrow(\mathscr{T}',\nabla_{\mathscr{T}'})\longrightarrow(\mathscr{O}_X,d)\longrightarrow 0.$$

Notice that the inflation of this sequence to $\mathrm{MIC}^{\mathrm{se}}(X/R)$ is ε. Hence, the induced connecting map is the identity map by construction of ε. □

With Theorem 7.1, we have a statement for cohomology comparison.

Corollary 7.2. *Let $(\mathscr{V},\nabla)$ be an object in $\mathrm{MIC}^{\circ}(X/k)$, and let $V=\eta^*(\mathscr{V})$ be the corresponding representation of $\pi(X/R)$. Then we have:*

$$\mathrm{H}^1(\pi^{\mathrm{geom}}(X/R),V)\cong \mathrm{H}^1(\pi(X/R),V).$$

Proof. The category $\mathrm{MIC}^{\mathrm{geom}}(X/R)\cong\mathrm{Rep}^{\mathrm{f}}(\pi^{\mathrm{geom}}(X/R))$ is a full subcategory of $\mathrm{MIC}^{\mathrm{se}}(X/R)$, so the homomorphism

$$\mathrm{H}^1(\pi^{\mathrm{geom}}(X/R),V)\longrightarrow \mathrm{H}^1(\pi(X/R),V)$$

is injective.

We prove the surjectivity, which, through Tannakian duality, amounts to saying that each exension

$$e:\quad 0\longrightarrow(\mathscr{V},\nabla_{/R})\longrightarrow(\mathscr{V}',\nabla_{\mathscr{V}'}))\longrightarrow\mathscr{O}_X\longrightarrow 0$$

is in $\mathrm{MIC}^{\mathrm{geom}}(X/R)$, meaning $(\mathscr{V}',\nabla_{\mathscr{V}'})\in\mathrm{Obj}(\mathrm{MIC}^{\mathrm{geom}}(X/R))$.

Let $\varepsilon \in \mathrm{H}^1(\pi(X/R), V) \cong \mathrm{H}^1_{\mathrm{dR}}(X/R, (\mathcal{V}, \nabla_{/R}))$ be the universal extension of Theorem 7.1. Then $e \in \mathrm{H}^1_{\mathrm{dR}}(X/R, (\mathcal{V}, \nabla_{/R}))$, seen as a map in $\mathrm{MIC}^{\mathrm{se}}(X/R)$

$$e : \mathcal{O}_X \longrightarrow \mathcal{O}_X \otimes \mathrm{H}^1_{\mathrm{dR}}(X/R, (\mathcal{V}, \nabla_{/R}),$$

fits in the commuative diagram

$$\begin{array}{cccccccccc}
e: & 0 & \longrightarrow & (\mathcal{V}, \nabla_{/R}) & \longrightarrow & (\mathcal{V}', \nabla') & \longrightarrow & (\mathcal{O}_X, d) & \longrightarrow & 0 \\
 & & & {\scriptstyle =}\downarrow & & \downarrow & & {\scriptstyle e}\downarrow & & \\
\tilde{\varepsilon}: & 0 & \longrightarrow & (\mathcal{V}, \nabla_{/R}) & \longrightarrow & (\mathcal{W}, \nabla_{/R}) & \longrightarrow & (\mathcal{Z}, \nabla_{/R}) & \longrightarrow & 0.
\end{array}$$

Applying long exact sequence of $\mathrm{H}^0_{\mathrm{dR}}(X/R, -)$, we obtain:

$$\begin{array}{cccc}
\longrightarrow & \mathrm{H}^1_{\mathrm{dR}}(X/R, \mathcal{O}_X) & \xrightarrow{\ d\ } & \mathrm{H}^1_{\mathrm{dR}}(X/R, (\mathcal{V}, \nabla_{/R})) \\
 & {\scriptstyle \epsilon}\downarrow & & {\scriptstyle \mathrm{id}}\downarrow \\
\longrightarrow & \mathrm{H}^1_{\mathrm{dR}}(X/R, (\mathcal{V}, \nabla_{/R})) & \xrightarrow{\ \mathrm{id}\ } & \mathrm{H}^1_{\mathrm{dR}}(X/R, (\mathcal{V}, \nabla_{/R})),
\end{array}$$

Thus, $(\mathcal{V}', \nabla)$ is a subconnection of $(\mathcal{W}, \nabla_{/R})$, hence it is an object of $\mathrm{MIC}^{\mathrm{geom}}(X/R)$. This completes the proof. □

7.2. **Comparison of cohomologies.** The objective of this subsection is to extend the result of Corollary 7.2. We prove that the group cohomology $\mathrm{H}^\bullet(\pi^{\mathrm{geom}}(X/A), V)$ coincides with the group cohomology $\mathrm{H}^\bullet(\pi(X/A), V)$ where A is a complete discrete valuation ring (see Corollary 7.9). To do so, we establish the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{H}^i(\pi^{\mathrm{geom}}(X/A), V) & \longrightarrow & \mathrm{H}^i(\pi(X/A), V) \\
\uparrow & & \downarrow {\scriptstyle \cong}\ \text{Theorem 6.9} \\
\mathrm{H}^i(\Pi(\mathfrak{X}/k)^{\Delta}, V) & & \mathrm{H}^i_{\mathrm{dR}}(X/A, (\mathcal{V}, \nabla_{/A})) \\
\uparrow & & \downarrow {\scriptstyle \cong}\ \text{Theorem 4.10} \\
{\scriptstyle \alpha_3} & & \mathrm{H}^i_{\mathrm{dR}}(\mathfrak{X}/\mathfrak{S}, (\widehat{\mathcal{V}}, \widehat{\nabla}_{/A})) \\
\uparrow & & \uparrow {\scriptstyle \alpha_1} \\
\mathrm{H}^i(\pi(\mathfrak{X}/k), V_s) \otimes A & \xrightarrow{\ \alpha_2\ } & \mathrm{H}^i_{\mathrm{dR}}(\mathfrak{X}/k, (\widehat{\mathcal{V}}, \widehat{\nabla})) \otimes_k A,
\end{array} \tag{24}$$

and then we show that the arrows α_1, α_2, and α_3 in the above diagram are bijective.

Lemma 7.3. *Let X be a projective curve with genus $g \geq 1$ over a complete discrete valuation ring A. Let $\mathfrak{X}$ be the completion of X with respect to the adic topology on A. Denote $\mathfrak{S} := \mathrm{Spf}(A)$. Let $(\mathcal{V}, \widehat{\nabla})$ be an object in $\mathrm{MIC}^{\circ}(\mathfrak{X}/k)$. Then*

$$\alpha_1 : \mathrm{H}^i_{\mathrm{dR}}(\mathfrak{X}/k, (\mathcal{V}, \widehat{\nabla})) \otimes_k A \longrightarrow \mathrm{H}^i_{\mathrm{dR}}(\mathfrak{X}/\mathfrak{S}, (\mathcal{V}, \widehat{\nabla}_{/A}))$$

is bijective for $i \geq 0$.

Proof. Using the Leray spectral sequence (see Theorem 4.12), we have

$$E^2_{p,q} = \mathrm{H}^p_{\mathrm{dR}}(\mathfrak{S}/k, \mathrm{H}^q_{\mathrm{dR}}(\mathfrak{X}/\mathfrak{S}, (\mathcal{V}, \widehat{\nabla}_{/A})) \Rightarrow \mathrm{H}^{p+q}_{\mathrm{dR}}(\mathfrak{X}/k, (\mathcal{V}, \widehat{\nabla})). \tag{25}$$

Since $\pi(\mathfrak{S}/k)$ is a trivial group, the above spectral sequence is degenerate, i.e., we have

$$\mathrm{H}^0_{\mathrm{dR}}(\mathfrak{S}/k, \mathrm{H}^q_{\mathrm{dR}}(\mathfrak{X}/\mathfrak{S}, (\mathcal{V}, \widehat{\nabla}_{/A})) \cong \mathrm{H}^q_{\mathrm{dR}}(\mathfrak{X}/k, (\mathcal{V}, \widehat{\nabla})).$$

Using the fact that $\pi(\mathfrak{S}/k)$ is trivial group again, the Gauss-Manin connection equipped on the de Rham cohomology $\mathrm{H}^q_{\mathrm{dR}}(\mathfrak{X}/\mathfrak{S},(\mathcal{V},\widehat{\nabla}_{/A}))$ is trivial. It means that this connection has enough horizontal sections, that is, we obtain

$$\mathrm{H}^q_{\mathrm{dR}}(\mathfrak{X}/k,(\mathcal{V},\widehat{\nabla}))\otimes_k A \cong \mathrm{H}^q_{\mathrm{dR}}(\mathfrak{X}/\mathfrak{S},(\mathcal{V},\widehat{\nabla}_{/A})).$$

□

Remark 7.4. We cannot use flat base change to prove the lemma above, since it is unclear whether $(\mathcal{V},\widehat{\nabla})\otimes A$ coincides with $(\mathcal{V},\nabla_{/A})$.

Lemma 7.5. *Let X be a proper smooth separated noetherian scheme over a complete discrete valuation A. Let $\mathfrak{X}$ be the completion of X with respect to the adic topology on A. The restriction functor from category* $\mathrm{MIC}^\circ(\mathfrak{X}/k)$ *to* $\mathrm{MIC}^\circ(X_s/k)$ *is equivalence.*

Proof. To prove this lemma, we reduce the problem to a local problem, and then we can use [DH18, Proposition 5.2.1]. □

Lemma 7.6. *Let X be a proper smooth separated noetherian scheme over a complete discrete valuation A. Let $\mathfrak{X}$ be the completion of X with respect to the adic topology on A. Let V be an object in* $\mathrm{Rep}^{\mathrm{f}}(\pi(\mathfrak{X}/k))$. *Then*

$$\mathrm{H}^i(\pi(\mathfrak{X}/k),V)\cong \mathrm{H}^i(\pi(X_s/k),V_s)$$

for $i\geq 0$.

Proof. This is straightforward from Lemma 7.5. □

To prove that the map α_2 in the Diagram (24) is bijective for $i\geq 0$, we need the following lemma.

Lemma 7.7. *Let X be a projective curve with genus $g\geq 1$ over a complete discrete valuation ring A. Let $\mathfrak{X}$ be the completion of X with respect to the adic topology on A. Let $(\mathcal{V},\widehat{\nabla})$ be an object in* $\mathrm{MIC}^\circ(\mathfrak{X}/k)$. *Then*

$$\mathrm{H}^i(\pi(\mathfrak{X}/k),V)\cong \mathrm{H}^i_{\mathrm{dR}}(\mathfrak{X}/k,(\mathcal{V},\widehat{\nabla}))$$

for $i\geq 0$.

Proof. We prove the latter equality by comparing two Leray spectral sequences.

Denote $\mathfrak{S}:=\mathrm{Spf}(A)$. We have the following diagram:

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{f} & \mathfrak{S} \\ & \searrow & \downarrow g \\ & & k \end{array}.$$

Since f is proper, we can define the functor

$$\mathrm{R}^0_{\mathrm{ind}}f_*:\mathrm{Ind}(\mathrm{MIC}^\circ(\mathfrak{X}/k))\to \mathrm{Ind}(\mathrm{MIC}^\circ(\mathfrak{S}/k))$$
$$(\mathcal{V},\widehat{\nabla})\mapsto f_*(\mathcal{V}^{\widehat{\nabla}}).$$

Consider the following diagram:

$$\begin{array}{ccc} \mathrm{Ind}(\mathrm{MIC}^\circ(\mathfrak{X}/k)) & \xrightarrow{\mathrm{R}^0_{\mathrm{ind}}f_*} & \mathrm{Ind}(\mathrm{MIC}^\circ(\mathfrak{S}/k)) \\ & \searrow & \downarrow \mathrm{R}^0_{\mathrm{ind}}g_* \\ & & \mathrm{Vect}_k \end{array}.$$

We can see that

$$\mathrm{R}^0_{\mathrm{ind}}(f\circ g)_* = \mathrm{R}^0_{\mathrm{ind}} f_* \circ \mathrm{R}^0_{\mathrm{ind}} g_*.$$

Let $\mathscr{I}$ be an injective object in $\mathrm{Ind}(\mathrm{MIC}^\circ(\mathfrak{X}/k))$. Since $\pi(\mathfrak{S}/k)$ is trivial, the object $\mathrm{R}^0_{\mathrm{ind}}(\mathscr{I})$ is $\mathrm{R}^0_{\mathrm{ind}} g_*$-acylic. Then the Grothendieck spectral sequence (see A.4.3) implies that

$$E^2_{p,q} = \mathrm{R}^p_{\mathrm{ind}} g_* \circ \mathrm{R}^q_{\mathrm{ind}} f_* \Longrightarrow \mathrm{R}^{p+q}_{\mathrm{ind}}(f\circ g)_*. \tag{26}$$

Using the triviality of $\pi(\mathfrak{S}/k)$ again, the above spectral sequence is degenerate; i.e, we obtain

$$\mathrm{R}^0_{\mathrm{ind}} g_* \circ (\mathrm{R}^q_{\mathrm{ind}} f_*(\mathcal{V},\widehat{\nabla})) \cong \mathrm{R}^q_{\mathrm{ind}}(f\circ g)_*(\mathcal{V},\widehat{\nabla})$$

where $(\mathcal{V},\widehat{\nabla}) \in \mathrm{Obj}(\mathrm{MIC}^\circ(\mathfrak{X}/k))$.

Since we have a natural map from $\mathrm{R}^i_{\mathrm{ind}}(f\circ g)_*(\mathcal{V},\widehat{\nabla})$ to $\mathrm{R}^i_{\mathrm{dR}}(f\circ g)_*(\mathcal{V},\widehat{\nabla})$, we can define the morphism from Spectral sequence (26) to Spectral sequence (25). Furthermore, Theorem 6.9 tells us that

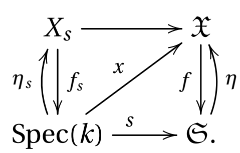
$$\mathrm{R}^q_{\mathrm{ind}} f_*(\mathcal{V},\widehat{\nabla}) \cong \mathrm{R}^q_{\mathrm{dR}} f_*(\mathcal{V},\widehat{\nabla}).$$

This implies that

$$\mathrm{H}^i(\pi(\mathfrak{X}/k), V) \cong \mathrm{H}^i_{\mathrm{dR}}(\mathfrak{X}/k,(\mathcal{V},\widehat{\nabla}))$$

where V is a representation corresponding to $(\mathcal{V},\widehat{\nabla})$ and $i \geq 0$. □

Denote $x = \eta(s)$. We have the following diagram:



Lemma 7.8. *Let X be a proper smooth separated noetherian scheme over a complete discrete valuation ring A. Let $\mathfrak{X}$ be the completion of X with respect to the adic topology on A. Denote $\mathfrak{S} := \mathrm{Spf}(A)$. Let V be an object in $\mathrm{Rep}^{\mathrm{f}}(\Pi(X/k)^\Delta)$. Then*

$$\alpha_3 : \mathrm{H}^i(\pi(\mathfrak{X}/k), V_s) \otimes A \longrightarrow \mathrm{H}^i(\Pi(X/k)^\Delta, V)$$

is bijective for $i \geq 0$.

Proof. Since the restriction functor:

$$\iota : \mathrm{MIC}^\circ(\mathfrak{X}/k) \longrightarrow \mathrm{MIC}^\circ(X_s/k)$$

is equivalence (see Lemma 7.5), we obtain

$$x^*(V_s) = \eta_s^*(V_s).$$

This implies that

$$\eta^*(V) = x^*(V_s) \otimes_k A.$$

Thus, the Hopf algebra $\mathcal{O}(\Pi(\mathfrak{X}/k)^\Delta)$ is given by

$$\mathcal{O}(\Pi(\mathfrak{X}/k)^\Delta) = \mathcal{O}(\pi(\mathfrak{X}/k)) \otimes_k A.$$

By applying [Ja87, 2.10-(7), p.34], we obtain

$$\mathrm{Hom}_{\pi(\mathfrak{X}/k)}(k, V_s) \otimes A \cong \mathrm{Hom}_{\pi(\mathfrak{X}/k)\otimes A}(k\otimes A, V_s \otimes A) = \mathrm{Hom}_{\Pi(\mathfrak{X}/k)^\Delta}(A, V),$$

and the result follows. □

We are now ready to prove the main result of this section.

Corollary 7.9. *Let X be a projective curve with genus $g \geq 1$ over a complete discrete valuation ring A. Let $(\mathcal{V}, \nabla)$ be an object in* $\mathrm{MIC}^\circ(X/k)$. *Denote $V := \eta^*(\mathcal{V}, \nabla_{/A})$. Then*

$$\mathrm{H}^i(\pi^{\mathrm{geom}}(X/A), V) \to \mathrm{H}^i(\pi(X/A), V) \overset{\text{Theorem6.9}}{\longrightarrow} \mathrm{H}^i_{\mathrm{dR}}(X/A, (\mathcal{V}, \nabla_{/A}))$$

is bijective for $i \geq 0$.

Proof. Thanks to Lemma 7.3, Lemma 7.7, and Lemma 7.8, the Diagram (24) becomes

$$\begin{array}{ccc}
\mathrm{H}^i(\pi^{\mathrm{geom}}(X/A), V) & \longrightarrow & \mathrm{H}^i(\pi(X/A), V) \\
\uparrow & & \text{Theorem 6.9} \downarrow \cong \\
\mathrm{H}^i(\Pi(\mathfrak{X}/k)^\Delta, V) & & \mathrm{H}^i_{\mathrm{dR}}(X/A, (\mathcal{V}, \nabla_{/A})) \\
\uparrow & & \text{Theorem 4.10} \downarrow \cong \\
\alpha_3 \; \cong & & \mathrm{H}^i_{\mathrm{dR}}(\mathfrak{X}/\mathfrak{S}, (\widehat{\mathcal{V}}, \widehat{\nabla}_{/A})) \\
\uparrow & & \alpha_1 \uparrow \cong \\
\mathrm{H}^i(\pi(\mathfrak{X}/k), V_s) \otimes A & \underset{\cong}{\overset{\alpha_2}{\longrightarrow}} & \mathrm{H}^i_{\mathrm{dR}}(\mathfrak{X}/k, (\widehat{\mathcal{V}}, \widehat{\nabla})) \otimes_k A.
\end{array}$$

It remains to prove that the map from $\mathrm{H}^i(\Pi(\mathfrak{X}/k)^\Delta, V)$ to $\mathrm{H}^i(\pi^{\mathrm{geom}}(X/A), V)$ is bijective for $i \geq 0$. Since $\pi(\mathfrak{S}/k)$ is trivial, the group scheme $\Pi(\mathfrak{S}/k)^\Delta$ is also trivial. Thus, the relative geometric group $\pi^{\mathrm{geom}}(\mathfrak{X}/\mathfrak{S})$ coincides with the discrete groupoid scheme $\Pi(\mathfrak{X}/k)^\Delta$.

On the other hand, by Lemma 4.9, we have

$$\mathrm{H}^i(\pi^{\mathrm{geom}}(\mathfrak{X}/\mathfrak{S}), V) \cong \mathrm{H}^i(\pi^{\mathrm{geom}}(X/A), V),$$

and the result follows ☐

Appendix A. Affine group schemes and groupoid schemes

A.1. **Affine group schemes and representations over a Dedekind domain.** Our reference is [DH18] and [De90], see also [Ja87, Sa72].

Let k be a field of characteristic 0. Let R be a k-algebra, which is a Dedekind ring.

A.1.1. *Representations (G-modules).* Let G be a flat affine group scheme over R. We denote by $\mathrm{Rep}_R(G)$ the category of finite G-representations in R-modules, meaning finite R-modules equipped with a (rational) action of G. Since G is flat, this is an abelian category, further, it is a tensor category. The full subcategory consisting of R-projective representations will be denoted by $\mathrm{Rep}^\circ_R(G)$. This subcategory is an R-linear, additive, rigid tensor category. Moreover, it is a subcategory of definition, i.e., each object of $\mathrm{Rep}_R(G)$ is a quotient of an object in $\mathrm{Rep}^\circ_R(G)$.

As R is a Dedekind ring, torsion free, flat and projective finite R-modules are the same. We say that $M \subset N$ is a special subobject in $\mathrm{Rep}_R(G)$ if the quotient N/M is R-flat. A special subquotient is *a special sub of a quotient*, or equivalently, *a quotient of a special sub.* This can be seen from the following diagram, where the left square is a push-out, or equivalently, a pull-back:

$$\begin{array}{ccccc} Q & \hookrightarrow & P & \longrightarrow & P/Q \\ \downarrow & \square & \downarrow & & \downarrow \simeq \\ M & \hookrightarrow & N & \longrightarrow & M/N. \end{array}$$

Thus, if $N/M \cong P/Q$ is R-flat then M is a special subquotient of P: it is a quotient of a special sub Q and a special sub of N.

A.1.2. *Morphisms of flat affine group schemes.* We study flat affine group schemes and morphisms between them in this subsection. Let $f: G \to G'$ be a homomorphism of flat affine group schemes over R. We say that f is surjective or a *quotient map* if it is faithfully flat.

Theorem A.1 (Theorem 4.1.2 [DH18])**.** *Let $f: G \longrightarrow G'$ be a homomorphism of affine flat groups over R, and ω_f° be the corresponding functor* $\mathrm{Rep}_R^\circ(G') \longrightarrow \mathrm{Rep}_R^\circ(G)$.

(1) *f is faithfully flat if and only if $\omega_f{}^\circ: \mathrm{Rep}_R^\circ(G') \longrightarrow \mathrm{Rep}_R^\circ(G)$ is fully faithful and its image is closed under taking subobjects.*
(2) *f is a closed immersion if and only if every object of* $\mathrm{Rep}_R^\circ(G)$ *is isomorphic to a special subquotient of an object of the form $\omega_f(X')$, for $X' \in \mathrm{Obj}(\mathrm{Rep}_R^\circ(G'))$.*

A.1.3. *Exact sequence of flat affine group schemes.* Let $G \longrightarrow A$ be a homomorphism of affine group schemes over R. The kernel of this map is defined to be

$$L := G \times_A \mathrm{Spec}(R).$$

This is a closed subgroup of G. Let I_A be the kernel of counit $\epsilon: \mathcal{O}(A) \longrightarrow R$, i.e., the augmentation ideal of $\mathcal{O}(A)$, and let $I_A\mathcal{O}(G)$ be the ideal generated by the image of I_A in $\mathcal{O}(G)$. Then coordinate ring of L is isomorphic to $\mathcal{O}(G)/I_A\mathcal{O}(G)$. The sequence

$$1 \longrightarrow L \xrightarrow{q} G \xrightarrow{p} A \longrightarrow 1$$

is said to be *exact* if p is a quotient map with kernel L. We will provide a criterion for the exactness in terms of the functors

$$\mathrm{Rep}_R^\circ(A) \xrightarrow{p^*} \mathrm{Rep}_R^\circ(G) \xrightarrow{q^*} \mathrm{Rep}_R^\circ(L). \tag{27}$$

Theorem A.2 (Theorem 4.2.2 [DH18])**.** *Let us be given a sequence of homomorphisms*

$$L \xrightarrow{q} G \xrightarrow{p} A$$

with q a closed immersion and p faithfully flat. Then this sequence is exact if and only if the following conditions are fulfilled:

(a) *For an object V in* $\mathrm{Rep}_R^\circ(G)$, *$q^*(V)$ in* $\mathrm{Rep}_R^\circ(L)$ *is trivial if and only if $V \cong p^*U$ for some object U in* $\mathrm{Rep}_R^\circ(A)$.
(b) *Let W_0 be the maximal trivial subobject of $q^*(V)$ in* $\mathrm{Rep}_R^\circ(L)$. *Then there exists* $V_0 \subset V \in \mathrm{Obj}(\mathrm{Rep}_R^\circ(G))$, *such that $q^*(V_0) \cong W_0$.*
(c) *Any object W in* $\mathrm{Rep}_R^\circ(L)$ *is a quotient in (hence, by taking duals, a subobject of) $q^*(V)$ for some* $V \in \mathrm{Obj}(\mathrm{Rep}_R^\circ(G))$.

A.2. **Groupoid schemes.** Our reference [De90], Section 3.

Let S be a k-scheme. An *affine k-groupoid scheme acting on S* is an $S\times_k S$-affine scheme G (with the structure maps being $s,t:G\to S$, which are called the source and the target maps), together with the following data:

(i) a map $m:G\,{}_s\!\times_t G\to G$, called the product of G, satisfying the following associativity property:
$$m(m\,{}_s\!\times_t \mathrm{id}_G)=m(\mathrm{id}_G\,{}_s\!\times_t m)$$
(ii) a map $\varepsilon:S\to G$, called the unit element map, satisfying:
$$m(\varepsilon\,{}_s\!\times_t \mathrm{id}_G)=m(\mathrm{id}_G\,{}_s\!\times_t m)=\mathrm{id}_G$$
(iii) a map $\iota:G\to G$, called the inverse map, satisfying:
$$\iota\circ s=t;\quad \iota\circ t=s$$
$$m(\iota\,{}_s\!\times_t \mathrm{id}_G)=\varepsilon\circ s,\quad m(\mathrm{id}_G\,{}_s\!\times_t \iota)=\varepsilon\circ t,$$

where ${}_s\!\times_t$ denotes the fiber product over S with respect to the maps s and t.

An affine k-groupoid scheme G acting on S is called *flat* if the source and target morphisms $s,t:G\longrightarrow S$ are flat. The groupoid scheme G said to be *acting transitively on S* if for any pair of morphism $(a,b):T\times U\to S$, if there is a faithfully flat quasi-compact map $\phi:W\to T\times U$ such that the set
$$\mathrm{Mor}_{S\times S}(W,G)\neq\emptyset.$$
This implies that $(s,t):G\to S\times S$ is a faithfully flat map.

Definition A.3. Let G be an affine k-groupoid schemes acting on S. We call H a *subgroupoid scheme* if H is a closed subscheme of G such that the morphisms $(m|_H,\varepsilon|_H,\iota|_H)$ makes H into k-groupoid scheme acting on S. We call H a *discrete subgroupoid scheme* if H is a closed subscheme of the diagonal group scheme G^Δ. A subgroupoid scheme H of G is said to be *normal* in G if $H(T)$ is a normal subgroupoid of $G(T)$ for every k-scheme T.

A.2.1. *The diagonal group scheme.* Define the diagonal group scheme G^Δ of G as the pull-back of G along the diagonal map $\Delta:S\longrightarrow S\times S$.
$$\begin{array}{ccc} G^\Delta & \longrightarrow & G\\ \downarrow & & \downarrow\\ S & \longrightarrow & S\times_k S.\end{array}$$

A.2.2. *Representations (G-modules).* Let V be a quasi-coherent sheaf on S. A *representation* of G in V (a *G-module*) is an operation ρ, that assigns to each k-schema T and each morphism $\phi:T\to G$ a T-isomorphism
$$\rho(\phi):a^*V\to b^*V \tag{28}$$
where $(a,b)=(s,t)\phi$, the source and the target of ϕ, and a^* (resp. b^*) denotes the pull-back of V along a (resp. b). One requires that this operation be compatible with the composition law of the groupoid $(S(T),G(T))$ and with the base change. The latter means: for any morphism $r:T'\to T$
$$\rho(r^*\phi)=r^*\rho(\phi). \tag{29}$$

Let $S = \mathrm{Spec}(R)$ be an affine scheme. Any R-module V defines the functor V_l and V_r from the category of R-algebras to category the of groups as follows:

$$\begin{aligned} V_r : \; R-\text{algebras} &\longrightarrow \text{groups} \\ B &\mapsto (B \otimes V, +) \;\; B \text{ consider as right } R-\text{module}, \end{aligned}$$

and

$$\begin{aligned} V_l : \; R-\text{algebras} &\longrightarrow \text{groups} \\ B &\mapsto (V \otimes B, +) \;\; B \text{ consider as left } R-\text{module}. \end{aligned}$$

From the perspective of functorial language, we have another way to define the representation of a groupoid scheme. Let G be a flat affine k-groupoid scheme acting on S. For any k-algebra B, consider ${}_sB, B_t$ are R-modules through the morphisms s and t, respectively. A *representation* of G on V (a *G-module*) is an action of G on the functor V_r, defined as follows:

$$G(B) \times V_r({}_sB) \longrightarrow V_l(B_t)$$

where $G(B)$ acts on $V_r({}_sB)$ by B-linear maps and this action commutes with base change.

A representation is called *finite* if the underlying sheaf is coherent. We denote this category by $\mathrm{Rep}(S:G)$ and denote the full subcategory of finite representations by $\mathrm{Rep}^{\mathrm{f}}(S:G)$.

Assume that G acts transitively on S, then finite representations of G are locally free as sheaves on S. Moreover, they form a k-linear rigid tensor abelian category. Being equipped with a fiber functor – the forgetful functor to quasi-coherent sheaves on S, $\mathrm{Rep}(S:G)$ is a (non-neutral) Tannakian category. Each object of $\mathrm{Rep}(S:G)$ is a filtered union of its finite rank subrepresentations.

A.2.3. *The coordinate ring.* Let S be an affine scheme, i.e., $S = \mathrm{Spec}(R)$ for some commutative ring R, and let G be an affine group scheme, with its coordinate ring denoted by $\mathcal{O}(G)$. The groupoid structure on G induces the structures of a Hopf algebroid on $\mathcal{O}(G)$. The source and the target map for G induce algebra maps $s, t : R \to \mathcal{O}(G)$. The transitivity of G on S can be rephrased by saying that $\mathcal{O}(G)$ is faithfully flat over $R \otimes_k R$ with respect to the base map $t \otimes_k s : R \otimes_k R \to \mathcal{O}(G)$.

The composition law for G induces an $R \otimes_k R$-algebra map

$$\Delta : \mathcal{O}(G) \longrightarrow \mathcal{O}(G) \,{}_s\!\otimes_t \mathcal{O}(G). \tag{30}$$

satisfying $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$. The unit element of G induces a $R \otimes_k R$-algebra map

$$\varepsilon : \mathcal{O}(G) \longrightarrow R \tag{31}$$

where $R \otimes_k R$ acts on R diagonally (i.e., $(\lambda \otimes_k \mu)\nu = \lambda\mu\nu$). One has

$$m(\varepsilon \otimes \mathrm{id})\Delta = m(\mathrm{id} \otimes \varepsilon)\Delta = \mathrm{id}. \tag{32}$$

Finally, the operation which consists of taking the inverse in G induces an automorphism ι of $\mathcal{O}(G)$ which interchanges the actions t and s:

$$\iota(t(\lambda)s(\mu)h) = s(\lambda)t(\mu)\iota(h), \tag{33}$$

and satisfies the following equations:

$$m(\iota \otimes \mathrm{id})\Delta = s \circ \varepsilon \quad m(\mathrm{id} \otimes \iota)\Delta = t \circ \varepsilon. \tag{34}$$

Since $S = \mathrm{Spec}(R)$, quasi-coherent sheaves on S are R-modules and coherent sheaves are finite R-modules. A representation ρ of G in V induces a map $\rho : V \to V \otimes_t \mathcal{O}(G)$, called coaction of $\mathcal{O}(G)$ on V, such that

$$(\mathrm{id}_V \otimes \Delta)\rho = (\rho \otimes \mathrm{id}_V)\rho, \quad (\mathrm{id}_V \otimes \varepsilon)\rho = \mathrm{id}_V. \tag{35}$$

An R-module equipped with such an action is called an $\mathcal{O}(G)$-comodule. Conversely, any coaction of $\mathcal{O}(G)$ on an R-module V defines a representation of G in V. In fact, we have an equivalence between the category of G-representations and the category of $\mathcal{O}(G)$-comodules. The discussion in the previous subsection shows that V is projective over R.

In particular, the coproduct on $\mathcal{O}(G)$ can be considered as a coaction of $\mathcal{O}(G)$ on itself and hence defines a representation of G in $\mathcal{O}(G)$, called the *(left) right regular representation.*

A.2.4. *Morphisms.* A morphism of k-groupoid schemes acting on a k-scheme S is a morphism of the underlying k-schemes which is compatible with all structure maps. We define the kernel of a homomorphism $f : G_1 \to G$ as the fiber product $\ker f := S \times_G G_1$:

$$\begin{array}{ccc} \ker f & \xrightarrow{f} & G \\ \downarrow & & \downarrow \\ S & \longrightarrow & S \times S. \end{array}$$

Thus, $\ker f$ is a group scheme over S. Taking the diagonal group schemes, we see that $\ker f$ is isomorphic to the kernel of the homomorphism $G_1^\Delta \to G^\Delta$ of group schemes:

$$\begin{array}{ccccc} \ker f & \longrightarrow & G_1 & \xrightarrow{f} & G \\ \| & & \uparrow & & \uparrow \\ \ker f^\Delta & \longrightarrow & G_1^\Delta & \xrightarrow{f^\Delta} & G^\Delta. \end{array}$$

Indeed, the canonical map $\ker f^\Delta \longrightarrow \ker f$ comes from the definition of $\ker f$ and the (outer) commutative diagram

$$\begin{array}{ccccc} \ker f^\Delta & \longrightarrow & G_1^\Delta & \longrightarrow & G_1 \\ \downarrow & & \downarrow & & \downarrow f \\ S & \longrightarrow & G^\Delta & \longrightarrow & G. \end{array}$$

And the canonical map $\ker f \longrightarrow \ker f^\Delta$ comes from the map $\ker f \longrightarrow G_1^\Delta$ which satisfies the commutative diagram

$$\begin{array}{ccc} \ker f & \longrightarrow & G_1^\Delta \\ \downarrow & & \downarrow \\ S & \longrightarrow & G^\Delta. \end{array}$$

A.3. **The induction of groupoid schemes.** In this subsection, we extend the result of [EH06].

A.3.1. *Fixed point functor.* Let G be a flat affine k-groupoid scheme acting on S, and let V be a G-module. We define the set of fixed points by

$$V^G = \{v \in V | \rho_V(v) = v \otimes 1 \in V \otimes_t \mathcal{O}(G)\}.$$

If $\phi : V \to V'$ is a homomorphism of G-modules, then $\phi(V^G) \subset V'^G$. Thus, we obtain the fixed point functor

$$\begin{gathered} \mathrm{Rep}(S : G) \longrightarrow \mathrm{Vec}_k \\ V \mapsto V^G. \end{gathered}$$

Lemma A.4. *Let G be a flat affine groupoid scheme acting over S, and let H be a normal discrete subgroupoid scheme of G. If V is a G-module, then V^H is an G-submodule of V.*

Proof. Let $\rho_V : V \to V \otimes_t \mathcal{O}(H)$ denote the comodule map of V, considered as an H-module. We regard $V \otimes_t \mathcal{O}(H)$ as an G-module via the given action on V and the conjugation action on $\mathcal{O}(H)$. Using Sweedler's notation, the conjugate action of G on $\mathcal{O}(H)$ can be rewritten as follows:

$$\begin{aligned}\Delta_c : \mathcal{O}(H) &\longrightarrow \mathcal{O}(H) \otimes_t \mathcal{O}(G)\\ \overline{v} &\mapsto \sum_{(v)} \overline{v_{(2)}} \otimes v_{(1)}\iota(v_{(3)}).\end{aligned}$$

To prove that $\mathcal{V} \otimes_t \mathcal{O}(G)$ is a G-module, we need to prove that the following diagram:

$$\begin{array}{ccc} V & \xrightarrow{\rho_H} & V \otimes \mathcal{O}(H) \\ \downarrow{\scriptstyle \rho_V} & & \downarrow{\scriptstyle \mathrm{id}\otimes\Delta_c} \\ V \otimes_t \mathcal{O}(G) & \xrightarrow{\rho_H \otimes \mathrm{id}} & V \otimes \mathcal{O}(H) \otimes_t \mathcal{O}(G) \end{array}$$

is commutative. We have

$$\begin{aligned}(\mathrm{id}\otimes\Delta)\circ\rho_H(v) &= (\mathrm{id}\otimes\Delta)(\sum_{(v)} v_{(0)} \otimes v_{(1)})\\ &= \sum_{(v)} v_{(0)(0)} \otimes \overline{v_{(1)(2)}} \otimes v_{(1)(1)}\iota(v_{(1)(3)})\\ &= \sum_{(v)} v_{(0)} \otimes \overline{v_{(3)}} \otimes v_{(2)}\iota(v_{(4)})\\ &= \sum_{(v)} v_{(0)} \otimes \overline{v_{(3)}} \otimes \iota(v_{(2)})v_{(4)}\\ &= \sum_{(v)} v_{(0)} \otimes \overline{v_{(1)}} \otimes v_{(2)}\\ &= \sum_{(v)} v_{(0)(0)} \otimes \overline{v_{(0)(1)}} \otimes v_{(1)}\\ &= (\rho_H \otimes \mathrm{id})\rho_V(v).\end{aligned}$$

Thus, the map $\rho_V : V \to V \otimes_t \mathcal{O}(H)$ is a homomorphism of G-modules. The same holds for the map $v \mapsto \rho_V(v) - v \otimes 1$. Since the category $\mathrm{Rep}^{\mathrm{f}}(R:G)$ is abelian, the space of H-invariants, V^H, which is the kernel of this map, is a G-submodule. □

A.3.2. *The functor* $\mathrm{Ind}_{G^\Delta}^G$. We continue to assume S and G are affine, $S = \mathrm{Spec}(R)$. For any representation $W \in \mathrm{Obj}(\mathrm{Rep}_R(G^\Delta))$, we have

(36) $$\mathrm{Ind}_{G^\Delta}^G(W) := (W \otimes_t \mathcal{O}(G))^{G^\Delta}.$$

The space $\mathrm{Ind}_{G^\Delta}^G(W)$ can also be given as the equalizer of the maps

(37) $$\begin{aligned} p &: W \otimes_t \mathcal{O}(G) \xrightarrow{\rho_W \otimes \mathrm{id}} W \otimes \mathcal{O}(G^\Delta) \otimes_t \mathcal{O}(G)\\ q &: W \otimes_t \mathcal{O}(G) \xrightarrow{\mathrm{id}\otimes\Delta} W \otimes_t \mathcal{O}(G)\ {}_s\!\otimes_t \mathcal{O}(G) \xrightarrow{\pi} W \otimes \mathcal{O}(G^\Delta) \otimes_t \mathcal{O}(G),\end{aligned}$$

where $\rho_W : W \to W \otimes \mathcal{O}(G^\Delta)$ is the coaction of $\mathcal{O}(G^\Delta)$ on W, and Δ is the coproduct on $\mathcal{O}(G)$.

A.3.3. *Frobenius Reciprocity.* For any R-module V, let $\varepsilon_V : V \otimes_t \mathcal{O}(G) \to V$ be the linear map as follows:

$$\epsilon_V : V \otimes_t \mathcal{O}(G) \longrightarrow V \tag{38}$$

$$v \otimes g \mapsto (\mathrm{id}_V \bar{\otimes} \varepsilon_G)(v \otimes g) = v.\varepsilon_G(g). \tag{39}$$

Lemma A.5 (Frobenius Reciprocity)**.** *Let G be a flat affine k-groupoid scheme acting on $S = \mathrm{Spec}(R)$. Let W be an G^Δ-module. For each G-module V the map $\varphi \mapsto \varepsilon_V \circ \varphi$ is an isomorphism*

$$\mathrm{Hom}_G(V, \mathrm{Ind}^G_{G^\Delta}(W)) \cong \mathrm{Hom}_{G^\Delta}(V, W), \tag{40}$$

i.e., the functor $\mathrm{Ind}^G_{G^\Delta}$ is the right adjoint to the functor restricting G-representations to G^Δ.

Proof. The converse map is given by $f \mapsto (f \otimes \mathrm{id})\rho_V$, where f belongs to $\mathrm{Hom}_{G^\Delta}(V, W)$. We first check that this map is well defined, meaning that $(f \otimes \mathrm{id})\rho_V$ is the morphism between V and $\mathrm{Ind}^G_{G^\Delta}(W)$. According to (37), we need to prove that

$$(\rho_W \otimes \mathrm{id})(f \otimes \mathrm{id})\rho_V = \pi(\mathrm{id} \otimes \Delta)(f \otimes \mathrm{id})\rho_V.$$

From the left hand side, we obtain the following commutative diagram:

$$\begin{array}{ccccccc}
V & \xrightarrow{\rho_V} & V \otimes_t \mathcal{O}(G) & \xrightarrow{f \otimes \mathrm{id}} & W \otimes_t \mathcal{O}(G) & \xrightarrow{\rho_W \otimes \mathrm{id}} & W \otimes \mathcal{O}(G^\Delta) \otimes_t \mathcal{O}(G) \\
\downarrow{\scriptstyle \rho_V} & & \downarrow{\scriptstyle \rho_V \otimes \mathrm{id}} & & & \nearrow{\scriptstyle f \otimes \mathrm{id} \otimes \mathrm{id}} & \\
V \otimes_t \mathcal{O}(G) & \xrightarrow{\mathrm{id} \otimes \Delta} & V \otimes_t \mathcal{O}(G)\ {}_s\otimes_t \mathcal{O}(G) & \xrightarrow{\mathrm{id} \otimes \pi \otimes \mathrm{id}} & V \otimes \mathcal{O}(G^\Delta) \otimes_t \mathcal{O}(G), & &
\end{array}$$

so

$$(\rho_W \otimes \mathrm{id})(f \otimes \mathrm{id})\rho_V = (f \otimes \mathrm{id} \otimes \mathrm{id})(\mathrm{id} \otimes \pi \otimes \mathrm{id})(\mathrm{id} \otimes \Delta)\rho_V.$$

Besides, we also have the folowing commutative diagram:

$$\begin{array}{ccccc}
W \otimes_t \mathcal{O}(G) & \xrightarrow{1 \otimes \Delta} & W \otimes_t \mathcal{O}(G)\ {}_s\otimes_t \mathcal{O}(G) & \xrightarrow{\mathrm{id} \otimes \pi \otimes \mathrm{id}} & W \otimes \mathcal{O}(G^\Delta) \otimes_t \mathcal{O}(G) \\
{\scriptstyle f \otimes \mathrm{id}}\uparrow & & & & {\scriptstyle f \otimes \mathrm{id} \otimes \mathrm{id}}\uparrow \\
V \otimes_t \mathcal{O}(G) & \xrightarrow{\mathrm{id} \otimes \Delta} & V \otimes_t \mathcal{O}(G)\ {}_s\otimes_t \mathcal{O}(G) & \xrightarrow{\mathrm{id} \otimes \pi \mathrm{id}} & V \otimes \mathcal{O}(G^\Delta) \times_t \mathcal{O}(G),
\end{array}$$

this implies that

$$\begin{aligned}
(\rho_W \otimes \mathrm{id})(f \otimes \mathrm{id})\rho_V &= (f \otimes \mathrm{id} \otimes \mathrm{id})(\mathrm{id} \otimes \pi \otimes \mathrm{id})(\mathrm{id} \otimes \Delta)\rho_V \\
&= \pi(\mathrm{id} \otimes \Delta)(f \otimes \mathrm{id})\rho_V.
\end{aligned}$$

We now prove that the the maps $\phi \mapsto \varepsilon \circ \phi$ and $f \mapsto (f \otimes \mathrm{id})\rho_V$ are inverse to each other. Let f be a G-morphism from V to $\mathrm{Ind}^G_{G^\Delta}(W)$. Then we have

$$\begin{aligned}
((\varepsilon_W \circ f) \otimes \mathrm{id}) \circ \rho_V &= \varepsilon_W \circ (f \otimes \mathrm{id}) \circ \rho_V \\
&= \varepsilon_W \circ (\rho_{\mathrm{Ind}^G_{G^\Delta}(W)} \circ f) \\
&= f.
\end{aligned}$$

On the other hand, let ϕ be a G^Δ-morphism between V and W. We also have

$$\begin{aligned}
\varepsilon_W((\phi \otimes \mathrm{id}) \circ \rho_V) &= \varepsilon_W(\rho_W \circ \phi) \\
&= \phi.
\end{aligned}$$

□

Theorem A.6 (Cf. [EH06, Remark 6.7])**.** *The functor* $\mathrm{Ind}_{G^\Delta}^G$ *is faithfully exact. Hence, the canonical map*

$$\mathrm{Ind}_{G^\Delta}^G(W) \longrightarrow W$$

is surjective for any G*-representation* W.

Proof. Before giving the proof of this theorem, we discuss the algebra $\mathcal{O}(G^\Delta)$. By definition of G^Δ, we have

$$\mathcal{O}\left(G^\Delta\right) \cong \mathcal{O}(G) \otimes_{R\otimes_k R} R$$

where $R \otimes_k R \to R$ is the product map. Then $J := \mathrm{Ker}(R \otimes_k R \to R)$ is generated by elements of the form $\lambda\otimes 1 - 1\otimes\lambda, \lambda \in R$. Since $\mathcal{O}(G)$ is faithful over $R\otimes_k R$, tensoring the exact sequence $0 \to J \to R\otimes_k R \to R \to 0$ with $\mathcal{O}(G)$ over $R \otimes_k R$, one obtains an exact sequence

$$0 \longrightarrow J\mathcal{O}(G) \longrightarrow \mathcal{O}(G) \xrightarrow{\pi} \mathcal{O}\left(G^\Delta\right) \longrightarrow 0. \tag{41}$$

That is, we can identify $J \otimes_{R\otimes_k R} \mathcal{O}(G)$ with its image $J\mathcal{O}(G)$ in $\mathcal{O}(G)$. In order to prove the faithfully exactness of $\mathrm{Ind}_{G^\Delta}^G$, we need the following claim.

Claim. Let us use the following notation of Sweedler for the coproduct on $\mathcal{O}(G)$:

$$\Delta(g) = \sum_{(g)} g_{(1)} \otimes g_{(2)}.$$

The following map:

$$\begin{aligned} \varphi : \mathcal{O}(G) \otimes_{R\otimes_k R} \mathcal{O}(G) &\longrightarrow \mathcal{O}\left(G^\Delta\right) \otimes_t \mathcal{O}(G), \\ g \otimes h &\longmapsto \sum_{(g)} \pi\left(g_{(1)}\right) \otimes g_{(2)} h, \end{aligned} \tag{42}$$

is an isomorphism, where π is defined in the formula 41.

Verification. We define the inverse map to this map. Let

$$\bar{\psi} : \mathcal{O}(G)_s \otimes_t \mathcal{O}(G) \longrightarrow \mathcal{O}(G) \otimes_{R\otimes_k R} \mathcal{O}(G)$$

be the map that maps $g \otimes h \mapsto \sum_{(g)} g_{(1)} \otimes \iota\left(g_{(2)}\right) h$. We have for $\lambda \in R$ and for $t, s : R \to \mathcal{O}(G)$

$$\begin{aligned} \bar{\psi}\left(t(\lambda) g\ {}_s\otimes_t h\right) &= \sum_{(g)} g_{(1)} \otimes \iota\left(t(\lambda) g_{(2)}\right) h \\ &= \sum_{g} g_{(1)} \otimes_{R\otimes_k R} s(\lambda)\iota\left(g_{(2)}\right) h \quad \text{by (33)} \\ &= s(\lambda) \sum_{g} g_{(1)} \otimes_{R\otimes_k R} \iota\left(g_{(2)}\right) h \\ &= \bar{\psi}\left(s(\lambda) g\ {}_s\otimes_t h\right). \end{aligned}$$

Thus, $\bar{\psi}$ maps $J\mathcal{O}(G)\ {}_s\otimes_t \mathcal{O}(G)$ to 0, hence factors through a map

$$\psi : \mathcal{O}\left(G^\Delta\right) \otimes_t \mathcal{O}(G) \to \mathcal{O}(G) \otimes_{R\otimes_k R} \mathcal{O}(G).$$

Checking $\varphi\psi = \mathrm{id}, \psi\varphi = \mathrm{id}$ can be easily done using the property (34) of ι.

We now prove that the functor $\mathrm{Ind}_{G^\Delta}^G$ is faithfully exact. Let $W \in \mathrm{Obj}(\mathrm{Rep}_R(G^\Delta))$. Tensoring the isomorphism in (42) with W and applying the functor $(-)^{\mathrm{G}^\Delta}$, we obtain the following map

$$\begin{aligned} \Phi : \mathrm{Ind}_{G^\Delta}^G(W) \otimes_{R\otimes_k R} \mathcal{O}(G) &\xrightarrow{\cong} W \otimes_t \mathcal{O}(G), \\ w \otimes g \otimes h &\longmapsto w \otimes gh. \end{aligned} \tag{43}$$

The above map is isomorphism since we have its inverse as follows:

$$W \otimes_t \mathcal{O}(G) \longrightarrow W \otimes \mathcal{O}\left(G^\Delta\right) \otimes \mathcal{O}(G) \xrightarrow{\Psi} W \otimes_t \mathcal{O}(G) \otimes_{R\otimes_k R} \mathcal{O}(G),$$
$$\Psi = \left(\mathrm{id}_W \otimes \psi\right)\left(\rho_W \otimes \mathrm{id}\right).$$

According to (43), the functor

$$\mathrm{Ind}_{G^\Delta}^G(-) \otimes_{R\otimes_k R} \mathcal{O}(G) \cong (-) \otimes_t \mathcal{O}(G)$$

is therefore exact. Since $\mathcal{O}(G)$ is faithfully flat over $R \otimes_k R$, the functor $\mathrm{Ind}_{G^\Delta}^G$ is faithfully exact.

We now prove the last part of the theorem. Setting $V = \mathrm{Ind}_{G^\Delta}^G(W)$ in (40), we define the canonical $u_W : \mathrm{Ind}_{G^\Delta}^G(W) \to W$ as follows:

$$\mathrm{Hom}_G(\mathrm{Ind}_{G^\Delta}^G(W), \mathrm{Ind}_{G^\Delta}^G(W)) \longrightarrow \mathrm{Hom}_{G^\Delta}(\mathrm{Ind}_{G^\Delta}^G(W), W)$$
$$\mathrm{id} \longmapsto u_W.$$

The map u_W is nonzero whenever W is nonzero. Indeed, since $\mathrm{Ind}_{G^\Delta}^G$ is faithfully exact, the G^Δ-module $\mathrm{Ind}_{G^\Delta}^G(W)$ is nonzero whenever W is nonzero. Thus, if $u_W = 0$, then both sides of (40) are zero for any V. On the other hand, the right-hand side contains the identity map. This shows that u_W cannot vanish.

We now turn to show that u_W is surjective. Let $\mathrm{U} = \mathrm{Im}(u_W)$ and let $\mathrm{T} = W/\mathrm{U}$. We have the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ind}_{G^\Delta}^G(U) & \longrightarrow & \mathrm{Ind}_{G^\Delta}^G(W) & \longrightarrow & \mathrm{Ind}_{G^\Delta}^G(T) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & U & \longrightarrow & W & \longrightarrow & T & \longrightarrow & 0. \end{array}$$

The composition $\mathrm{Ind}_{G^\Delta}^G(W) \to \mathrm{Ind}_{G^\Delta}^G(T) \to T$ is 0. Therefore, $\mathrm{Ind}_{G^\Delta}^G(T) \to T$ is a zero map, implying $T = 0$. □

Corollary A.7. *Any G^Δ-representation is a quotient of a G-representation. Consequently, any R-projective finite representation of G^Δ is also a special subobject of a finite G-representation considered as representation of G^Δ.*

Proof. Let W be a representation of G^Δ and $u_W : \mathrm{Ind}_{G^\Delta}^G(W) \longrightarrow W$ be the canonical map in Theorem A.6. This theorem implies that W is a quotient of $\mathrm{Ind}_{G^\Delta}^G(W)$.

We now prove the rest of the corollary. Since $\mathrm{Ind}_{G^\Delta}^G(W)$ is a union of its finite subrepresentations, we can find a finite G-subrepresentation $W_0(W)$ of $\mathrm{Ind}_{G^\Delta}^G(W)$, which still maps surjectively on W. In order to obtain the statement on the embedding of R-projective representation W, one repeats the above argument for $W^\vee$ to get the surjective map $W_0(W^\vee) \twoheadrightarrow W^\vee$, and then dualizes to $W \hookrightarrow (W_0(W^\vee))^\vee$. □

A.4. **Cohomology of groupoid schemes.** Let G be a flat affine k-groupoid scheme acting on S. We define an *injective* G-module to be an injective object in the category of all G-modules.

A.4.1. *The injective object in the category* $\mathrm{Rep}(S:G)$.

Lemma A.8. *Let G be a flat affine k-groupoid scheme acting on S. The following statements are true:*

(1) *The category* $\mathrm{Rep}(S:G)$ *have enough injectives.*

(2) *A G-module V is injective if and only if there is an injective R-module I such that V is isomorphic to a direct summand of $I \otimes_t \mathscr{O}(G)$ with I regards as a trivial G-module.*

Proof. Before giving the proof of (1), we require the following claims.

Claim. Let V be an R-module and U be a G-module. We have the following functorial isomorphism

$$\mathrm{Hom}_G(U, V \otimes_t \mathscr{O}(G)) \cong \mathrm{Hom}_R(U, V).$$

Verification. We have the following map

$$\begin{aligned}\mathrm{Hom}_G(U, V \otimes_t \mathscr{O}(G) &\to \mathrm{Hom}_R(U, V)\\ f &\mapsto \varepsilon_V f,\end{aligned}$$

where ε_V is defined as in (38). On the other hand, we have the inverse map as follows

$$\begin{aligned}\mathrm{Hom}_R(U, V) &\to \mathrm{Hom}_G(U, V \otimes_t \mathscr{O}(G)\\ g &\mapsto (g \otimes \mathrm{id})\rho_U,\end{aligned}$$

where ρ_V is the coaction of $\mathscr{O}(G)$ on V. ▷

Proof of (1). Let V be a G-module. Since V is quasi-coherent as an R-module, we can embed the R-module V into some injective module I. We consider I as a G-module with trivial G-action, there is an injective G-map as follows:

$$V \xrightarrow{\rho_V} V \otimes_t \mathscr{O}(G) \hookrightarrow I \otimes_t \mathscr{O}(G).$$

We show that $I \otimes_t \mathscr{O}(G)$ is an injective G-module. Consider the following diagram of G-modules:

$$\begin{array}{ccccc} 0 & \longrightarrow & M & \xrightarrow{i} & M' \\ & & \downarrow f & \swarrow g & \\ & & I \otimes_t \mathscr{O}(G) & & \end{array}.$$

We will show that there exists a map g. By Claim, we have the following commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_G(M', I \otimes_t \mathscr{O}(G)) & \xrightarrow{i^*} & \mathrm{Hom}_G(M, I \otimes_t \mathscr{O}(G) \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{Hom}_R(M', I) & \xrightarrow{i^*} & \mathrm{Hom}_R(M, I). \end{array} \tag{44}$$

Since the R-module I is injective, the map i^* at the bottom is surjective. Hence, for $f \in \mathrm{Hom}_G(N, I \otimes_t \mathscr{O}(G))$, we can find g that satisfies Diagram (44). Therefore, $I \otimes_t \mathscr{O}(G)$ is an injective G-module, and the result follows.

Proof of (2). We have proved that V can be embedded into some $I \otimes_t \mathscr{O}(G)$. Since V is an injective G-module, it follows that V is a direct summand of $I \otimes_t \mathscr{O}(G)$. The reverse direction holds because every direct summand of an injective G-module is also an injective G-module. □

A.4.2. *Group cohomology.* Let G be a flat affine k-groupoid scheme acting on S, and let V be a G-module. Consider the fixed point functor

$$\begin{aligned}\mathrm{Rep}(S : G) &\longrightarrow \mathrm{Vec}_k\\ V &\mapsto V^G,\end{aligned}$$

see A.3.1. We observe that if G is flat, then the fixed point functor is left exact. The category $\operatorname{Rep}(S:G)$ has enough injectives (Lemma A.8), which allows us to define the derived functor

$$V \mapsto \mathrm{H}^n(G,V),$$

and we call $\mathrm{H}^n(G,V)$ the *n-cohomology group* of V.

A.4.3. *Shapiro's lemma.* Grothendieck's spectral sequence is standard in Homological algebra We recall it here for the reader's sake. One can find proof in the book of Weilbel [We94].

Grothendieck's spectral sequence. Let $\mathscr{C},\mathscr{C}',\mathscr{C}''$ be abelian categories with $\mathscr{C},\mathscr{C}'$ having enough injectives. Suppose now that $\mathscr{F}:\mathscr{C}\to\mathscr{C}'$ and $\mathscr{F}':\mathscr{C}'\to\mathscr{C}''$ are additive (covariant) functors. If $\mathscr{F}'$ is left exact and if $\mathscr{F}$ maps injectite objects in $\mathscr{C}$ to objects acyclic for $\mathscr{F}'$, then there is a spectral sequence for each object M in $\mathscr{C}$ with differentials d_r of bidegree $(r,1-r)$, and

$$E_2^{p,q} = (R^p\mathscr{F}')(R^q\mathscr{F})M \Rightarrow R^{p+q}(\mathscr{F}'\circ\mathscr{F})M. \tag{45}$$

Remark A.9. We remark on two facts.

(1) If $\mathscr{F}'$ is exact, then $\mathscr{F}'\circ R^q\mathscr{F} \simeq R^q\left(\mathscr{F}'\circ\mathscr{F}\right)$ for all $n\in\mathbb{N}$.
(2) If $\mathscr{F}$ is exact and maps injective objects to objects acyclic for $\mathscr{F}'$, then

$$(R^n\mathscr{F}')\circ\mathscr{F} \simeq R^n(\mathscr{F}'\circ\mathscr{F})$$

for all $n\in\mathbb{N}$.

Remark A.10. Let G be a flat affine k-groupoid scheme acting on $S=\operatorname{Spec}(R)$ and H be its flat subgroupoid scheme. Then $\operatorname{Ind}_H^G(-)$ is left exact functor. Thus, we can take the right derived functor $R^n\operatorname{Ind}_H^G(-)$.

Proposition A.11. *Let G be a flat affine k-groupoid scheme acting on $S=\operatorname{Spec}(R)$ and H be its flat discrete subgroupoid scheme. Let W be an H-module. There is a spectral sequence with*

$$E_2^{p,q} = \mathrm{H}^p\left(G, R^q\operatorname{Ind}_H^G W\right) \Rightarrow \mathrm{H}^{p+q}(H,W).$$

Proof. Lemma A.5 can be interpreted as an isomorphism of functors (choose $V=R$)

$$\operatorname{Hom}_G(R,-)\circ\operatorname{Ind}_H^G \simeq \operatorname{Hom}_H(R,-).$$

Since Ind_H^G is the right adjoint to the exact functor restricting G-module to H, it maps injective H-modules to injective G-modules. Hence, we can apply Grothendieck's spectral sequence. □

Definition A.12. Let G be an affine k-groupoid scheme acting on $S=\operatorname{Spec}(R)$ and H be its flat subgroupoid scheme. We call *H exact* in G if Ind_H^G is an exact functor.

Using Proposition A.11, we obtain the following result.

Corollary A.13 (Shapiro's lemma)**.** *Let G be a flat affine k-groupoid scheme acting on $S=\operatorname{Spec}(R)$ and H be its flat discrete subgroupoid scheme. Suppose that H is exact in G. Let W be an H-module. For each $n\in\mathbb{N}$, there is an isomorphism*

$$\mathrm{H}^n\left(G,\operatorname{Ind}_H^G W\right) \simeq \mathrm{H}^n(H,W).$$

Since $\operatorname{Ind}_1^G = -\otimes_t\mathscr{O}(G)$ is an exact functor, we obtain the following result.

Lemma A.14. *Let G be a flat affine k-groupoid scheme acting on $S = \operatorname{Spec}(R)$ and H be its flat discrete subgroupoid scheme. Let $n \in \mathbb{N}$. We have for each G-module V :*

$$\mathrm{H}^n(G, V \otimes_t \mathscr{O}(G)) \simeq \begin{cases} V & \text{if} \quad n = 0, \\ 0 & \text{if} \quad n > 0. \end{cases}$$

Proposition A.15. *Let G be a flat affine k-groupoid scheme acting on $S = \operatorname{Spec}(R)$ and H be its flat subgroupoid scheme. We have for each H-module V and each $n \in \mathbb{N}$ an isomorphism of k-vector spaces*

$$\mathrm{H}^n(H, V \otimes_t \mathscr{O}(G)) \simeq \left(R^n \mathrm{Ind}_H^G\right) V.$$

Proof. The proof is based on the definition of the induction functor. Indeed, the definition of Ind_H^G yields an isomorphism of functors

$$\mathrm{For} \circ \mathrm{Ind}_H^G \simeq (-)^H \circ (- \otimes_t \mathscr{O}(G)),$$

where For is the forgetful functor from $\operatorname{Rep}(R : G)$ to R-modules. Since the functor $(- \otimes_t \mathscr{O}(G))$ is exact and maps injective H-modules to modules that are acyclic for the fixed points functor (see Lemma A.14-(1)), we can apply Remark A.9-(1),(2) and the result follows. ☐

A.5. **Tannakian duality over a field.** Reference for this subsection is [DM82] and [De90].

Definition A.16. A rigid abelian tensor category $\mathscr{C}$ equipped with an exact faithful k-linear tensor functor $\omega : \mathscr{C} \to \mathrm{Vec}_k$ is called a *neutral Tannakian category* over k. The functor ω is called a *fibre functor* for $\mathscr{C}$.

Theorem A.17. *Let $(\mathscr{C}, \omega)$ be a neutral Tannakian category. Then, there exists a k-group scheme G, such that ω induces an equivalence between $\mathscr{C}$ and $\mathrm{Rep}_k(G)$.*

The group scheme G above is called the Tannakian group of the category $(\mathscr{C}, \omega)$.

An example of a Tannakian category is the category of finite dimensional representations of an affine group scheme G over k, equipped with the forgetful functor of k-vector spaces. The resulting Tannakian group is isomorphic to G.

Another example, which is an object of this work, is the category of connections defined subsequently.

Definition A.18. A rigid abelian tensor category $\mathscr{C}$ equipped with an exact faithful k-linear tensor functor $\omega : \mathscr{C} \to \mathrm{mod} R$, R is a k-algebra, is called a *(general) Tannakian category* over k if. The functor ω is called a *fibre functor* for $\mathscr{C}$ with values in R-modules.

Theorem A.19. *Let $(\mathscr{C}, \omega)$ be a general Tannakian category with values in R-modules. Then, there exists a k-groupoid scheme $\mathscr{C}$, acting transitively upon $\operatorname{Spec} R \times_k \operatorname{Spec} R$ such that ω induces an equivalence between $\mathscr{C}$ and $\operatorname{Rep}(R : \mathscr{G})$.*

The groupoid $\mathscr{G}$ is called the Tannakian groupoid of $(\mathscr{C}, \omega)$. Conversely, if we start from a groupoid scheme $\mathscr{G}$ acting transitively upon a ring R, then $\operatorname{Rep}(R : \mathscr{G})$ equipped with the forgetful functor is a Tannakian category. The corresponding Tannakian groupoid is isomorphic to $\mathscr{G}$.

A.6. **Tannakian duality over a Dedekind ring.** Reference for this subsection is [Sa72], see also [DH18].

Assume that $\mathscr{C}$ is an R-linear abelian tensor category. Denote by $\mathscr{C}^o$ the full subcategory of $\mathscr{C}$ consisting of rigid objects. We say that $\mathscr{C}$ is *dominated* by $\mathscr{C}^o$ if each object of $\mathscr{C}$ is a quotient of a rigid object.

Definition A.20. A (neutral) Tannakian category over a Dedekind ring R is an R-linear abelian tensor category $\mathscr{C}$, dominated by $\mathscr{C}^o$, together with an exact faithful tensor functor $\omega : \mathscr{C} \to \mathsf{Mod}(R)$.

Theorem A.21 ([Sa72, Thm. II.4.1.1])**.** *Let* $(\mathscr{C}, \omega)$ *be a neutral Tannakian category over a Dedekind ring* R*. Then the group functor* $A \mapsto \mathsf{Aut}^{\otimes}_A(\omega \otimes A)$ *is representable by a flat group scheme* G *and* ω *factors through an equivalence between* $\mathscr{C}$ *and* $\mathrm{Rep}_R(G)$.

Appendix B. Morphisms between affine flat group schemes

Let R be a Dedekind domain. We denote $\mathbf{FGSch}/R$ be the full subcategory of the category of affine group scheme over R whose objects are R-flat. Let $\Pi \in \mathbf{FGSch}/R$. We recall some concepts:

(1) A subcomodule M of an $R[\Pi]$-comodule N is said to be *special* if N/M is flat over R; a *special subquotient* M of an $R[\Pi]$-comodule N is a special submodule of a quotient of N, or, equivalently, a quotient of a special submodule of N.
(2) Let N be an $R[\Pi]-$comodule. Then N_{tor}, the R-torsion submodule of N is an $R[\Pi]$-subcomodule. Hence for any $R[\Pi]$-subcomodule M, the preimage of $(N/M)_{\mathrm{tor}}$ in N, denoted M^{sat}, is an $R[\Pi]$-comodule. Since R is a Dedekind ring, the quotient N/M^{sat} is flat, being torsion-free. Thus M^{sat} is the smallest special subcomodule of N, containing M. It is called the *saturation* of M in N.

Definition B.1. Let H′ be a flat Hopf algebra over R. A Hopf subalgebra H of H′ is an R-submodule equipped with a Hopf algebra structure such that the inclusion H → H′ is a homomorphism of Hopf algebras. We say that H is *saturated* in H′ if H′/H is flat as an R-module.

Let $\rho : \Pi \to \mathrm{G}$ be a morphism in $\mathbf{FGSch}/R$. We describe the "images" of ρ in two ways.

Definition B.2 (The diptych)**.** Define Ψ_ρ as the group scheme whose Hopf algebra is the image of $R[G]$ in $R[\Pi]$. Define $R[\Psi'_\rho]$ as the saturation of the latter inside $R[\Pi]$. The obvious commutative diagram

$$\begin{array}{ccc} \psi'_\rho & \longrightarrow & \psi_\rho \\ \uparrow & & \downarrow \\ \Pi & \xrightarrow[\rho]{} & G \end{array}$$

is called the diptych of ρ.

Remark B.3. We have

(1) The image of a Hopf algebra homomorphism is a Hopf subalgebra (of the target).
(2) Implicit in the above definition is the fact that $\mathrm{R}[\Psi'_\rho]$ is a Hopf algebra. Indeed, we have the filtration
$$R[\Psi'_\rho]^{\mathrm{sat}} \otimes R[\Psi'_\rho]^{\mathrm{sat}} \subset R[\Psi'_\rho]^{\mathrm{sat}} \otimes R[\Pi] \subset R[\Pi] \otimes R[\Pi],$$
the successive quotients of which are flat, hence $(R[\Pi] \otimes R[\Pi])/(R[\Psi'_\rho]^{\mathrm{sat}} \otimes R[\Psi'_\rho]^{\mathrm{sat}})$ is also flat. Thus
$$(R[\Psi'_\rho] \otimes R[\Psi'_\rho])^{\mathrm{sat}} \subset R[\Psi'_\rho]^{\mathrm{sat}} \otimes R[\Psi'_\rho]^{\mathrm{sat}}.$$

Hence, by the definition of $R[\Psi'_\rho]^{\mathrm{sat}}$, we have

$$\Delta\left(R[\Psi'_\rho]^{\mathrm{sat}}\right) \subset (R[\Psi'_\rho]\otimes R[\Psi'_\rho])^{\mathrm{sat}} \subset R[\Psi'_\rho]^{\mathrm{sat}}\otimes R[\Psi'_\rho]^{\mathrm{sat}}.$$

Lemma B.4. *The morphism* $\Pi \to \Psi'_\rho$ *is faithfully flat.*

Proof. See [DH18, Theorem 4.1.1]. □

Proposition B.5. *Let s be a closed point of* $\mathrm{Spec}(R)$. *If* $\Psi'_{\rho,\mathrm{k}_s} \to \Psi_{\rho,\mathrm{k}_s}$ *is faithfully flat for every residue field* k_s, *then* $\Psi'_\rho \to \Psi_\rho$ *is an isomorphism.*

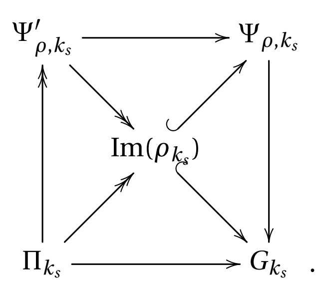
Proof. By construction, $R\left[\Psi_\rho\right] \to \mathrm{R}[\Psi'_\rho]$ is injective and $K[\Psi_\rho] = K[\Psi'_\rho]$. Since $\Psi'_{\rho,\mathrm{k}}$ is faithfully flat over $\Psi_{\rho,\mathrm{k}}$ for k being either the fraction field K or any residue field k_s of R, then Ψ'_ρ is faithfully flat over Ψ_ρ by Proposition 3.2 in [DHH17]. The faithful flatness of $R[\Psi'_\rho]$ over $R[\Psi_\rho]$ implies that $R[\Psi_\rho]$ is saturated in $R[\Psi'_\rho]$. Indeed, tensoring the exact sequence

$$0 \to R[\Psi_\rho] \to R[\Psi'_\rho] \to R[\Psi'_\rho]/R[\Psi_\rho] \to 0$$

on the right with $R[\Psi'_\rho]$ over $R[\Psi_\rho]$ we obtain a split exact sequence (the splitting is given by the map $R[\Psi'_\rho]\otimes_{R[\Psi_\rho]} R[\Psi'_\rho] \to R[\Psi'_\rho], m\otimes n \mapsto mn$). By assumption $R[\Psi'_\rho]$ is R-flat, hence so is $R[\Psi'_\rho]\otimes_{R[\Psi_\rho]} R[\Psi'_\rho]$. Consequently $R[\Psi'_\rho]/R[\Psi_\rho]\otimes_{R[\Psi_\rho]} R[\Psi'_\rho]$ is R-flat. Now the faithful flatness of $R[\Psi'_\rho]$ over $R[\Psi_\rho]$ implies that $R[\Psi'_\rho]/R[\Psi_\rho]$ is R-flat, that is, $R[\Psi_\rho]$ is saturated in $R[\Psi'_\rho]$ as an R-module. □

Over the residue field k_s, there is yet another interesting group scheme in sight: the image of ρ_{k_s}. We then have the *triptych* of ρ_{k_s}, which is the commutative diagram

(46)



Together with Proposition B.5, diagram (46) proves the following:

Corollary B.6. *The following claims are true.*

i) *If* $\mathrm{Im}\left(\rho_{\mathrm{k}_s}\right) \to \Psi_{\rho,\mathrm{k}_s}$ *is an isomorphism for every residue field* k_s, *then* $\Psi'_\rho \to \Psi_\rho$ *is an isomorphism.*
ii) *If* $\Psi'_\rho \to \Psi_\rho$ *is an isomorphism, then* $\mathrm{Im}\left(\rho_{\mathrm{k}_s}\right) \to \Psi_{\rho,\mathrm{k}_s}$ *is an isomorphism.*
iii) *The image of* $\Psi'_{\rho,\mathrm{k}_s}$ *in* Ψ_{ρ,k_s} *is none other than* $\mathrm{Im}\left(\rho_{\mathrm{k}_s}\right)$.

We have some notations, conventions and standard terminology.

(1) If V is a finite R-projective module, we write $\mathrm{GL}(V)$ for the general linear group scheme representing $A \longrightarrow \mathrm{Aut}_A(V\otimes A)$. If $V = R^n$, then $\mathrm{GL}(V) = \mathrm{GL}_n$.

(2) An object V of $\mathrm{Rep}^{\circ}_R(G)$ is said to be a *faithful representation* if the resulting morphism $G \longrightarrow \mathrm{GL}(V)$ is a closed immersion. Similar conventions are in force for group schemes over k. We admonish the reader that this is not the terminology of the authoritative [SGA3], where a faithful action is decreed to be one having no kernel (see Definition 2.3.6.2 of exposé I).

Let $\Pi \in \mathbf{FGSch}/R$. This group scheme admits a closed embedding into some $\mathrm{GL}_{n,R}$ as J.S. Milne mention in Aside 9.4 of his book [Mi12], or, according to the above notions, Π possesses a faithful representation. We remark on the concept of special subquotient which we introduced in Subsection A.1.1.

Definition B.7. Let $\Pi \in \mathbf{FGSch}/\mathrm{R}$ and $V \in \mathrm{Rep}^{\circ}_R(\Pi)$. Call an object $V'' \in \mathrm{Rep}^{\circ}_R(\Pi)$ a *special sub-quotient* of V if there exists a special monomorphism $V' \to V$ and an epimorphism $V' \to V''$. The category of all special sub-quotients of various $T^{a_1,b_1}(\mathrm{V}) \oplus \cdots \oplus T^{a_m,b_m}(\mathrm{V})$ is denoted by $\langle V\rangle^s_{\otimes}$.

Let V be a finite R- projective module and assume that our G (in the diptych) equals GL(V). We now interpret Ψ_ρ and Ψ'_ρ in terms of their representation categories.

Proposition B.8. *Let* V *be an object of* $\mathrm{Rep}^{\circ}_{\mathrm{R}}(\Pi)$ *and* ρ *be the natural homomorphism* $\Pi \to \mathrm{G} := \mathrm{GL(V)}$.

(1) *The obvious functor* $\mathrm{Rep}_R(\Psi_\rho) \to \mathrm{Rep}_R(\Pi)$ *defines an equivalence of categories between* $\mathrm{Rep}^{\circ}_R(\Psi_\rho)$ *and* $\langle V\rangle^s_{\otimes}$.
(2) *The obvious functor* $\mathrm{Rep}_R(\Psi'_\rho) \to \mathrm{Rep}_R(\Pi)$ *defines an equivalence between* $\mathrm{Rep}_R(\Psi'_\rho)$ *and* $\langle \mathrm{V}\rangle_{\otimes}$.

Proof. Proof of (1). Let V be a finite projective $R[\Psi_\rho]$-comodule. The coaction $\rho : \mathrm{V} \longrightarrow \mathrm{V} \otimes R[\Psi_\rho]$ induces a map

$$\mathrm{Cf}: V^\vee \otimes V \longrightarrow R[\Psi_\rho], \quad \varphi \otimes m \mapsto \sum \varphi(m_i)\, m'_i, \quad \varphi \in V^\vee, m \in V, \Delta(m) = \sum_i m_i \otimes m'_i.$$

The image of this map, denoted by Cf(V), is called the *coefficient space* of V.

Since V is projective R-module, there exist an R-module I such that $M = V \oplus I$ is a free R-module. The R-module M can be considered as a Π-module if we regard I as a trivial Π-module. We consider the diptych of ρ as follows:

$$\begin{array}{ccccc} \Psi'_\rho & \longrightarrow & \Psi_\rho & & \\ \uparrow & & & \searrow & \\ \Pi & \xrightarrow{\rho} & \mathrm{GL}(V) & \longrightarrow & \mathrm{GL}(M) \end{array} .$$

Since M is a faithful ψ_ρ-module, we obtain $\mathrm{Cf}(M) = R[\Psi_\rho]$. By the construction of M, we can see that $\mathrm{Cf}(V) = \mathrm{M}$, so we obtain

$$\mathrm{Rep}^{\circ}_R(\Psi_\rho) \cong \langle V\rangle^s_{\otimes}$$

via Lemma 1.1.7 of [DH18].

Proof of (2). The main idea of this proof is Theorem A.1. We have the morphism $\Pi \longrightarrow \Psi'_\rho$ is faithfully flat (Proposition B.4) since $\mathrm{Rep}^{\circ}_R(\Psi'_\rho) \longrightarrow \mathrm{Rep}^{\circ}_R(\Pi)$ is fully faithful and its image closed under taking subobject. On the other hand, since $\langle V\rangle_{\otimes}$ is the Tannakian category (we can check that it satisfies the Definition A.20), we have

$$\langle V\rangle_{\otimes} \cong \mathrm{Rep}_R(H),$$

where $H \in \mathbf{FGSch}/R$ (see Theorem A.21). According to (1), we have the following composition map

$$\mathrm{Rep}^{\circ}_R(\Psi_\rho) \longrightarrow \mathrm{Rep}^{\circ}_R(\Psi'_\rho) \longrightarrow \langle V\rangle^s_{\otimes}$$

is isomorphism. Combine with $\mathrm{Rep}^{\circ}_R(H) = \langle V\rangle^{\circ}_{\otimes}$ (the full subcategory of subobjects of finite direct sums of copies of tensor generated of V) and Claim 1, we see that the natural map $H \to \Psi'_\rho$ is closed immersion via Theorem A.1-(2). ☐

Note that, in general, $\mathrm{Rep}_R\left(\Psi_\rho\right)$ is not a full subcategory of $\mathrm{Rep}_R(\Pi)$. This means that we have the following interpretation of the diptych (Definition B.2) in terms of representation categories:

$$\begin{tikzcd} \langle V\rangle_{\otimes} \arrow[d] & \langle V\rangle^{s}_{\otimes} \arrow[l] \arrow[d] \\ \mathrm{Rep}_R(\Pi) & \mathrm{Rep}^{\circ}_R(\mathrm{GL}(V)). \arrow[l] \end{tikzcd}$$

We now deal with the representation theoretic interpretation of the triptych (Diagram (46)) of ρ. For that, given an R-linear category C, we define the special fiber C_s at a closed point s of $S := \mathrm{Spec}(R)$ to be the full subcategory whose objects W are annihilated by $\mathfrak{m}_s$, i.e. $m_s.id_W = 0$ in $\mathrm{Hom}_C(W,W) = 0$, where $\mathfrak{m}_s$ is the maximum ideal of R, which determines s. One can show that C_s is equivalent to the scalar extension C_{k_s} where $k_s = R/m_s$ is the residue field of R at s. We then have a commutative diagram of solid arrows between k-linear abelian categories:

$$\begin{tikzcd} \mathrm{Rep}_R(\Psi'_\rho)_{(k_s)} \arrow[dd] & & \mathrm{Rep}_R(\Psi_\rho)_{(k_s)} \arrow[ll] \arrow[dl, dotted] \\ & \mathrm{Im}(\rho_{k_s}) \arrow[ul] \arrow[dl] & \\ \mathrm{Rep}_R(\Pi)_{(k_s)}. & & \end{tikzcd} \tag{47}$$

From [Ja87, Part I, 10.1, 162] the categories $\mathrm{Rep}_R(-)_{(k_s)}$ are simply the corresponding representations categories of the group schemes obtained by base change $R \to k_s$. Since V is a faithful representation of Ψ_ρ (recall that $\Psi_\rho \to \mathrm{GL(V)}$ is a closed immersion by construction), $V \otimes k_s$ is a faithful representation of $\Psi_\rho \otimes k_s$, so that each object of $\mathrm{Rep}_R\left(\Psi_\rho\right)_{(k_s)}$ is a sub-quotient of some $\bigoplus T^{a_i, b_i}(\mathrm{V}\otimes k_s)$. This means that the upper horizontal arrow in Diagram (47) factors through $\langle \mathrm{V}\otimes k_s\rangle_{\otimes}$, i.e. the dotted arrow exists and still produces a commutative diagram. We conclude that Diagram (47) captures the essence of Diagram (46) as the former can easily be completed by introducing the representation category of the general linear group in the lower right corner.

Theorem B.9 (Lemma 3.5)**.** *Let $(\mathcal{V},\nabla)$ be an absolute connection on X/k. Then each locally free relative connection in $\langle \inf(\mathcal{V},\nabla)\rangle_{\otimes}$ is indeed a* special *subobject of a tensor generated object from* $\inf(\mathcal{V},\nabla)$.

Proof. The R-point η induces an equivalence of abelian tensor categories

$$\eta^* : \langle \inf(\mathcal{V},\nabla)\rangle_{\otimes} \longrightarrow \mathrm{Rep}_R(G'), \tag{48}$$

where G' is a flat group scheme over R (see Theorem A.21). Let $\rho : G' \to \mathrm{GL}(\eta^*\inf(\mathcal{V},\nabla))$ be the associated representation of the group G. Since $\eta^*\inf(\mathcal{V},\nabla)$ is projective R-module, there exist an R-module I such that $M = V \oplus I$ is a free R-module. The R-module M can be considered as a Π-module if we regard I as a trivial Π-module. We consider the diptych of ρ (Definition B.2) as follows:

$$\begin{array}{ccccc} \Psi'_\rho & \longrightarrow & G & & \\ \uparrow & & & \searrow & \\ G' & \xrightarrow[\rho]{} & \mathrm{GL}(\eta^* \inf(\mathcal{V},\nabla)) & \longrightarrow & \mathrm{GL}(M) \end{array}$$

where G is the group scheme Ψ_ρ of Definition B.2. Theorem A.1 and Proposition B.8-(2) show that the leftmost arrow above is an isomorphism. Moreover, according to Proposition B.8-(1), the functor η^* induces an equivalence

$$\langle \inf(\mathcal{V},\nabla)\rangle_\otimes^s \xrightarrow{\sim} \mathrm{Rep}_R^\circ(G). \tag{49}$$

Thus, the statement of this lemma can be proved if we can show that G isomorphic to G'.

To prove isomorphic between G and G', we only need to treat the local property, that is, we only need to prove that G'_{k_s} is fully faithful over G_{k_s} thanks to Proposition B.5. Moreover, we still prove that the functor

$$\mathrm{Rep}_k(G_{k_s}) \longrightarrow \langle \inf(\mathcal{V},\nabla)|_{X_{k_s}}\rangle_\otimes$$

is an equivalence based on the Corollary B.6 (i) and Diagram (47). The proof is a consequence of the following claims which we use the notion of k instead of k_s for convenience.

Claim 1. Let $\mathcal{N}$ and $\mathcal{N}'$ be objects in $\mathrm{MIC}^\circ(X/k)$ and let

$$\mathcal{N}|_{X_k} = \inf(\mathcal{N})|_{X_k} \xrightarrow{\theta} \inf(\mathcal{N}')|_{X_k} = \mathcal{N}'|_{X_k}$$

be an arrow of $\mathrm{MIC}^{\mathrm{se}}(X_k/k)$. Then there exists a morphism of $\mathrm{MIC}^{\mathrm{se}}(X/R)$

$$\tilde{\theta} : \inf(\mathcal{N}) \longrightarrow \inf(\mathcal{N}')$$

lifting θ.

Verification. We have an arrow of $\mathrm{MIC}^\circ\ (X_k/k)$

$$\theta : \mathcal{N}|_{X_k} \longrightarrow \mathcal{N}'|_{X_k}$$

which gives us an arrow of $\mathrm{MIC}^\circ\ (X_k/k)$

$$\sigma : \mathcal{O}_{X_k} \longrightarrow \mathcal{N}|_{X_k}^\vee \otimes \mathcal{N}'|_{X_k} =: \mathcal{E}|_{X_k}.$$

We will show that σ is the restriction of an arrow $\mathcal{O}_X \to \mathcal{E}$ of $\mathrm{MIC}^{\mathrm{se}}(X/R)$. Now, let

$$\iota : \mathrm{T} \longrightarrow \mathcal{E}|_{\mathrm{X_k}}$$

be the maximal trivial subobject; the arrow σ can therefore be written as a composition in $\mathrm{MIC}^\circ\ (X_k/\mathrm{k})$

$$\mathcal{O}_{X_k} \xrightarrow{\tau} \mathrm{T} \xrightarrow{\iota} \mathcal{E}|_{X_k}. \tag{50}$$

According to Lemma 3.3-(1), it is possible to find $\mathcal{T} \in \mathrm{Obj}(\mathrm{MIC}^\circ(S/k))$ and a morphism of $\mathrm{MIC}^{\mathrm{se}}(X/k)$

$$\tilde{\iota} : f^*(\mathcal{T}) \longrightarrow \mathcal{E}$$

such that ι is the restriction to X_k of $\tilde{\iota}$. As f is geometrically connected fibers, we have $f_*\mathcal{O}_\mathrm{X} = \mathcal{O}_\mathrm{S}$; it then follows that the functor f^* from vector bundles on S to vector bundles on X is full and faithful. As S is affine, we conclude that the morphism of $\mathcal{O}_{X_k}$-modules

$$f^*(\mathcal{O}_S)|_{X_k} = \mathcal{O}_{X_k} \xrightarrow{\tau} \mathrm{T} = f^*(\mathcal{T})|_{X_k}$$

appearing in (50) is the restriction of a morphism

$$\tilde{\tau} = f^*(\delta) : f^*(\mathcal{O}_S) \longrightarrow f^*(\mathcal{T}).$$

Of course, δ need not be a morphism of $\mathrm{MIC}^\circ(S/k)$, but $f^*(\delta)$ is surely a morphism of $\mathrm{MIC}^{\mathrm{se}}(X/R)$, that is, an arrow between inflations

$$\inf(\mathscr{O}_X) \longrightarrow \inf\big(f^*\mathscr{T}\big).$$

In conclusion, we have proved that σ is the restriction of $\tilde{\imath}\circ\tilde{\tau}$.

Claim 2. For each $\mathscr{W}$ belonging to $\langle \inf(\mathscr{V})|_{X_k}\rangle_\otimes$, there exist $\mathscr{E}$ and $\mathscr{E}'$ in $\langle\inf(\mathscr{V})\rangle_\otimes^s$ and an arrow in $\mathrm{MIC}^{\mathrm{se}}(X/R)$

$$\tilde\theta : \mathscr{E} \longrightarrow \mathscr{E}'$$

such that

$$\mathscr{W} \cong \mathrm{Coker}\big(\tilde\theta|_{X_k} : \mathscr{E}|_{X_k} \longrightarrow \mathscr{E}'|_{X_k}\big).$$

Verification. According to Lemma 3.3-(2) (applied to the dual of $\mathscr{W}$), we can find $\mathscr{N}$ and $\mathscr{N}'$ in $\langle\inf(\mathscr{V})\rangle_\otimes$ fitting into an exact sequence of $\mathrm{MIC}^\circ(X/k)$-modules:

$$\mathscr{N}|_{X_k} \stackrel{\theta}{\longrightarrow} \mathscr{N}'|_{X_k} \longrightarrow \mathscr{W} \longrightarrow 0.$$

Using Claim 1, θ is the restriction to X_k of an arrow in $\mathrm{MIC}^{\mathrm{se}}(X/R)$

$$\tilde\theta : \inf(\mathscr{N}) \longrightarrow \inf\big(\mathscr{N}'\big).$$

We then take $\mathscr{E} = \inf(\mathscr{N})$ and $\mathscr{E}' = \inf\big(\mathscr{N}'\big)$, and the proof is finished since these do belong to $\langle\mathscr{V}\rangle_\otimes^{\mathrm{s}}$.

Claim 3. Denote by η the composition of functors

$$\mathrm{Rep}_R(G) \longrightarrow \mathrm{Rep}_R(G') \stackrel{\sim}{\longrightarrow} \langle\inf(\mathscr{V})\rangle_\otimes.$$

For each $V \in \mathrm{Obj}(\mathrm{Rep}_k(G_k))$, there exists $N \in \mathrm{Obj}(\mathrm{Rep}_R^\circ(G))$ such that

(1) V is a quotient of N_k and
(2) there exists some $\mathscr{N} \in \mathrm{Obj}(\langle\mathscr{V}\rangle_\otimes)$ such that $\eta(N) = \inf(\mathscr{V})$.

Verification. According to [Se68, Proposition 3, p.41] we can "almost lift" V. Precisely, there exists $E \in \mathrm{Rep}_R^\circ(G)$ and a surjection $E_k \to V$. By means of the equivalence

$$\eta : \mathrm{Rep}_{\mathrm{R}}^\circ(\mathrm{G}) \stackrel{\sim}{\longrightarrow} \langle\inf(\mathscr{V})\rangle_\otimes^{\mathrm{s}}$$

of (49), we can find a diagram in $\mathrm{MIC}^{\mathrm{se}}(X/R)$:

$$\begin{array}{ccc} \mathscr{F} & \longrightarrow & \mathscr{T} \\ \downarrow & & \\ \eta(E), & & \end{array}$$

where $\mathscr{T}$ is some tensor power of $\inf(\mathscr{V})$, the vertical arrow is an epimorphism (in $\mathrm{MIC}^{\mathrm{se}}(X/R)$), and the horizontal arrow is special. In particular, $\mathscr{T} = \inf(\mathscr{T}')$ for some tensor power $\mathscr{T}'$ of $\mathscr{V}$. According to Lemma 3.4, there exists $\mathscr{N} \in \mathrm{MIC}^\circ(X/k)$ and an epimorphism

$$\inf(\mathscr{N}) \longrightarrow \mathscr{F}.$$

Since $\inf(\mathscr{N})$ in fact belongs to $\langle\inf(\mathscr{V})\rangle_\otimes^{\mathrm{s}}$; the above equivalence then produces the desired N, viz. any object of $\mathrm{Rep}_R^\circ(G)$ which is taken by η to $\inf(\mathscr{N})$. Indeed, (2) is verified by construction, and (1) follows from the fact that if $\eta(\theta) : \eta(N) \to \eta(E)$ is an epimorphism of $\mathrm{MIC}^{\mathrm{se}}(X/R)$, then θ is an epimorphism in $\mathrm{Rep}_R(G)$ (between objects of $\mathrm{Rep}_R^\circ(G)$).

Claim 4. The functor

$$\eta_k : \mathrm{Rep}_k(G_k) = \mathrm{Rep}_R(G)_{(k)} \longrightarrow \langle \inf(\mathcal{V})\rangle_{\otimes,(k)}$$

is full.

Verification. Let $\varphi : \eta_k(\mathrm{V}) \to \eta_k\left(V'\right)$ be a morphism in $\mathrm{MIC}^{\circ}(X_k/k)$. It then fits into a commutative diagram

$$\begin{array}{ccc} \eta(N)\otimes k & \xrightarrow{\theta} & \eta(N')\otimes k \\ \downarrow & & \downarrow \\ \eta_k(V) & \xrightarrow[\phi]{} & \eta_k(V'), \end{array}$$

where $\eta(N) = \inf(\mathcal{N})$ and $\eta\left(N'\right) = \inf\left(\mathcal{N}'\right)$ are constructed from Claim 3. Claim 1 gives us a lift

$$\tilde{\theta} : \eta(\mathrm{N}) \longrightarrow \eta\left(\mathrm{N}'\right)$$

of θ. Since $\mathcal{N}$ and $\mathcal{N}'$ belong to $\langle\mathcal{V}\rangle_{\otimes}$, both $\inf(\mathcal{N})$ and $\inf\left(\mathcal{N}'\right)$ lie in $\langle\inf(\mathcal{V})\rangle^{s}_{\otimes}$. Since η is an equivalence between $\mathrm{Rep}^{\circ}_R(G)$ and $\langle\mathcal{V}\rangle^{s}_{\otimes}$, there exists $\sigma : \mathrm{N} \to \mathrm{N}'$ such that $\eta(\sigma) = \tilde{\theta}$. Since the vertical arrows in the above diagram also belong to the image of η_k, the proof of the claim is finished. □

APPENDIX C. RELATIVE DE RHAM COHOMOLOGY FOR FORMAL SCHEMES

C.1. **Formal schemes.** We can express formal schemes and formal morphisms as follows [EGA I, 10.6]:

- Let $\mathfrak{X}$ be a formal scheme and $\mathscr{I} \subset \mathscr{O}_{\mathfrak{X}}$ an ideal of definition ([EGA I, 10.5]). For each $n \in \mathbb{N}$, we set $X_n := (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}/\mathscr{I}^{n+1}})$ and indicate that $\mathfrak{X}$ is the direct limit of the schemes X_n by
$$\mathfrak{X} = \lim_{n\in\mathbb{N}} X_n.$$
The topologically ringed spaces $\mathfrak{X}$ and X_n have the same underlying topological space, so we will not distinguish between a point in $\mathfrak{X}$ or X_n.
- If $f : \mathfrak{X} \to \mathfrak{S}$ is a map of two formal schemes, $\mathscr{I} \subset \mathscr{O}_{\mathfrak{X}}$ and $\mathscr{J} \subset \mathscr{O}_{\mathfrak{S}}$ are ideals of definition such that $f^*(\mathscr{J})\mathscr{O}_{\mathfrak{X}} \subset \mathscr{I}$ and
$$f_n : X_n := (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{I}^{n+1}) \to S_n := (\mathfrak{S}, \mathscr{O}_{\mathfrak{S}}/\mathscr{J}^{n+1}) \tag{51}$$
is the morphism induced by f for each $n \in \mathbb{N}$, then f is expressed as
$$f = \lim_{n\in\mathbb{N}} f_n. \tag{52}$$

A formal scheme admits the largest ideal of definition $\mathscr{I}_0$ [EGA I, 10.5.4], we recall that the *reduced subscheme of definition* X_0 is the ordinary scheme $(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{I}_0)$. Let us recall some definitions from Sections 1 and 2 of [TLR07] as follows:

- A morphism $f : \mathfrak{X} \to \mathfrak{S}$ of formal schemes is of *pseudo finite type* if the induced map on reduced subschemes of definition $f_0 : X_0 \to Y_0$ is of finite type.
- We say that f is *adic* if for some ideal of definition $\mathscr{J} \subset \mathscr{O}_{\mathfrak{S}}$, $f^*(\mathscr{J})\mathscr{O}_{\mathfrak{X}}$ is an ideal of definition of $\mathfrak{X}$. We say that f is *of finite type* if f is pseudo finite type and adic. (Note that every morphism of ordinary schemes of finite type is of finite type according to this definition)

- The morphism f is *smooth* if it is of pseudo finite type and satisfies the following lifting condition: for any affine $\mathfrak{S}-$scheme Z and for each closed subscheme $T \hookrightarrow Z$ given by a square zero ideal $\mathscr{I} \subset \mathscr{O}_Z$ the induced map
$$\mathrm{Hom}_{\mathfrak{S}}(Z, \mathfrak{X}) \longrightarrow \mathrm{Hom}_{\mathfrak{S}}(T, \mathfrak{X})$$
is surjective.

C.2. **Kahler differential on formal schemes.** We give a quick review of differentials on formal schemes in [LNS05, 2.5-2.6].

Let $A \longrightarrow B$ be a continuous homomorphism of adic rings. Let $\mathfrak{b}$ be any ideal of definition in B. Let $\Omega^1_{B/A}$ be the relative $B-$module of differentials and $\Omega^m_{B/A}$ its $m-$th exterior power ($m \geq 0$), we define the $\mathfrak{b}$-adic completion of $\Omega^m_{B/A}$ as follows

$$\widehat{\Omega}^m_{B/A} := \lim(\Omega^m_{B/A} \otimes_B B_i) \tag{53}$$

where $B_i := B/\mathfrak{b}^{i+1}$.

Let $f : \mathfrak{X} \to \mathfrak{S}$ be a morphism of locally noetherian formal schemes. Suppose $\mathscr{I} \subset \mathscr{O}_{\mathfrak{X}}$ and $\mathscr{J} \subset \mathscr{O}_{\mathfrak{S}}$ are the definition ideals such that $f^*(\mathscr{J})\mathscr{O}_{\mathfrak{X}} \subset \mathscr{I}$. By (51) we have the following morphism of ordinary schemes
$$f_n : X_n := (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{I}^{n+1}) \to S_n := (\mathfrak{S}, \mathscr{O}_{\mathfrak{S}}/\mathscr{J}^{n+1}).$$

Let $j_n : X_n \to \mathfrak{X}$ be the canonical closed immersion. For all $i \in \mathbb{Z}$, the *sheaf of i-differentials* of $\mathfrak{X}$ over $\mathfrak{S}$ is the sheaf
$$\widehat{\Omega}^m_{\mathfrak{X}/\mathfrak{S}} := \lim j_{n*}\Omega^m_{X_n/S_n}.$$
This $\widehat{\Omega}^m_{\mathfrak{X}/\mathfrak{S}}$ is independent of the choice of $\mathscr{I}, \mathscr{J}$. For every n there is a natural sheaf homomorphism $\mathscr{O}_{X_n} \to \Omega^1_{X_n/S_n}$, and hence applying j_{n*} and taking inverse limits results in a natural sheaf homomorphism $\widehat{d}_{\mathfrak{X}/\mathfrak{S}} : \mathscr{O}_{\mathfrak{X}} \to \widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{S}}$ (see (52)). If f is of pseudo finite type, then for all i, the sheaf of i-differentials $\widehat{\Omega}^m_{\mathfrak{X}/\mathfrak{S}}$ is a coherent $\mathscr{O}_{\mathfrak{X}}$-module.

C.3. **Some results.**

Lemma C.1. *Let $f : \mathfrak{X} \to \mathfrak{S}$ be a smooth morphism of noetherian formal schemes and $(\mathscr{M}, \widehat{\nabla})$ be an object in* $\mathrm{MIC}(\mathfrak{X}/\mathfrak{S})$. *The functors $\mathrm{H}^i_{\mathrm{dR}}(\mathfrak{X}/\mathfrak{S}, -)$ are the right derived functors of the left exact functor*
$$\mathrm{R}^0_{\mathrm{dR}} f_*(-) : \mathrm{MIC}(\mathfrak{X}/\mathfrak{S}) \longrightarrow \mathrm{MIC}(\mathfrak{S}/\mathfrak{S}) = \text{(quasicoherent sheaves on } \mathfrak{S})$$
$$(\mathscr{M}, \widehat{\nabla}) \mapsto f_*(\mathscr{M}^{\widehat{\nabla}}).$$

Proof. To prove this fact we adapt the proof in [ABC20, 23.2.5].

Claim. Let $(\mathscr{I}, \widehat{\nabla})$ be an injective object in $\mathrm{MIC}(\mathfrak{X})$. Then $\widehat{\Omega}^\bullet_{\mathfrak{X}/\mathfrak{S}} \otimes \mathscr{I}$ is an injective resolution of $\mathscr{I}^{\nabla_{\mathfrak{X}/\mathfrak{S}}}$, and $\mathscr{I}^{\nabla_{\mathfrak{X}/\mathfrak{S}}}$ is a flabby $\mathscr{O}_{\mathfrak{X}}$-module.

Verification. Indeed, since $(\mathscr{I}, \widehat{\nabla})$ is injective, the functor $\mathrm{Hom}_{\mathscr{D}_{\mathfrak{X}}}(-, \mathscr{I})$ is exact, this implies that $\mathscr{H}om_{\mathscr{D}_{\mathfrak{X}}}(-, \mathscr{I})$ is exact. Applying this functor to the Spencer resolution of $f^*\mathscr{D}_{\mathfrak{S}}$
$$Sp^\bullet_{\mathfrak{X}/\mathfrak{S}}(\mathscr{D}_{\mathfrak{X}}) \longrightarrow f^*\mathscr{D}_{\mathfrak{S}} \longrightarrow 0$$

we obtain the exact sequence

$$0 \longrightarrow \mathscr{H}om_{\mathscr{D}_{\mathfrak{X}}}\left(f^*\mathscr{D}_{\mathfrak{S}}, \mathscr{I}\right) \longrightarrow \mathscr{H}om_{\mathscr{D}_{\mathfrak{X}}}\left(Sp^{\bullet}_{\mathfrak{X}/\mathfrak{S}}(\mathscr{D}_X), \mathscr{I}\right)$$

and $\widehat{\Omega}^{\bullet}_{\mathfrak{X}/\mathfrak{S}} \otimes \mathscr{I} \cong \mathscr{H}om_{\mathscr{D}_{\mathfrak{X}}}\left(Sp^{\bullet}_{\mathfrak{X}/\mathfrak{S}}(\mathscr{D}_{\mathfrak{X}}), \mathscr{I}\right)$ is a resolution of $\mathscr{I}^{\nabla_{\mathfrak{X}/\mathfrak{S}}} \cong \mathscr{H}om_{\mathscr{D}_{\mathfrak{X}}}\left(f^*\mathscr{D}_{\mathfrak{S}}, \mathscr{I}\right)$.

To see the injectivity, observe that

$$\mathrm{Hom}_{\mathscr{O}_{\mathfrak{X}}}\left(-, \widehat{\Omega}^{i}_{\mathfrak{X}/\mathfrak{S}} \otimes \mathscr{I}\right) = \mathrm{Hom}_{\mathscr{D}_{\mathfrak{X}}}\left(- \otimes_{\mathscr{O}_{\mathfrak{X}}} Sp^{i}_{\mathfrak{X}/\mathfrak{S}}(\mathscr{D}_{\mathfrak{X}}), (\mathscr{I}, \nabla)\right)$$

is an exact functor, since $Sp^{i}_{\mathfrak{X}/\mathfrak{S}}(\mathscr{D}_{\mathfrak{X}})$ is locally free over $\mathscr{D}_{\mathfrak{X}}$ and $\mathscr{I}$ is injective. Finally, since $\mathscr{I}$ is injective, for any left $\mathscr{D}_{\mathfrak{X}}$-module $\mathscr{G}$ and any open inclusion $j : \mathfrak{U} \longrightarrow \mathfrak{X}$, the canonical inclusion $\mathscr{G}_{\mathfrak{U}} \to \mathscr{G}$ gives a surjective morphism $\mathrm{Hom}_{\mathscr{D}_{\mathfrak{X}}}(\mathscr{G}, \mathscr{I}) \to \mathrm{Hom}_{\mathscr{D}_{\mathfrak{X}}}(\mathscr{G}_{\mathfrak{U}}, \mathscr{I})$, which shows that the sheaf $\mathscr{H}om_{\mathscr{D}_{\mathfrak{X}}}(\mathscr{G}, \mathscr{I})$ is flabby. In particular, using $\mathscr{G} = f^*\mathscr{D}_{\mathfrak{S}}$, we have that $\mathscr{I}^{\nabla_{\mathfrak{X}/\mathfrak{S}}} = \mathscr{H}om_{\mathscr{D}_{\mathfrak{X}}}\left(f^*\mathscr{D}_{\mathfrak{S}}, \mathscr{I}\right)$ is flabby. ▷

We are now ready to prove the lemma, it is sufficient to observe that the functors $\mathrm{H}^{i}_{\mathrm{dR}}(\mathfrak{X}/\mathfrak{S}, (\mathscr{M}, \widehat{\nabla}))$ form an effaceable δ-functor ($\mathscr{I}^{\nabla_{\mathfrak{X}/\mathfrak{S}}}$ is f_*-acyclic, being flabby) and $\mathrm{H}^{0}_{\mathrm{dR}}(\mathfrak{X}/\mathfrak{S}, (\mathscr{M}, \widehat{\nabla}))$ coincides with $\mathrm{R}^{0}_{dR} f_*(\mathscr{M}, \widehat{\nabla})$. Applying [Ha77, III, 1.3A and 1.4] completes the proof. □

Lemma C.2. *Let X be a proper smooth scheme over complete discrete valuation ring A. Let $\mathfrak{X}$ be the completion of X with respect to the adic topology on A. Denote $\mathfrak{S} := \mathrm{Spf}(A)$. The functor*

$$(-)^{\wedge} : \mathrm{MIC}^{\mathrm{coh}}(X/A) \longrightarrow \mathrm{MIC}^{\mathrm{coh}}(\mathfrak{X}/\mathfrak{S}) \qquad (54)$$
$$(\mathscr{M}, \nabla) \mapsto (\widehat{\mathscr{M}}, \widehat{\nabla})$$

yields an equivalence of category. Moreover, the functor $(-)^{\wedge}$ from $\mathrm{MIC}^{\mathrm{se}}(X/A)$ to $\mathrm{MIC}^{\mathrm{se}}(\mathfrak{X}/\mathfrak{S})$ is an equivalence.

Proof. Let $\mathscr{M}$ be an object in $\mathrm{MIC}^{\mathrm{coh}}(X/A)$. Using Grothendieck's variant of connection in [ABC20, Definition 4.2.1], the connection $\mathscr{M}$ can be considered as $(\mathscr{M}, \psi_{\mathscr{M}})$ where $\psi_{\mathscr{M}}$ is an isomorphism of $\mathscr{P}_{X/A}$-module

$$\psi_{\mathscr{M}} : \mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{P}^1_{X/A} \longrightarrow \mathscr{P}^1_{X/A} \otimes_{\mathscr{O}_X} \mathscr{M} \qquad (55)$$

which is identity modulo $\Omega^1_{X/A}$. Thus, the morphism $f : (\mathscr{M}, \psi_{\mathscr{M}}) \to (\mathscr{N}, \psi_{\mathscr{N}})$ is the map such that the following diagram commmutes

$$\begin{array}{ccc} \mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{P}^1_{X/A} & \xrightarrow{\psi_{\mathscr{M}}} & \mathscr{P}^1_{X/A} \otimes_{\mathscr{O}_X} \mathscr{M} \\ \big\downarrow{\scriptstyle f\otimes \mathrm{id}} & & \big\downarrow{\scriptstyle \mathrm{id}\otimes f} \\ \mathscr{N} \otimes_{\mathscr{O}_X} \mathscr{P}^1_{X/A} & \xrightarrow{\psi_{\mathscr{N}}} & \mathscr{P}^1_{X/A} \otimes_{\mathscr{O}_X} \mathscr{N} \end{array} .$$

Let $0 \to (\mathscr{M}, \psi_{\mathscr{M}}) \xrightarrow{f} (\mathscr{N}, \psi_{\mathscr{N}}) \xrightarrow{g} (\mathscr{P}, \psi_{\mathscr{P}}) \to 0$ be an exact sequence in $\mathrm{MIC}^{\mathrm{coh}}(X/A)$. We have the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{P}^1_{X/A} \otimes_{\mathscr{O}_X} \mathscr{M} & \xrightarrow{\mathrm{id}\otimes f} & \mathscr{P}^1_{X/A} \otimes_{\mathscr{O}_X} \mathscr{N} & \xrightarrow{\mathrm{id}\otimes g} & \mathscr{P}^1_{X/A} \otimes_{\mathscr{O}_X} \mathscr{P} & \longrightarrow & 0 \\ & & {\scriptstyle \psi_{\mathscr{M}}}\big\uparrow & & {\scriptstyle \psi_{\mathscr{N}}}\big\uparrow & & {\scriptstyle \psi_{\mathscr{P}}}\big\uparrow & & \\ 0 & \longrightarrow & \mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{P}^1_{X/A} & \xrightarrow{f\otimes \mathrm{id}} & \mathscr{N} \otimes_{\mathscr{O}_X} \mathscr{P}^1_{X/A} & \xrightarrow{g\otimes \mathrm{id}} & \mathscr{P} \otimes_{\mathscr{O}_X} \mathscr{P}^1_{X/A} & \longrightarrow & 0 \end{array} .$$

Since the completion functor is exact (as exactness is a local problem, we can assume X is affine and use the fact that the functor $M \longrightarrow \widehat{M}$ is exact; see 8.1.2 in [FGIKNV05]), we have the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widehat{\mathscr{P}}^1_{X/A}\widehat{\otimes}_{\mathscr{O}_{\mathfrak{X}}}\widehat{\mathscr{M}} & \xrightarrow{\mathrm{id}\widehat{\otimes}\widehat{f}} & \widehat{\mathscr{P}}^1_{X/A}\widehat{\otimes}_{\mathscr{O}_{\mathfrak{X}}}\widehat{\mathscr{N}} & \xrightarrow{\mathrm{id}\widehat{\otimes}\widehat{g}} & \widehat{\mathscr{P}}^1_{X/A}\widehat{\otimes}_{\mathscr{O}_{\mathfrak{X}}}\widehat{\mathscr{P}} & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle\widehat{\psi_{\mathscr{M}}}} & & \uparrow{\scriptstyle\widehat{\psi_{\mathscr{N}}}} & & \uparrow{\scriptstyle\widehat{\psi_{\mathscr{P}}}} & & \\ 0 & \longrightarrow & \widehat{\mathscr{M}}\widehat{\otimes}_{\mathscr{O}_{\mathfrak{X}}}\widehat{\mathscr{P}}^1_{X/A} & \xrightarrow{\widehat{f}\widehat{\otimes}\mathrm{id}} & \widehat{\mathscr{N}}\widehat{\otimes}_{\mathscr{O}_{\mathfrak{X}}}\widehat{\mathscr{P}}^1_{X/A} & \xrightarrow{\widehat{g}\widehat{\otimes}\mathrm{id}} & \widehat{\mathscr{P}}\widehat{\otimes}_{\mathscr{O}_{\mathfrak{X}}}\widehat{\mathscr{P}}^1_{X/A} & \longrightarrow & 0 \end{array} . \tag{56}$$

We remark from [EGA I, Proposition 10.11.7] that the completion $\widehat{\mathscr{P}^1_{X/A}\otimes_{\mathscr{O}_X}\mathscr{M}}$ is completed tensor product $\widehat{\mathscr{P}}^1_{X/A}\widehat{\otimes}_{\mathscr{O}_{\mathfrak{X}}}\widehat{\mathscr{M}}$, and hence the map $\widehat{\mathrm{id}\otimes f}$ is also $\mathrm{id}\widehat{\otimes}\widehat{f}$.

Since $\mathfrak{X} \longrightarrow \mathfrak{S}$ is smooth, we have $\widehat{\mathscr{P}}^1_{X/A} = \mathscr{P}^1_{\mathfrak{X}/\mathfrak{S}}$. Combined with the fact that the functor $\widehat{(-)}$ is exact, this implies that

$$\widehat{\psi_{\mathscr{M}}} : \widehat{\mathscr{M}}\widehat{\otimes}_{\mathscr{O}_{\mathfrak{X}}}\mathscr{P}^1_{\mathfrak{X}/\mathfrak{S}} \longrightarrow \mathscr{P}^1_{\mathfrak{X}/\mathfrak{S}}\widehat{\otimes}_{\mathscr{O}_{\mathfrak{X}}}\widehat{\mathscr{M}}$$

is an isomorphism of $\mathscr{P}^1_{\mathfrak{X}/\mathfrak{S}}$-module, which is the identity modulo $\widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{S}}$. This show that the corresponding connection $(\widehat{\mathscr{M}}, \widehat{\nabla})$ is an object of $\mathrm{MIC}(\mathfrak{X}/\mathfrak{S})$ and the functor 12 is exact. Finally, by applying Grothendieck's existence theorem (Theorem 8.4.2 in [FGIKNV05]) we conclude that the functor (12) yields an equivalence between two categories.

It remains to prove the rest of the lemma. We observe that the functor $(-)^\wedge$ from $\mathrm{MIC}^\circ(X/A)$ to $\mathrm{MIC}^\circ(\mathfrak{X}/\mathfrak{S})$ also yields an equivalence of category, the result follows. □

Theorem C.3. *Let X be a projective smooth variety over complete discrete valuation ring A. Let $\mathfrak{X}$ be the completion of X with respect to the adic topology on A. Denote $\mathfrak{S} := \mathrm{Spf}(A)$. Let $\mathscr{V}$ be an object in $\mathrm{MIC}^{\mathrm{coh}}(X/A)$. Then*

$$\mathrm{H}^i_{\mathrm{dR}}(X/A, \mathscr{V}) \cong \mathrm{H}^i_{\mathrm{dR}}(\mathfrak{X}/\mathfrak{S}, \widehat{\mathscr{V}})$$

for $i \geq 0$.

Proof. The map ϕ also induces the morphism from the spectral sequence

$$E_1^{p,q} = \mathrm{H}^q(X, \Omega^p_{X/A} \otimes \mathscr{V}) \Rightarrow \mathrm{H}^{p+q}_{\mathrm{dR}}(X/A, \mathscr{V})$$

to the spectral sequence

$$E_1^{p,q} = \mathrm{H}^q(\mathfrak{X}, \widehat{\Omega}^p_{\mathfrak{X}/\mathfrak{S}}\widehat{\otimes}\widehat{\mathscr{V}}) \Rightarrow \mathrm{H}^{p+q}_{\mathrm{dR}}(\mathfrak{X}/\mathfrak{S}, \widehat{\mathscr{V}}).$$

Thanks to (3.1.2) in the argument in [TLL99, Proposition 3.1.1], we can see that

$$\mathrm{H}^i(X, \Omega^\bullet_{X/A} \otimes \mathscr{V}) \cong \mathrm{H}^i(\mathfrak{X}, \widehat{\Omega}^\bullet_{\mathfrak{X}/\mathfrak{S}}\widehat{\otimes}\widehat{\mathscr{V}}),$$

and the result follows. □

C.4. **Gauss-Manin connection on de Rham cohomology of relative formal schemes.** Let $f : \mathfrak{X} \to \mathfrak{S}$ be a smooth morphism of smooth $\mathfrak{Z}$-formal scheme in category of locally noetherian formal schemes. Following Katz and Oda ([Ka70], [OK68]) we will define the canonical Gauss-Manin connection on de Rham cohomology sheaf $\mathrm{H}^i_{\mathrm{dR}}(\mathfrak{X}/\mathfrak{S}, (\mathscr{V}, \widehat{\nabla}))$, where $(\mathscr{V}, \widehat{\nabla})$ comes from absolute connection. The exact sequence (see [LNS05, Proposition 2.6.5])

$$0 \to f^*\widehat{\Omega}^1_{\mathfrak{Y}/\mathfrak{Z}} \xrightarrow{\Phi} \widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Z}} \xrightarrow{\Psi} \widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}} \to 0 \tag{57}$$

of sheaves on $\mathfrak{X}$ give rise to a filtration of the complex $\widehat{\Omega}^\bullet_{\mathfrak{X}/\mathfrak{Z}} \otimes \mathscr{V}$. Indeed, we have the filtration of $\widehat{\Omega}^\bullet_{\mathfrak{X}/\mathfrak{Z}}$ as follows

$$\widehat{\Omega}^\bullet_{\mathfrak{X}/\mathfrak{Z}} = F^0(\widehat{\Omega}^\bullet_{\mathfrak{X}/\mathfrak{Z}}) \supset F^1(\widehat{\Omega}^\bullet_{\mathfrak{X}/\mathfrak{Z}}) \supset \dots$$

where $F^i(\widehat{\Omega}^\bullet_{\mathfrak{X}/\mathfrak{Z}})$ is defined by

$$F^i(\widehat{\Omega}^\bullet_{\mathfrak{X}/\mathfrak{Z}}) := \operatorname{image}(f^*\widehat{\Omega}^i_{\mathfrak{S}/\mathfrak{Z}} \otimes \widehat{\Omega}^\bullet_{\mathfrak{X}/\mathfrak{Z}}[-i] \xrightarrow{\wedge} \widehat{\Omega}^\bullet_{\mathfrak{X}/\mathfrak{Z}})$$

Since the sheaves $\widehat{\Omega}^i_{\mathfrak{X}/\mathfrak{Z}}$ and $\widehat{\Omega}^i_{\mathfrak{S}/\mathfrak{Z}}$ on $\mathfrak{X}$ and $\mathfrak{S}$ respectively are locally free (see [LNS05][Proposition 2.6.1]), the local splitting of Sequence (57) give us that

$$(F^i/F^{i+1})(\widehat{\Omega}^\bullet_{\mathfrak{X}/\mathfrak{Z}}) = f^*\widehat{\Omega}^i_{\mathfrak{S}/\mathfrak{Z}} \otimes \widehat{\Omega}^\bullet_{\mathfrak{X}/\mathfrak{Z}}[-i].$$

We define a filtration of the complex $\widehat{\Omega}^\bullet_{\mathfrak{X}/\mathfrak{Z}}$ as follows

$$F^i(\widehat{\Omega}^\bullet_{\mathfrak{X}/\mathfrak{Z}} \otimes \mathcal{V}) := F^i(\widehat{\Omega}^\bullet_{\mathfrak{X}/\mathfrak{Z}}) \otimes \mathcal{V}, \tag{58}$$

the associated graded modules are

$$(F^i/F^{i+1})(\widehat{\Omega}^\bullet_{\mathfrak{X}/\mathfrak{Z}} \otimes \mathcal{V}) = f^*\widehat{\Omega}^i_{\mathfrak{S}/\mathfrak{Z}} \otimes (\widehat{\Omega}^\bullet_{\mathfrak{X}/\mathfrak{Z}}[-i] \otimes \mathcal{V}).$$

Consider now the spectral sequence of the derived functors of f_* applied to filtered complex 58. We have

$$\begin{aligned} E_1^{p,q} = \mathbf{R}^{p+q} f_*(F^p/F^{p+1}) &= \mathbf{R}^{p+q} f_*(f^*(\widehat{\Omega}^p_{\mathfrak{S}/\mathfrak{Z}}) \otimes (\widehat{\Omega}^\bullet_{\mathfrak{X}/\mathfrak{S}}[-p]) \otimes \mathcal{V}) \\ &= \widehat{\Omega}^p_{\mathfrak{S}/\mathfrak{Z}} \otimes \mathbf{R}^q f_*(\widehat{\Omega}^\bullet_{\mathfrak{X}/\mathfrak{S}} \otimes \mathcal{V}) \\ &= \widehat{\Omega}^p_{\mathfrak{S}/\mathfrak{Z}} \otimes \mathrm{H}^q_{\mathrm{dR}}(\mathfrak{X}/\mathfrak{S}, (\mathcal{V}, \widehat{\nabla})). \end{aligned}$$

The second equality follows from the projection formula.

To define cup product for formal schemes, we take the canonical flasque resolutions of the sheaves $\widehat{\Omega}^i_{\mathfrak{X}/\mathfrak{S}} \widehat{\otimes} \mathcal{V}$. Since the canonical resolution is a functor, we get a double complex, and we also get the associated total complex. Thus, the exterior algebra structure on the de Rham complex, allows us to construct products in de Rham cohomology sheaves. Since the filtration of $\widehat{\Omega}_{\mathfrak{X}/\mathfrak{Z}}$ is compatible with the exterior product and the sequence of functor $\mathbf{R}^q f_*$ is multiplicative, it implies that this spectral sequence has a product structure. Namely, we have a family of bilinear maps

$$\begin{aligned} E_r^{p,q} \times E_r^{p',q'} &\longrightarrow E_r^{p+p',q+q'} \\ (e, e') &\mapsto e.e' \end{aligned} \tag{59}$$

such that

i) $e.e' = (-1)^{(p+q)(p'+q')} e'.e$
ii) $d_r(e.e') = d_r(e).e' + (-1)^{p+q} e.d_r(e')$.

Now the map

$$d_1^{0,q} : \mathrm{H}^q_{\mathrm{dR}}(\mathfrak{X}/\mathfrak{S}, (\mathcal{V}, \widehat{\nabla})) \longrightarrow \widehat{\Omega}^p_{\mathfrak{X}/\mathfrak{Z}} \otimes \mathrm{H}^q_{\mathrm{dR}}(\mathfrak{X}/\mathfrak{S}, (\mathcal{V}, \widehat{\nabla}))$$

satisfies i), ii) of (59), then $d_1^{0,q}$ is integrable connection.

In conclusion, we have the following result.

Proposition C.4. *Let $f : \mathfrak{X} \to \mathfrak{S}$ be a smooth morphism of smooth $\mathfrak{Z}$-formal scheme in the category of locally noetherian formal schemes. There exist a canonical integrable connection $\widehat{\nabla}^q_{GM}$ on the de Rham cohomology sheaf* $\mathrm{H}^q_{\mathrm{dR}}(\mathfrak{X}/\mathfrak{S}, (\mathcal{V}, \widehat{\nabla}))$.

Theorem C.5 (Leray spectral sequence)**.** *Let $g : \mathfrak{X} \longrightarrow \mathfrak{S}$ be a smooth morphism of smooth k-formal schemes and $f : \mathfrak{S} \longrightarrow \mathfrak{Z}$ be a smooth morphism of smooth k-formal schemes. Let $(\mathcal{V}, \widehat{\nabla})$ be an object in* $\mathrm{MIC}(\mathfrak{X}/k)$. *Then there exists a spectral sequence of de Rham sheaf cohomology of relative formal schemes:*

$$E_2^{p,q} = R^p_{\mathrm{dR}} f_*\left(R^q_{\mathrm{dR}} g_*\left(\mathcal{V}, \widehat{\nabla}\right), \nabla^q_{\mathfrak{S}}\right) \Longrightarrow R^{p+q}_{\mathrm{dR}}(f \circ g)_*(\mathcal{V}, \widehat{\nabla}),$$

where ∇_X^q *indicates the Gauss-Manin connection on* $R^q_{\mathrm{dR}} g_* \left(\mathcal{V}, \widehat{\nabla}\right)$.

Proof. To prove this lemma we adapt the proof in [ABC20, 23.3.1]. Since $R^q_{\mathrm{dR}} f_*$ is the q-th derived functor of $R^0_{\mathrm{dR}} f_*$, we can express the spectral sequence for the composition of two maps $g : Y \to X$ and $f : X \to S$. Specifically, we have $R^0_{\mathrm{dR}}(f \circ g)_* = R^0_{\mathrm{dR}} f_* \circ R^0_{\mathrm{dR}} g_*$, and $R^0_{\mathrm{dR}} g_*$ maps injective sheaves into flabby sheaves, acyclic for $R^0_{\mathrm{dR}} f_*$. Therefore, we obtain the composition of derived functors:

$$RR^0_{\mathrm{dR}}(f \circ g)_* = RR^0_{\mathrm{dR}} f_* \circ RR^0_{\mathrm{dR}} g_*$$

yielding the corresponding spectral sequence

$$R^p_{\mathrm{dR}} f_* \circ R^q_{\mathrm{dR}} g_* \Longrightarrow R^{p+q}_{\mathrm{dR}}(f \circ g)_*,$$

which completes the argument. □

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