LOG CONTINUITY OF SOLUTIONS OF COMPLEX MONGE-AMPÈRE EQUATIONS

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ABSTRACT. Let X be a compact Kähler manifold with semipositive anticanonical line bundle. Let L be a big and semi-ample line bundle on X and α be the Chern class of L. We give a sufficient condition ensuring that the solution of the complex Monge-Ampère equations in α with L^p right-hand side $(p > 1)$ is $\log^M$-continuous for every constant $M > 0$. As an application, we show that every singular Ricci-flat metric in a semi-ample class in a projective Calabi-Yau surface X is globally $\log^M$-continuous with respect to a smooth metric on X.

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1. INTRODUCTION

Let (X, ω) be a compact Kähler manifold. A cohomology $(1,1)$-class α is said to be semi-positive if α contains a semi-positive smooth form. Let θ be a smooth closed $(1,1)$-form in a big and semi-positive cohomology class. We consider the following complex Monge-Ampère equation

$$(1.1) \qquad (dd^c u + \theta)^n = f\omega^n, \quad \sup_X u = 0,$$

where $f \in L^p$ $(p > 1)$ is a nonnegative function so that $\int_X f\omega^n = \int_X \theta^n$. The regularity of solutions of (1.1) is well-known if θ is Kähler thanks to pioneering works by Yau [45] and Kołodziej [30], and many subsequent papers. We refer to [17, 19, 20, 32, 31, 36, 37, 38, 41, 42, 43] and references therein for details on Hölder continuity of solutions when θ is Kähler.

The focus of our work is the case where θ belongs to a semi-positive and big cohomology class. In this general setting, it is well-known by [8] that the solution u is smooth outside the non-Kähler locus of the cohomology class of θ. By [22, 9] or [18], we know that the equation (1.1) admits a unique continuous solution u on X if the cohomology class of θ is integral (see [24] for more information). The aim of this paper is to quantify this continuity property of solutions. The methods in [22, 9] or [18] seem to be only qualitative. To state our results, we need to introduce some notions.

Let $M > 0$ be a constant. We fix a smooth Riemannian metric $\mathrm{dist}(\cdot, \cdot)$ on X. A function u on X is said to be $\log^M$-continuous if there exists a constant $C_M > 0$ such that

$$|u(x) - u(y)| \leq \frac{C_M}{|\log \mathrm{dist}(\cdot, \cdot)|^M},$$

for every $x, y \in X$. Let K_X be the canonical line bundle of X. Recall that X is Calabi-Yau if $c_1(K_X) = 0$, and X is Fano if $K_X < 0$. A line bundle L on X is said to be semi-ample if L^k

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is base-point free for some large enough integer $k \geq 1$. It is a well-known fact that (see [39, Section 2] for a summary), L is automatically semi-ample if X is a projective Calabi-Yau manifold and L is big and nef.

Let L be a big and semi-ample line bundle on X (hence X is forced to be projective by Moishezon's theorem). Let $d_k := \dim H^0(X, L^k)$ and $\{s_1, \ldots, s_{d_k}\}$ be a basis of $H^0(X, L^k)$. We define $\Phi_k : X \to \mathbb{CP}^{d_k - 1}$ by putting

$$\Phi_k(x) := [s_1(x) : \cdots : s_{d_k}(x)].$$

Observe that Φ_k is a well-defined map outside $B(kL) := \cap_{s \in H^0(X, L^k)} \{s = 0\}$.

We recall $dd^c := i/\pi \partial\bar{\partial}$. Since L is semi-ample, there is $k' > 0$ sufficiently large so that $B(k'L) = \varnothing$. Hence $\Phi_{k'} : X \to \mathbb{CP}^{d_k}$ is a holomorphic map. Since L is big, we can find $k'' > 0$ so that $\Phi_{k''}$ is of maximal rank. Let $k_L := k'k''$. It follows that Φ_{k_L} is a holomorphic map of maximal rank. Let $X' := \Phi_{k_L}(X)$ which is an irreducible analytic subset of dimension n in $\mathbb{CP}^{d_{k_L} - 1}$. By increasing k_L if necessary, we also have that Φ_{k_L} is an algebraic fibre space, *i.e*, the fibers of Φ_{k_L} are connected, and X' is a normal variety (see [33, Theorem 2.1.27]), moreover Φ_{k_L} is bihomorphic outside the non-Kähler locus N of $c_1(L)$, see [7, Theorem A].

Here is our main result in this work giving a partial answer to the above question.

Theorem 1.1. *Assume that K_X^* is semi-positive (where K_X^* is the dual of the canonical line bundle K_X) and X' has only isolated singularities. Let $\theta \in c_1(L)$ be a smooth form. Then the unique solution u of (1.1) is $\log^M$-continuous for every constant $M > 0$.*

Since X' is normal, if $n = 2$, then X' always has isolated singularities. Theorem 1.1 is probably the first known quantitative (global) regularity for solutions of complex Monge-Ampère equations in a semi-positive class. We would like to notice that it was proved in [28] that the solution of the equation $(dd^c u + \omega)^n = e^F \omega^n$ for $e^F \in L^1(\log L)^p$ is $\log^M$-continuous for $M := \min\{\frac{p-n}{n}, \frac{p}{n+1}\}$; see also [25] for a recent development. As far as we can see, the method in [28] or [25] uses crucially the fact that ω is Kähler and it is not clear if this can be extended to semi-positive classes to obtain a $\log^M$-continuity for solutions of (1.1).

Assume that X, L, ω are as in the statement of Theorem 1.1. Hence the non-Kähler locus N of $c_1(L)$ is a proper analytic subset in X; see [6]. Let F be a smooth function on X such that $\int_X e^F \omega^n = \int_X (c_1(L))^n$ and denote by ω_F the (singular) positive $(1,1)$-form on X such that $\omega_F^n = e^F \omega^n$. Recall that ω_F is a genuine Kähler metric on $X \backslash N$.

Corollary 1.2. *Let X, L, ω, N and F be as above and the hypothesis in Theorem 1.1 holds. Then for every constant $M > 0$, there exists a constant $C_M > 0$ such that*

$$d_{\omega_F}(x, y) \leq C_M |\log \text{dist}(x, y)|^{-M},$$

for every $x, y \in X \backslash N$, where d_{ω_F} is the distance induced by ω_F on $X \backslash N$.

In the case where θ is in a Kähler class, one has better estimates; see [25, 34, 44] for details. We are not aware of any previous result similar to Corollary 1.2 for merely semi-ample and big classes. As an immediate consequence of Corollary 1.2, we get the following.

Corollary 1.3. *Let X, L, ω and F be as above and the hypothesis in Theorem 1.1 holds. Assume ω_0 is a (singular) Kähler-Einstein metric in $c_1(L)$. Then ω_0 has a $\log^M$-continuous potential.*

Moreover, if d_{ω_0} denotes the distance induced by ω_0 on $X \backslash N$ then for every constant $M > 0$ there is a constant $C_M > 0$ so that

$$d_{\omega_0}(x, y) \leq C_M |\log \operatorname{dist}(x, y)|^{-M},$$

for every $x, y \in X \backslash N$.

One can apply Corollary 1.3 to the case where X is a Calabi-Yau surface for which the hypothesis of Corollary 1.3 is automatically satisfied. In this case ω_0 is the Ricci-flat metric in $c_1(L)$ which always exists uniquely (see [22]).

We now explain main ideas in the proof of Theorem 1.1. We will need to approximate our smooth solution u by smooth quasi-psh function $(u_\epsilon)_\epsilon$. Using [15] or [16, Theorem 4.12] (analytic approximation for general closed positive $(1,1)$-currents), one obtains $(\theta + \epsilon\omega)$-psh functions u_ϵ so that u_ϵ converges to u in a quantitative way in L^1. However $\|\nabla u_\epsilon\|_{L^\infty}$ grows like $e^{1/\epsilon}$. The fact that u_ϵ is only $(\theta + \epsilon\omega)$-psh and a bad control on $\|\nabla u_\epsilon\|_{L^\infty}$ is not usable in our approach. For this reason, we have to restrict ourselves to the line bundle setting for which a more precise approximation procedure is available. Precisely we will need a modified version of Demailly's analytic approximation of singular (not necessarily Kähler) Hermitian metrics for a line bundle (Theorem 5.8), this is the place where we need to use the hypothesis that X' has only isolated singularities. This regularisation together with Kołodziej's capacity technique will give us a weak Log continuity property for u (see Lemma 6.2). Our second ingredient (Sections 2 and 3) is to say that a function satisfying this weak Log continuity property is indeed Log continuous as desired.

The paper is organized as follows. In Sections 2 and 3, we present important facts about log continuity of functions. In Section 4, we recall some facts about Hölder continuous measures. In Section 5, we present a modified version of Demailly's analytic approximation. The rest of the paper is devoted to the proof of main results.

2. LOG CONTINUITY OF PSEUDOMETRICS

Let Z be a topological space and $d : Z^2 \to \mathbb{R}_{\geq 0}$ be a function. Let $B \geq 1$ be a constant. We say that d is *a B-pseudometric on* Z if the following holds:

(i) d is symmetric, i.e, $d(x, y) = d(y, x)$,

(ii) d is continuous on Z^2,

(iii) for every $x_1, \ldots, x_m \in Z$, one has

$$d(x_1, x_m) \leq B \sum_{j=1}^{m-1} d(x_j, x_{j+1}).$$

Lemma 2.1. *Let $U \subset \mathbb{R}^m$ be a bounded convex domain ($m \geq 2$). Let $B \geq 1$ be a constant. Let $d : U \times U \to [0, \infty)$ be a B-pseudometric satisfying the following condition: there exist constants $\alpha > 0$, $D > 1$ and $C_0 > 0$ such that*

$$(2.1) \qquad d(x, y) \leq \frac{C_0}{|\log|x - y||^\alpha},$$

for every $x, y \in U$ with $|x - y|^D \leq \min\{\operatorname{dist}(x, \partial U), \operatorname{dist}(y, \partial U)\}$, where

$$\operatorname{dist}(w, \partial U) = \inf\{|w - \xi| : \xi \in \partial U\}.$$

Then, there exists a constant $C > 0$ depending only on B, C_0, α, D and U such that

$$d(x, y) \leq \frac{C}{|\log|x - y||^\alpha},$$

for every $x, y \in U$.

Proof. Without loss of generality, we can assume that $diam(U) \leq 1$. In particular, $|x-y|^D \leq |x - y|$ for every $x, y \in U$.

Fix $a \in U$ and denote $r = \operatorname{dist}(a, \partial U)$. The desired assertion is clear if we have either $|x - y| \geq r/2$ or

$$\min\{\operatorname{dist}(x, \partial U), \operatorname{dist}(y, \partial U)\} \geq r/2 \geq |x - y|$$

(by (2.1)). Consider now the case where

$$|x - y| \leq r/2 \quad \text{and} \quad \min\{\operatorname{dist}(x, \partial U), \operatorname{dist}(y, \partial U)\} < r/2.$$

Thus, we have $\max\{|x - a|, |y - a|\} > |x - y|$. Without loss of generality, we can assume that $|x - a| > |x - y| := \delta > 0$. Set

$$x_0 = \frac{(|x - a| - \delta)x}{|x - a|} + \frac{\delta a}{|x - a|}.$$

In other words, x_0 is a point in $[x, a]$ satisfying $|x - x_0| = \delta$. Since U is convex, we have

$$(2.2) \qquad \operatorname{dist}(x_0, \partial U) \geq \frac{(|x - a| - \delta)\operatorname{dist}(x, \partial U)}{|x - a|} + \frac{\delta \operatorname{dist}(a, \partial U)}{|x - a|} \geq r\delta.$$

For every $k \in \mathbb{Z}^+$, we denote by x_k the point in $[x, x_0]$ satisfying $|x - x_k| = \delta^{D^k}$. Then, we have

$$\operatorname{dist}(x_k, \partial U) \geq \frac{\operatorname{dist}(x_0, \partial U)|x_k - x|}{|x - x_0|} \geq r\delta^{D^k} \quad \text{and} \quad |x_k - x_{k-1}| \leq \delta^{D^{k-1}}.$$

Put $M = \left[\frac{1}{r}\right] + 1$, where $[\cdot]$ is the greatest integer function. For every $l = 0, ..., M$ and $k \in \mathbb{Z}^+$, we denote

$$x_{k,l} = x_{k-1} + \frac{l(x_k - x_{k-1})}{M}.$$

Then $|x_{k,l} - x_{k,l+1}| \leq \frac{\delta^{D^{k-1}}}{M}$. Moreover, since $\operatorname{dist}(., \partial U)$ is a concave function on U, we have

$$\operatorname{dist}(x_{k,l}, \partial U) \geq \min\{\operatorname{dist}(x_k, \partial U), \operatorname{dist}(x_{k-1}, \partial U)\} \geq r\delta^{D^k} \geq \frac{\delta^{D^k}}{M}.$$

Therefore, by the condition (2.1), we get

$$d(x_{k,l}, x_{k,l+1}) \leq \frac{C_0}{|\log|x_{k,l} - x_{k,l+1}||^\alpha} \leq \frac{C_0}{D^{(k-1)\alpha}|\log\delta|^\alpha}.$$

Thus, we have

$$d(x_k, x_{k-1}) \leq B \sum_{l=0}^{M-1} d(x_{k,l}, x_{k,l+1}) \leq \frac{BC_0 M}{D^{(k-1)\alpha}|\log\delta|^\alpha}.$$

Hence

$$d(x_k, x_0) \leq B \sum_{j=1}^{k} d(x_j, x_{j-1}) \leq \frac{B^2 C_0 M}{|\log\delta|^\alpha} \sum_{j=1}^{k} D^{-(j-1)\alpha} \leq \frac{C_1}{|\log\delta|^\alpha},$$

where $C_1 = \dfrac{B^2 C_0 M \, D^{-\alpha}}{1 - D^{-\alpha}} = \dfrac{B^2 C_0 M}{D^\alpha - 1}$.

Since d is continuous on $U \times U$, one gets

(2.3) $$d(x, x_0) = \lim_{k \to \infty} d(x_k, x_0) \leq \frac{C_1}{|\log\delta|^\alpha}.$$

Since $|y - x_0| \leq |x - y| + |x - x_0| \leq 2\delta$, by using the same argument as above, we also have

(2.4) $$d(y, x_0) \leq \frac{C_2}{|\log\delta|^\alpha},$$

where $C_2 > 0$ depends only on B, C_0, M, D and α.

Combining (2.3) and (2.4), we get

$$d(x, y) \leq B(d(x, x_0) + d(y, x_0)) \leq \frac{B(C_1 + C_2)}{|\log\delta|^\alpha} = \frac{B(C_1 + C_2)}{|\log|x - y||^\alpha}.$$

The proof is completed. $\qquad\square$

Lemma 2.2. *Let $U \subset \mathbb{R}^m$ be a bounded convex domain ($m \geq 2$). Let $B \geq 1$ be a constant. Assume $d : U \times U \to [0, \infty)$ is a B-pseudometric satisfying the following condition: there exist constants $\alpha > 1$ and $C_0 > 0$ such that*

(2.5) $$d(x, y) \leq \frac{C_0}{|\log|x - y||^\alpha},$$

for every $x, y \in U$ with $|x - y| \leq \min\{\mathrm{dist}(x, \partial U), \mathrm{dist}(y, \partial U)\}$, where

$$\mathrm{dist}(w, \partial U) = \inf\{|w - \xi| : \xi \in \partial U\}.$$

Then, there exists a constant $C > 0$ depending only on B, C_0, α, D and U such that

$$d(x, y) \leq \frac{C}{|\log|x - y||^{\alpha-1}},$$

for every $x, y \in U$.

Proof. We will use the same method as in the proof of Lemma 2.1. Without loss of generality, we can assume that there exists $a \in U$ such that $r = \mathrm{dist}(a, \partial U) \geq 1$. We only need to consider the case where $|x - y| \leq r/2 \quad \text{and} \quad \min\{\mathrm{dist}(x, \partial U), \mathrm{dist}(y, \partial U)\} < r/2$. In this case, we have $\max\{|x - a|, |y - a|\} > |x - y|$. We can assume that $|x - a| > |x - y| := \delta > 0$. Set

$$x_0 = \frac{(|x - a| - \delta)x}{|x - a|} + \frac{\delta a}{|x - a|}.$$

In other words, x_0 is a point in $[x, a]$ satisfying $|x - x_0| = \delta$. Since U is convex, we have

$$(2.6) \qquad \operatorname{dist}(x_0, \partial U) \geq \frac{(|x - a| - \delta)\operatorname{dist}(x, \partial U)}{|x - a|} + \frac{\delta \operatorname{dist}(a, \partial U)}{|x - a|} \geq r\delta.$$

For every $k \in \mathbb{Z}^+$, we denote by x_k the point in $[x, x_0]$ satisfying $|x - x_k| = 2^{-k}\delta$. Then, we have

$$\operatorname{dist}(x_k, \partial U) \geq \frac{\operatorname{dist}(x_0, \partial U)|x_k - x|}{|x - x_0|} \geq r2^{-k}\delta \quad \text{and} \quad |x_k - x_{k-1}| \leq 2^{-k}\delta.$$

By the condition (2.5), we have

$$d(x_k, x_{k-1}) \leq \frac{C_0}{|\log|x_k - x_{k-1}||^\alpha} \leq \frac{C_0}{(|\log \delta| + k \log 2)^\alpha},$$

for every $k \in \mathbb{Z}^+$. Hence

$$d(x_k, x_0) \leq B \sum_{j=1}^{k} d(x_j, x_{j-1}) \leq \sum_{j=1}^{k} \frac{BC_0}{(|\log \delta| + j \log 2)^\alpha} \leq \frac{BC_0}{\log 2} \int_{\log|\delta|}^{\infty} \frac{dt}{t^\alpha} \leq \frac{C_1}{|\log \delta|^{\alpha-1}},$$

where $C_1 = \frac{BC_0}{(\alpha-1)\log 2}$.

Since d is continuous on $U \times U$, one has

$$(2.7) \qquad d(x, x_0) = \lim_{k \to \infty} d(x_k, x_0) \leq \frac{C_1}{|\log \delta|^{\alpha-1}}.$$

Since $|y - x_0| \leq |x - y| + |x - x_0| \leq 2\delta$, by using the same argument as above, we also have

$$(2.8) \qquad d(y, x_0) \leq \frac{C_2}{|\log \delta|^{\alpha-1}},$$

where $C_2 > 0$ depends only on B, C_0 and α. Combining (2.7) and (2.8), we get

$$d(x, y) \leq B(d(x, x_0) + d(y, x_0)) \leq \frac{B(C_1 + C_2)}{|\log \delta|^{\alpha-1}} = \frac{B(C_1 + C_2)}{|\log|x - y||^{\alpha-1}}.$$

The proof is completed. $\qquad \square$

Proposition 2.3. *Let $N_1, N_2..., N_p$ be affine subspaces of $\mathbb{R}^m$ such that $\operatorname{codim}(N_j) \geq 2$ for every $j = 1, ..., p$. Denote $N = \cup_{j=1}^{p} N_j$. Let $B \geq 1$ be a constant. Let $\alpha > 0$, $D \geq 1$ and $C_0 > 0$ be constants. Let d be a B-pseudometric on $\mathbb{B}^m \setminus N$ satisfying one of the following conditions*

(i) *$D > 1$ and*

$$(2.9) \qquad d(x, y) \leq \frac{C_0}{|\log|x - y||^\alpha},$$

for every $x, y \in \mathbb{B}^m \setminus N$ with $|x - y|^D \leq \min\{\operatorname{dist}(x, N), \operatorname{dist}(y, N)\}$.

(ii) *$D = 1$ and*

$$(2.10) \qquad d(x, y) \leq \frac{C_0}{|\log|x - y||^{\alpha+1}},$$

for every $x, y \in \mathbb{B}^m \setminus N$ with $|x - y| \leq \min\{\operatorname{dist}(x, N), \operatorname{dist}(y, N)\}$.

Then, there exists $C > 0$ depending only on B, C_0, α, D, N and m such that

$$d(x, y) \leq \frac{C}{|\log|x - y||^{\alpha}},$$

for every $x, y \in \mathbb{B}^m \setminus N$.

Proof. We will give the proof for the first case where (i) is satisfied. The second case is similar (use Lemma 2.2 in place of Lemma 2.1). Recall that d is a continuous functionon $(\mathbb{B}^m \setminus N) \times (\mathbb{B}^m \setminus N)$. Let H_j be a hyperplane containing N_j for $j = 1, ..., p$, and denote $H = \cup_{j=1}^{p} H_j$. Observe that the connected components of $\mathbb{B}^m \setminus H$ are bounded convex subsets of $\mathbb{R}^m$. Moreover, if U is a connected component of $\mathbb{B}^m \setminus H$ then by (2.9), u satisfies the condition (2.1) in Lemma 2.1.

Let $x, y \in \mathbb{B}^m \setminus N$. We distinguish into three cases.

Case 1: there exists a connected component U of $\mathbb{B}^m \setminus H$ such that $x, y \in \overline{U} \setminus N$.
In this case, by Lemma 2.1 and by the continuity of d, we have

$$d(x, y) \leq \frac{C_U}{|\log|x - y||^{\alpha}},$$

where $C_U > 0$ is a constant depending only on B, C_0, α, D and U.

Case 2: $[x, y] \cap H \neq \emptyset$ but $[x, y] \cap N = \emptyset$.
In this case, there exist connected components $U_1, U_2, ..., U_k$ of $\mathbb{B}^m \setminus H$ and $x_0, x_1, x_2, ..., x_k \in [x, y]$ such that $x_0 = x \in \overline{U_1}$, $x_k = y \in \overline{U_k}$ and $x_j \in \partial U_j \cap \partial U_{j+1}$ for every $j = 1, ..., k - 1$. Using the result in Case 1, we have

$$\begin{aligned}
d(x, y) \leq B \sum_{j=1}^{k} d(x_j, x_{j-1}) &\leq \sum_{j=1}^{k} \frac{BC_{U_j}}{|\log|x_j - x_{j-1}||^{\alpha}} \\
&\leq \frac{kBC_1}{|\log|x - y||^{\alpha}} \\
&\leq \frac{(p + 1)BC_1}{|\log|x - y||^{\alpha}},
\end{aligned}$$

where $C_1 = \sup\{C_U : U \text{ is a connected component of } \mathbb{B}^m \setminus N\}$.

Case 3: $[x, y] \cap N \neq \emptyset$.
Denote $f(t) = tx + (1 - t)y$, $0 \leq t \leq 1$. Then, there exist $0 < k \leq p$ and $0 < t_1 < t_2 < ... < t_k < 1$ such that

$$[x, y] \cap N = \{f(t_j) : j = 1, ..., k\}.$$

By Lemma 2.4 below, for every $j = 1, ..., k$ and for every $0 < \epsilon \ll 1$, there exists a pieceweise linear curve $l = a_0 a_1 ... a_{4^p}$ with $a_0 = f(t_j + \epsilon)$ and $a_{4^p} = f(t_j - \epsilon)$ such that l does not intersect N and

$$L(l) \leq C_2 |f(t_j + \epsilon) - f(t_j - \epsilon)| = 2C_2\epsilon|x - y|,$$

where $C_2 \geq 1$ is a constant depending only on p. Therefore, by the result in Case 2, we have

$$(2.11) \qquad\qquad d(f(t_j + \epsilon), f(t_j - \epsilon)) = O(\epsilon).$$

Denote $t_0 = 0$ and $t_{k+1} = 1$. By Case 2 and by (2.11), we have

$$d(x, y) = \lim_{\epsilon \to 0+} d(f(t_0 + \epsilon), f(t_{k+1} - \epsilon))$$

$$\leq \limsup_{\epsilon \to 0+} B \sum_{j=0}^{k} d(f(t_j + \epsilon), f(t_{j+1} - \epsilon)) + \limsup_{\epsilon \to 0+} B \sum_{j=1}^{k} d(f(t_j + \epsilon), f(t_j - \epsilon))$$

$$\leq \limsup_{\epsilon \to 0+} \sum_{j=0}^{k} \frac{B^2 C_1 (p+1)}{|\log|f(t_j + \epsilon) - f(t_{j+1} - \epsilon)||^\alpha}$$

$$\leq \frac{B^2 C_1 (p+1)^2}{|\log|x - y||^\alpha}.$$

The proof is completed. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

The following lemma plays also an important role in our proof later.

Lemma 2.4. *Let $N_1, N_2, ..., N_k$ be affine subspaces of $\mathbb{R}^m$ such that $\operatorname{codim}(N_j) \geq 2$ for every $j = 1, ..., k$. Denote $N = \cup_{j=1}^{k} N_j$. Then, there is a constant $C \geq 1$ depending only on k (and m) satisfying the following property: for every $x, y \in \mathbb{R}^m \setminus N$, there is a polygonal chain $l = a_0 a_1 ... a_{4^k}$ with $a_0 = x$ and $a_{4^k} = y$ such that*

$$C \operatorname{dist}(\xi, N) \geq \min\{\operatorname{dist}(x, N), \operatorname{dist}(y, N)\},$$

for every $\xi \in \cup_{s=0}^{4^k-1}[a_s, a_{s+1}]$, and

$$L(l) \leq C|x - y|,$$

where $L(l) = |a_0 - a_1| + |a_1 - a_2| + ... + |a_{4^k-1} - a_{4^k}|$ is the length of l.

In order to prove Lemma 2.4, we need the following elementary lemma:

Lemma 2.5. *Let N be an affine subspace of $\mathbb{R}^m$ with $\operatorname{codim} N \geq 2$. Let $r \geq 1$ be a constant. Then, for every $x, y \in \mathbb{R}^m \setminus N$, there exists $w \in \mathbb{R}^m \setminus N$ such that*

$$|x - w| + |w - y| \leq 3|x - y|,$$

and

$$2r \operatorname{dist}(\xi, N) \geq \min\{\operatorname{dist}(x, N), \operatorname{dist}(y, N)\} \geq r \operatorname{dist}(\xi, [x, y]),$$

for every $\xi \in [x, w] \cup [w, y]$.

Proof. Observe that the function $\operatorname{dist}(\cdot, N)$ is convex on $\mathbb{R}^m$. Indeed, for every $a, b \in \mathbb{R}^m$, there exist $a_0, b_0 \in \mathbb{N}$ such that $|a - a_0| = \operatorname{dist}(a, N)$ and $|b - b_0| = \operatorname{dist}(b, N)$. Hence, if $\eta = \alpha a + (1 - \alpha)b$ for some $\alpha \in [0, 1]$ then

$$\alpha \operatorname{dist}(a, N) + (1 - \alpha) \operatorname{dist}(b, N) = \alpha|a - a_0| + (1 - \alpha)|b - b_0|$$
$$\geq |\alpha(a - a_0) + (1 - \alpha)(b - b_0)|$$
$$= |\eta - (\alpha a_0 + (1 - \alpha)b_0)| \geq \operatorname{dist}(\eta, N).$$

Let $R := \min\{\operatorname{dist}(x, N), \operatorname{dist}(y, N)\}$. If $\operatorname{dist}(\eta, N) \geq R/(2r)$ for every $\eta \in [x, y]$, then $w := x$ satisfies the desired property. Assume, from now on, that there is a point $\eta \in [x, y]$ such that $\operatorname{dist}(\eta, N) \leq R/(2r) \leq R/2$. We deduce that

(2.12) $\quad |x - y| = |x - \eta| + |\eta - y| \geq \operatorname{dist}(x, N) - \operatorname{dist}(\eta, N) + \operatorname{dist}(y, N) - \operatorname{dist}(\eta, N) \geq R.$

We distinguish into three cases

Case 1: Either $[x, y]$ is parallel to N or the line passing through x, y intersects N but $[x, y] \cap N = \varnothing$.
In this case, we can take $w := x$.

Case 2: $[x, y] \cap N \neq \varnothing$.
Since $\operatorname{codim} N \geq 2$, there exists a hyperplane $\tilde{N}$ containing x, y, N. Let $w_0 = [x, y] \cap N$. Let w be a point in $\mathbb{R}^m \backslash \tilde{N}$ so that $|w - w_0| = \frac{R}{r}$ and $[w, w_0]$ is orthogonal to $\tilde{N}$. We have

$$|w - x| + |w - y| \leq |x - w_0| + 2|w - w_0| + |y - w_0| \leq |x - y| + 2R \leq 3|x - y|,$$

where the last estimate holds due to (2.12).
Moreover, if $\xi \in [x, w] \cup [y, w]$ then

$$\operatorname{dist}(\xi, [x, y]) \leq \operatorname{dist}(w, [x, y]) = |w - w_0| = \frac{R}{r}.$$

Let $x_0 \in N$ such that $|x - x_0| = \operatorname{dist}(x, N)$. If $\xi \in [x, w]$ then $\xi = \alpha x + (1 - \alpha)w$ for some $\alpha \in [0, 1]$. Since $x - x_0 \perp N$ and $w - w_0 \perp \tilde{N}$, we have

$$\operatorname{dist}(\xi, N) = |\xi - \alpha x_0 - (1 - \alpha)w_0| = \sqrt{\alpha^2 |x - x_0|^2 + (1 - \alpha)^2 |w - w_0|^2} \geq \frac{R}{2r}.$$

Similarly, if $\xi \in [y, w]$ then we also have $\operatorname{dist}(\xi, N) \geq \frac{R}{2r}$. Then, w satisfies the desired properties.

Case 3: $[x, y]$ is not parallel to N and the line d passing through x, y does not intersect N.
In this case, there exist $w_1 \in N$, $w_2 \in d$ such that $[w_1, w_2]$ is orthogonal to N and d, and $|w_1 - w_2| = \min\{|z - z'| : z \in d, z' \in N\}$. Using the convexity of $d(\cdot, N)$ and the fact that there exists $\eta \in [x, y]$ with $d(\eta, N) < R/2$, we deduce that $d(\xi, N) > R/2$ for every $\xi \in d \backslash [x, y]$. Consequently $w_2 \in [x, y]$.

Let w be the point in the line passing through w_1, w_2 such that w_2 lies between w_1 and w, and $|w - w_2| = R/r$. We check that w satisfies the required properties. Let $\xi \in [w, x]$. Write $\xi = \alpha x + (1 - \alpha)w$ for some constant $\alpha \in [0, 1]$. Let ξ_0, x_0 be points in N such that $[x, x_0]$ and $[\xi, \xi_0]$ are orthogonal to N. We have

$$\xi_0 = \alpha x_0 + (1 - \alpha)w_1.$$

Compute

$$\begin{aligned} |\xi - \xi_0|^2 &= |\alpha(x - x_0) + (1 - \alpha)(w - w_1)|^2 \\ &= \alpha^2 |x - x_0|^2 + (1 - \alpha)^2 |w - w_1|^2 + 2\alpha(1 - \alpha)\langle x - x_0, w - w_1 \rangle. \end{aligned}$$

Recall that $w - w_1$ is both orthogonal to N and d. It follows that

$$\langle x - x_0, w - w_1 \rangle = \langle x - w_2, w - w_1 \rangle + \langle w_2 - w_1, w - w_1 \rangle + \langle w_1 - x_0, w - w_1 \rangle$$

which is equal to $\langle w_2 - w_1, w - w_1 \rangle \geq 0$. Hence we obtain

$$|\xi - \xi_0|^2 \geq \alpha^2 |x - x_0|^2 + (1 - \alpha)^2 |w - w_1|^2$$

$$\geq \alpha^2 R^2 + (1 - \alpha)^2 R^2/r^2 \geq \frac{R^2}{2r^2}.$$

Since $\operatorname{dist}(\xi, N) = |\xi - \xi_0|$, we infer

$$2 \operatorname{dist}(\xi, N) \geq R/r.$$

On the other hand, we have

$$\operatorname{dist}(\xi, [x, y]) \leq \operatorname{dist}(w, [x, y]) \leq |w - w_2| = R/r.$$

We obtain a similar inequalities if $\xi \in [w, y]$. Finally, observe

$$|w - x| + |w - y| \leq |w - w_2| + |x - w_2| + |w - w_2| + |w_2 - y| = 2R + |x - y| \leq 3R \leq 3|x - y|,$$

because $w_2 \in [x, y]$ and we used here (2.12). Thus w satisfies the desired properties.

This finishes the proof. $\qquad \square$

Proof of Lemma 2.4. We will use induction in k. The case $k = 1$ is an immediate corollary of Lemma 2.5. Assume that Lemma 2.4 is true for $k = k_0$. We will show that it is also true for $k = k_0 + 1$.

Denote $N' = N_1 \cup N_2 \cup ... \cup N_{k_0}$ and $N = N_1 \cup N_2 \cup ... \cup N_{k_0+1}$. Let $x, y \in \mathbb{R}^m \setminus N$, $x \neq y$. By the induction assumption, there exists a polygonal chain $l_0 = a_0 a_1 ... a_{4^{k_0}}$ with $a_0 = x$ and $a_{4^{k_0}} = y$ such that

$$(2.13) \qquad\qquad L(l_0) \leq C_0 |x - y|,$$

and

$$(2.14) \qquad\qquad C_0 \operatorname{dist}(\xi, N') \geq \min\{\operatorname{dist}(x, N'), \operatorname{dist}(y, N')\},$$

for every $\xi \in \cup_{s=1}^{4^{k_0}} [a_{s-1}, a_s]$, where $C_0 \geq 1$ is a constant depending only on k_0 and m.

We will construct a polygonal chain $l = b_0 b_1 ... b_{4^{k_0+1}}$ satisfying the conditions as in Lemma 2.4. Denote

$$A := \min\{\operatorname{dist}(x, N), \operatorname{dist}(y, N)\}.$$

If $2C_0 \operatorname{dist}(\xi, N) \geq A$ for every $\xi \in \cup_{s=1}^{4^{k_0}} [a_{s-1}, a_s]$ then we can choose $l = l_0$ and $C = 2C_0$. It remains to consider the case where $2C_0 \operatorname{dist}(\xi_0, N) < A$ for some $\xi_0 \in \cup_{s=1}^{4^{k_0}} [a_{s-1}, a_s]$. In this case, we have

$$(2.15) \quad L(l_0) \geq |x - \xi_0| + |y - \xi_0| \geq (\operatorname{dist}(x, N) - \operatorname{dist}(\xi_0, N)) + (\operatorname{dist}(y, N) - \operatorname{dist}(\xi_0, N)) \geq A.$$

For every $s = 0, ..., 4^{k_0}$, we define b_{4s} as follows

- If $2C_0 \operatorname{dist}(a_s, N) \geq A$ then we put $b_{4s} = a_s$;
- If $2C_0 \operatorname{dist}(a_s, N) < A$ then we choose $b_{4s} \in \mathbb{R}^m$ such that the vector $b_{4s} - a_s$ is perpendicular to N_{k_0+1} and

$$(2.16) \qquad \operatorname{dist}(b_{4s}, N_{k_0+1}) = |a_s - b_{4s}| + \operatorname{dist}(a_s, N_{k_0+1}) = \frac{A}{2C_0}.$$

Thus we have

$$(2.17) \qquad |b_{4s} - b_{4s+4}| \leq |a_s - a_{s+1}| + |a_s - b_{4s}| + |a_{s+1} - b_{4s+4}| \leq |a_s - a_{s+1}| + \frac{A}{C_0},$$

and

$$(2.18) \qquad \operatorname{dist}(\xi, [a_s, a_{s+1}]) \leq \max\{|b_{4s} - a_s|, |b_{4s+4} - a_{s+1}|\} \leq \frac{A}{2C_0},$$

for each $\xi \in [b_{4s}, b_{4s+4}]$ and for every $s = 0, 1, ..., 4^{k_0} - 1$.

Combining (2.14) and (2.18), we get

$$(2.19) \qquad \text{dist}(\xi, N') \geq \inf_{\eta \in [a_s, a_{s+1}]} \text{dist}(\eta, N') - \text{dist}(\xi, [a_s, a_{s+1}]) \geq \frac{A}{2C_0},$$

for each $\xi \in [b_{4s}, b_{4s+4}]$ and for every $s = 0, 1, ..., 4^{k_0} - 1$.

We will find b_{4s+1}, b_{4s+2} and b_{4s+3} such that

(i) $\sum_{j=4s}^{4s+3} |b_j - b_{j+1}| \leq 3|a_s - a_{s+1}| + \frac{2A}{C_0}$;
(ii) $\text{dist}(\xi, N) \geq \frac{A}{8C_0}$ for every $\xi \in \cup_{j=4s}^{4s+3} [b_j, b_{j+1}]$.

We distinguish into three cases.

Case 1: $\text{dist}(\xi, N_{k_0+1}) \geq \frac{A}{4C_0}$ for all $\xi \in [b_{4s}, b_{4s+4}]$.
In this case, we put $b_{4s+1} = b_{4s+2} = b_{4s+3} = b_{4s+4}$. It follows from (2.17) and (2.19) that the conditions (i) and (ii) are satisfied.

Case 2: $\text{dist}(\xi_0, N_{k_0+1}) < \frac{A}{4C_0}$ for some $\xi_0 \in [b_{4s}, b_{4s+4}]$ and either $a_s \neq b_{4s}$ or $a_{s+1} \neq b_{4s+4}$.
In this case, we have

$$\min\{\text{dist}(b_{4s}, N_{k_0+1}), \text{dist}(b_{4s+4}, N_{k_0+1})\} = \frac{A}{2C_0}.$$

By Lemma 2.5, we can choose $q \in \mathbb{R}^n$ such that

$$(2.20) \qquad |b_{4s} - q| + |b_{4s+4} - q| \leq 3|b_{4s} - b_{4s+4}|$$

and

$$(2.21) \qquad 4\,\text{dist}(\xi, N_{k_0+1}) \geq \frac{A}{2C_0} \geq 2\,\text{dist}(\xi, [b_{4s}, b_{4s+4}]),$$

for every $\xi \in [b_{4s}, q] \cup [q, b_{4s+4}]$.

By (2.19) and (2.21), we have

$$(2.22) \qquad \text{dist}(\xi, N') \geq \inf_{\eta \in [b_{4s}, b_{4s+4}]} \text{dist}(\eta, N') - \text{dist}(\xi, [b_{4s}, b_{4s+4}]) \geq \frac{A}{4C_0}.$$

for every $\xi \in [b_{4s}, q] \cup [q, b_{4s+4}]$.

Put $b_{4s+1} = b_{4s+2} = b_{4s+3} = q$. It follows from (2.17) and (2.20) that (i) is satisfied. By (2.21) and (2.22), we also get (ii).

Case 3: $\text{dist}(\xi_0, N_{k_0+1}) < \frac{A}{4C_0}$ for some $\xi_0 \in [b_{4s}, b_{4s+4}]$ and $a_j = b_{4j}$ for $j = s, s + 1$.
In this case, we choose $b_{4s+2} \in \mathbb{R}^m$ such that the vector $b_{4s} - a_s$ is perpendicular to N_{k_0+1} and

$$(2.23) \qquad \text{dist}(b_{4s+2}, N_{k_0+1}) = |b_{4s+2} - b_{4s}| + \text{dist}(\xi_0, N_{k_0+1}) = \frac{A}{4C_0}.$$

Similar to (2.17) and (2.18) (and note that $a_j = b_{4j}$ for $j = s, s + 1$), we have

$$(2.24) \qquad |b_{4s} - b_{4s+2}| + |b_{4s+2} - b_{4s+4}| \leq |b_{4s} - b_{4s+4}| + \frac{A}{2C_0} = |a_s - a_{s+1}| + \frac{A}{2C_0},$$

and

$$(2.25) \qquad \text{dist}(\xi, [a_s, a_{s+1}]) = \text{dist}(\xi, [b_{4s}, b_{4s+4}]) \leq \frac{A}{4C_0},$$

for every $\xi \in [b_{4s}, b_{4s+2}] \cup [b_{4s+2}, b_{4s+4}]$.

Combining (2.14) and (2.25), we get

$$(2.26) \qquad \operatorname{dist}(\xi, N') \geq \frac{3A}{4C_0},$$

for every $\xi \in [b_{4s}, b_{4s+2}] \cup [b_{4s+2}, b_{4s+4}]$.

By using Lemma 2.5 for $[b_{4s}, b_{4s+2}]$ and $[b_{4s+2}, b_{4s+4}]$, we can choose b_{4s+1} and b_{4s+3} such that

$$(2.27) \qquad \sum_{j=4s}^{4s+3} |b_j - b_{j+1}| \leq 3(|b_{4s} - b_{4s+2}| + |b_{4s+2} - b_{4s+4}|),$$

and

$$(2.28) \qquad 2\operatorname{dist}(\xi, N_{k_0+1}) \geq \frac{A}{4C_0} \geq \operatorname{dist}(\xi, [b_{4s}, b_{4s+2}] \cup [b_{4s+2}, b_{4s+4}]),$$

for every $\xi \in \cup_{j=4s}^{4s+3} [b_j, b_{j+1}]$.

By (2.26) and (2.28), we have

$$(2.29) \qquad \operatorname{dist}(\xi, N') \geq \inf_{\eta \in I} \operatorname{dist}(\eta, N') - \operatorname{dist}(\xi, I) \geq \frac{A_0}{2C_0},$$

for every $\xi \in \cup_{j=4s}^{4s+3} [b_j, b_{j+1}]$, where $I = [b_{4s}, b_{4s+2}] \cup [b_{4s+2}, b_{4s+4}]$. It follows from (2.28) and (2.29) that (ii) is satisfied. By (2.24) and (2.27), we also obtain (i).

Now, let $l_0 = b_0 b_1 ... b_{4^{k_0}+1}$. By (ii), we have

$$\operatorname{dist}(\xi, N) \geq \frac{1}{8C_0} \min\{\operatorname{dist}(x, N), \operatorname{dist}(y, N)\}.$$

By (2.13), (2.15) and (i), we have

$$\begin{aligned}
L(l) = \sum_{j=0}^{4^{k_0+1}-1} |b_j - b_{j+1}| &\leq \sum_{s=0}^{4^{k_0}-1} \left(3|a_s - a_{s+1}| + \frac{2A}{C_0} \right) \\
&= 3L(l_0) + 4^{k_0} \frac{2A}{C_0} \\
&\leq 3 \left(1 + \frac{4^{k_0}}{C_0} \right) L(l_0) \\
&\leq 3 \left(C_0 + 4^{k_0} \right) |x - y|.
\end{aligned}$$

Choosing $C = 8 \left(C_0 + 4^{k_0} \right)$, we see that l and C satisfy the desired conditions. Thus, Lemma 2.4 is true in the case $k = k_0 + 1$. This completes the proof. $\qquad \square$

3. LOG CONTINUITY PRESERVED UNDER BLOWUPS

Let $f : X \to Y$ be a smooth surjective map between compact differentiable manifolds. Let g_X, g_Y be Riemannian metrics on X, Y respectively. Let d_{g_X}, d_{g_Y} denote the distances induced by g_X, g_Y on X, Y respectively. For $E \subset X$, let $d_X(a, E) := \inf_{b \in E} d_X(a, b)$. For every $a, b \in Y$, we define

$$d_{g_X, f}(a, b) := \inf_{a' \in f^{-1}(a), b' \in f^{-1}(b)} d_{g_X}(a', b').$$

We note the last function is in general not a metric on Y. Observe

$$(3.1) \qquad d_{g_Y} \leq C d_{g_X, f}$$

for some constant $C > 0$ because the differential Df is bounded uniformly on X.

Lemma 3.1. *Let $X_0, \ldots, X_m$ be compact complex manifolds and $f : X_j \to X_{j-1}$ be the blow up along a smooth submanifold $V_{j-1} \subset X_{j-1}$ in X_{j-1} for $1 \leq j \leq m$. Let $f := f_m \circ \cdots \circ f_0 : X_m \to X_0$. Let g_j be a Riemannian metric on X_j for $1 \leq j \leq m$. Let $A > 0, \beta \in (0, 1]$ be constants. Let u be a function on X_0 and $M > 0$ be a constant. Then if $u \circ f_m$ is a $\log^M$-continuous function, then so is u.*

We note that a similar property for Hölder continuity was proved in [27]. The following proof is more or less similar.

Proof. This is indeed implicitly in the proof of Lemma 3.4 in [27] if $m = 1$. The general case follows from an immediate induction on m. For readers' convenience, we reprove below the case where $m = 1$.

Let $f_1 : X_1 \to X_0$ be the blow up along a smooth submanifold V in X_0. Let $n := \dim X_0$ and $l := \dim V$. Let $a \in V$ and let $(U, x = (x_1, \ldots, x_n))$ be a local chart around a such that V is given by $\{x_j = 0, 1 \leq j \leq n - l\}$. Thus $f_1^{-1}(U)$ is biholomorphic to the submanifold of $U \times \mathbb{CP}^{n-l-1}$ defined by the equations $x_j v_s = v_j x_s$ for $1 \leq j, s \leq n - l$, where $v := [v_1 : \cdots : v_{n-l}]$ are the homogeneous coordinates on $\mathbb{CP}^{n-l-1}$. One can cover $f_1^{-1}(U)$ by $(n - l)$ open subsets

$$U_j := \{(x, v) \in f_1^{-1}(U) : v_j \neq 0\}.$$

In U_j, we have

$$f_1(x, v) = (v_1 x_j / v_j, v_2 x_j / v_j, \ldots, v_{n-l} x_j / v_j, x_{n-l+1}, \ldots, x_n).$$

Now let $a, b \in X_0$. It suffices to consider a, b close to each other and both close to V (because f_1 is biholomorphic outside V). We split the proof into several cases. Firstly observe that if $a, b \in V$, then since $f_1 : f_1^{-1}(V) \to V$ is a submersion, one gets

$$C d_{g_0}(a, b) \geq d_{g_1, f_1}(a, b),$$

for some constant $C > 0$ independent of a and b. Hence

$$(3.2) \qquad |u(a) - u(b)| = \inf_{a' \in f_1^{-1}(a), b' \in f_1^{-1}(b)} |u \circ f_1(a') - u \circ f_1(b')|$$

$$\lesssim |\log d_{g_1, f_1}(a, b)|^{-M} \lesssim |\log d_{g_0}(a, b)|^{-M}.$$

Note that in the last inequality, we only consider a and b close to each other, hence $\log d_{g_0}(a, b) < 0$.

Case 1. Consider now $b \in V$ and $a \notin V$ but close to b. Then there is a local chart (U, x) on X_0 containing b, a such that V is given by $\{x_j = 0, 1 \leq j \leq n - l\}$. We use now the Euclidean distance on that local chart.

Without loss of generality, we can assume that $b = 0$, $a = (x'_1, \ldots, x'_n)$ with $x'_1 \neq 0$ and $h(t) := (tx'_1, \ldots, tx'_n) \in U$ for every $t \in [0, 1]$. We see that

$$\hat{h}(t) := f_1^{-1} \circ h(t) = (tx'_1, \ldots, tx'_n, [x'_1 : \ldots : x'_n]),$$

for $t > 0$. Letting $t \to 0$ gives

$$\hat{h}(0) := \lim_{t \to 0} f_1^{-1} \circ h(t) = (0, \ldots, 0, [x_1' : \ldots : x_n']) \in f_1^{-1}(b).$$

We infer that

$$d_{g_1}\big(\hat{h}(1), \hat{h}(0)\big) \lesssim |x_1'| + \cdots + |x_n'| \lesssim |a - b|.$$

It follows that

$$d_{g_1}\big(\hat{h}(1), f^{-1}(b)\big) \lesssim |a - b| \lesssim d_{g_0}(a, b).$$

Hence $d_{g_1, f_1}(a, b) \lesssim d_{g_0}(a, b)$. Thus we get an estimate similar to (3.2).

Case 2. Consider now $a, b \notin V$ but close to V. Direct computations show that $|Df_1^{-1}(a)| \lesssim |d_{g_0}(a, V)|^{-2}$. Thus we get

$$d_{g_1, f_1}(a, b) \lesssim \max\{d_{g_0}(a, V)^{-2}, d_{g_0}(b, V)^{-2}\} d_{g_0}(a, b).$$

Hence if $\min\{d_{g_0}(a, V)^2, d_{g_0}(b, V)^2\} \geq d_{g_0}(a, b)^{1/2}$, then

$$d_{g_1, f_1}(a, b) \lesssim d_{g_0}(a, b)^{1/2}.$$

We treat now the case where $\min\{d_{g_0}(a, V)^2, d_{g_0}(b, V)^2\} \leq d_{g_0}(a, b)^{1/2}$. Without loss of generality, we can assume that $d_{g_0}(b, V) \leq d_{g_0}(a, b)^{1/4}$. Then

$$d_{g_0}(a, V) \leq d_{g_0}(a, b) + d_{g_0}(b, V) \lesssim d_{g_0}(a, b)^{1/4}.$$

Now we consider a local chart (U, x) containing a, b. We use now the Euclidean metric. Let a_V, b_V be the projection of a, b to V respectively.

$$|a_V - b_V| \lesssim |a - b|$$

and

$$|a - a_V| \lesssim |a - b|^{1/4}, |b - b_V| \lesssim |a - b|^{1/4}.$$

Now applying Case 1 to $(a, a_V), (b, b_V)$ and (a_V, b_V), one obtains

(3.3)
$$d_{g_1, f_1}(a, a_V) + d_{g_1, f_1}(a_V, b_V) + d_{g_1, f_1}(b_V, b) \lesssim |a - a_V| + |a_V - b_V| + |b - b_V| \lesssim |a - b|^{1/4}.$$

Put $x_1 := |u(a) - u(a_V)|$, $x_2 := |u(a_V) - u(b_V)|$ and $x_3 := |u(b_V) - u(b)|$. By previous parts of the proof, we see that

$$d_{g_0}(a, a_V) \gtrsim e^{-x_1^{-\frac{1}{M}}}, \quad d_{g_0}(a_V, b_V) \gtrsim e^{-x_2^{-\frac{1}{M}}},$$

and

$$d_{g_0}(b_V, b) \gtrsim e^{-x_3^{-\frac{1}{M}}}.$$

This combined with (3.1) and (3.3) gives

$$|a - b|^{1/4} \gtrsim \sum_{j=1}^{3} e^{-x_j^{-\frac{1}{M}}} \gtrsim \exp\left\{-\left(\frac{x_1 + x_2 + x_3}{3}\right)^{-1/M}\right\}.$$

It follows that

$$x_1 + x_2 + x_3 \lesssim \left|\log \frac{|a - b|}{C}\right|^{-M} \lesssim |\log|a - b||^{-M},$$

for some constant $C > 0$ independent of a and b. The left-hand side of the last inequality is $\geq |u(a) - u(b)|$. Hence $|u(a) - u(b)| \lesssim |\log|a - b||^{-M}$. This finishes the proof. $\qquad\square$

4. Hölder continuous measures

Let η be a closed smooth semi-positive $(1,1)$-form in a big (semi-positive) cohomology class. Let K be a Borel subset of X. The *capacity* of K is given by

$$\mathrm{cap}_\eta(K) := \sup \Big\{ \int_K \eta_\varphi^n : 0 \leq \varphi \leq 1, \varphi \ \eta\text{-psh} \Big\}.$$

The above notion was introduced in [22] generalizing those in [2, 30]; see [12, 35] and references therein for various generalizations of capacity.

Let $A, \beta > 0$. We say that a Borel measure μ on X satisfies the condition $\mathcal{H}(\beta, A, \eta)$ if

$$\mu(K) \leq A \left(\mathrm{cap}_\eta(K)\right)^{1+\beta},$$

for every Borel set $K \subset X$.

Fix a Kähler form ω on X. Let μ be a measure on X. Recall that μ is said to be a Hölder continuous measure with the Hölder constant A and the Hölder exponent $\gamma \in (0, 1]$ if for every ω-psh function φ_1, φ_2 with $\int_X \varphi_j \omega^n = 0$ for $j = 1, 2$ there holds

$$\int_X (\varphi_1 - \varphi_2) d\mu \leq A\|\varphi_1 - \varphi_2\|_{L^1(\omega^n)}.$$

Let $\mathcal{M}(A, \gamma)$ be the set of Hölder continuous measures with the Hölder constant A and the Hölder exponent $\gamma \in (0, 1]$. By [20, Lemma 3.3], a measure $\mu \in \mathcal{M}(A, \gamma)$ if there is a constant $C > 0$ depending only on A such that for every ω-psh function φ_1, φ_2, we have

$$(4.1) \qquad \int_X |\varphi_1 - \varphi_2| d\mu \leq C \max \left\{ \|\varphi_1 - \varphi_2\|_{L^1(X)}^\gamma, \|\varphi_1 - \varphi_2\|_{L^1(X)} \right\}.$$

Note that if μ is a Hölder continuous measure then it follows from [20, Proposition 2.4 and Proposition 4.4] that for every constant $\beta > 0$, there exists a constant $A_\beta > 0$ such that μ satisfies the condition $\mathcal{H}(\beta, A_\beta, \omega)$. Therefore, by the comparison of capacities (see [21, Theorem 3.17]), for every $\beta > 0$, there exists $A_\beta > 0$ such that μ satisfies the condition $\mathcal{H}(\beta, A_\beta, \eta)$. Alternatively, one can prove the last property by using results in [12].

The following proposition is a special case of [26, Proposition 5.3] (replace φ and ψ by w_1 and w_2, respectively):

Proposition 4.1. *Let η be a big semi-positive closed smooth $(1,1)$-form and let w_1, w_2 be negative η-psh functions such that w_1 is of full Monge-Ampère mass (i.e, $\int_X \eta_{w_1}^n = \int_X \eta^n$). Denote $\mu_1 = (\eta + dd^c w_1)^n$. Assume that the following conditions hold*

  (i) *there exists a constant $M > 0$ such that*

$$-M \leq \max\{w_1, w_2\} \leq 0;$$

  (ii) *there exist constants $A, \beta > 0$ such that μ_1 satisfies the condition $\mathcal{H}(\beta, A, \eta)$.*

Then, for every constant $r > 0$, there exists a constant $C > 0$ depending on $\omega, \eta, M, A, \beta$ and r such that

$$w_1 - w_2 \geq -C\|w_1 - w_2\|_{L^r(\mu_1)}^{\frac{\beta r}{n + \beta(n+r)}}.$$

In particular, if $\mu_1 \in \mathcal{M}(B, \alpha)$ for some $B > 0$ and $0 < \alpha \leq 1$ then for every $r, \gamma > 0$, there exists a constant $C' > 0$ depending on $\omega, \eta, B, \alpha, \gamma$ and r such that

$$w_1 - w_2 \geq -C' \|w_1 - w_2\|_{L^r(\mu_1)}^{\frac{\gamma r}{n + \gamma(n+r)}}.$$

We will apply Proposition 4.1 to the case where r is large enough, this means the exponent $\frac{\beta r}{n + \beta(n+r)}$ is close to be 1.

Corollary 4.2. *Let η be a big semi-positive closed smooth $(1,1)$-form and let w be a negative η-psh function of full Monge-Ampère mass with $\sup_X w = 0$. Assume that $(\eta + dd^c w)^n \in \mathcal{M}(B, \alpha)$ for some $B > 0$ and $0 < \alpha \leq 1$. Then $\|w\|_{L^\infty} \leq C$, where $C > 0$ is a constant depending on ω, η, B and α.*

By [20], a measure μ of mass $\int_X \omega^n$ is Hölder continuous if and only if $\mu = (dd^c u + \omega)^n$ for some Hölder continuous ω-psh function u on X. The following will be important for us.

Corollary 4.3. *Let $X_0, \ldots, X_m$ be compact complex manifolds and $f_j : X_j \to X_{j-1}$ be the blow up along a smooth submanifold $V_{j-1} \subset X_{j-1}$ in X_{j-1} for $1 \leq j \leq m$. Let $f := f_m \circ \cdots \circ f_0 : X_m \to X_0$. Let μ be a Hölder continuous measures on X_m. Then $f_*\mu$ is also Hölder continuous.*

Proof. By induction, it suffices to prove the desired assertion for $m = 1$. Let u_1, u_2 be ω_0-psh functions on X_0 for $j = 1, 2$, where ω_0 is a Kähler form on X_0. Put $u'_j := f_1^* u_j$. Let ω_1 be a Kähler form on X_1. Using Hölder continuity of μ, we obtain

$$\|u_1 - u_2\|_{L^1((f_1)_*\mu)} = \|u'_1 - u'_2\|_{L^1(\mu)} \lesssim \|u'_1 - u'_2\|_{L^1(\omega_1^n)}^\gamma + \|u'_1 - u'_2\|_{L^1(\omega_1^n)}.$$

Standard computations using local coordinates for blowups show that there exists a function $g \in L^p(\omega_0^n)$ for some constant $p > 1$ satisfying $(f_1)_* \omega_1^n = g \omega_0^n$. Hence

$$\|u'_1 - u'_2\|_{L^1(\omega_1^n)} = \int_{X_0} |u_1 - u_2| g \omega_0^n \lesssim \|u_1 - u_2\|_{L^q(\omega_0^n)}$$

where $1/q + 1/p = 1$. By [20, Lemma 2.2], one has

$$\|u_1 - u_2\|_{L^q(\omega_0^n)} \lesssim \|u_1 - u_2\|_{L^1(\omega_0^n)}^{1/(2q)}.$$

Hence $(f_1)_*\mu$ is Hölder continuous. □

5. REGULARIZATION OF PSH FUNCTIONS

5.1. L^2-estimates. We recall first the L^2-estimates for $\bar{\partial}$ and discuss some of its variants.

Theorem 5.1. *(see [16, Corollary 5.3]) Let (X, ω) be a compact Kähler manifold. Let $\epsilon > 0$ be a constant. Let L be a holomorphic line bundle on X together with a singular Hermitian metric h satisfying*

$$c_1(L, h) \geq \epsilon \omega.$$

Then for every $g \in L^2_{n,1}(X, L)$ with $\bar{\partial} g = 0$, there exists $u \in L^2_{n,0}(X, L)$ such that $\bar{\partial} u = g$, and

$$\int_X |u|_{h,\omega}^2 \omega^n \leq \epsilon^{-1} \int_X |g|_{h,\omega}^2 \omega^n,$$

where $|g(x)|_{h,\omega}$ denotes the norm of g with respect to the norm induced by the Hermitian metric h on L and the Riemannian metric on X associated to ω.

We deduce from the above result the following more or less standard consequence.

Theorem 5.2. *Let (X, ω) be a compact Kähler manifold. Let $\epsilon > 0$ be a constant. Let K_X^* be the dual of the canonical line bundle, and let $h_{K_X^*}$ denote the metric induced by ω on K_X^*. Let L be a holomorphic line bundle on X together with a singular Hermitian metric h. Assume that there exists a singular metric $\tilde{h}_{K_X^*}$ on K_X^* so that*

$$c_1(L, h) + c_1(K_X^*, \tilde{h}_{K_X^*}) \geq \epsilon\omega.$$

Then for every $g \in L_{0,1}^2(X, L)$ with $\bar{\partial}g = 0$, there exists $u \in L_{0,0}^2(X, L)$ such that $\bar{\partial}u = g$, and

$$\int_X |u|_h^2 e^{-2\vartheta}\omega^n \leq \epsilon^{-1}\int_X |g|_{h,\omega}^2 e^{-2\vartheta}\omega^n,$$

where ϑ is a quasi-psh function defined by $\tilde{h}_{K_X^*} = e^{-2\vartheta}h_{K_X^*}$.

Proof. Set $L' := L \otimes K_X^*$. Thus $L = L' \otimes K_X$. Let h' be the singular Hermitian metric on L' given by $h' = h \otimes \tilde{h}_{K_X^*}$. For every $0 \leq q \leq n$, we have a natural isometry

$$\Psi_q : \Lambda^{0,q}(T^*X) \otimes L \to \Lambda^{n,q}(T^*X) \otimes L',$$

e.g., see the proof of [10, Corollary 4.3]), where we use the metric h on L, and $h \otimes h_{K_X^*}$ on L'. The map Ψ commutes with $\partial, \bar{\partial}$ operators. Thus $\Psi_1(g) \in L_{n,1}^2(X, L')$ with $\bar{\partial}\Psi_1(g) = 0$. Since Ψ_1 is an isometry, one gets

$$|\Psi_1(g)|_{h'}^2 = |\Psi_1(g)|_{h \otimes h_{K_X^*}}^2 e^{-2\vartheta} = |g|_h^2 e^{-2\vartheta}.$$

The desired assertion now follows from Theorem 5.1. $\square$

In particular we obtain the following.

Corollary 5.3. *Let (X, ω) be a compact Kähler manifold so that the Chern class of K_X^* contains a closed positive $(1, 1)$-current of bounded potentials, i.e, there exists a bounded η_ω-psh function ϑ on X, where η_ω is the Chern form of the metric on K_X^* induced by ω. Let L be a holomorphic line bundle on X together with a singular Hermitian metric h such that*

$$c_1(L, h) \geq \epsilon\omega.$$

Then for every $g \in L_{0,1}^2(X, L)$ with $\bar{\partial}g = 0$, there exists $u \in L_{0,0}^2(X, L)$ such that $\bar{\partial}u = g$, and

$$\int_X |u|_h^2 \omega^n \leq \frac{e^{4\|\vartheta\|_{L^\infty}}}{\epsilon}\int_X |g|_{h,\omega}^2 \omega^n.$$

We recall now a special case of the Ohsawa-Takegoshi extension theorem (see [16, Theroem 13.6]).

Theorem 5.4. *Let X be a weakly pseudoconvex n-dimensional manifold with a Kähler metric ω. Let y be a point in X. Let (L, h) be a line bundle with a smooth positively curved Hermitian metric h on X and let E be the trivial holomorphic vector bundle of rank n equipped with the trivial Hermitian metric such that there exists a global section s of E with $y = \{s = 0\}$ so that $\Lambda^n ds(y) \neq 0$ and $|s| \leq e^{-1}$. Then for every $(n, 0)$-form f with values in L at y, there exist a $\bar{\partial}$-closed $(n, 0)$-form F with values in L on X such that $F(y) = f(y)$ and*

$$\int_X \frac{|F|_{h,\omega}^2}{|s|^{2n}(-\log|s|)^2}\omega^n \leq C_n \frac{|f|_{h,\omega}^2}{|\Lambda^n ds(y)|_\omega^2},$$

where C_n is a numerical constant depending only on n.

We note that since E and its Hermitian metric are trivial, the curvature of the metric of E vanishes everywhere, and s is nothing but a tuple of n holomorphic functions on X.

Corollary 5.5. *Let X, ω, L, h, E, y, s be as in Theorem 5.4. Assume furthermore that $\Lambda^n ds(y) \neq 0$ for every $y \in X \backslash N$, where N is an analytic set in X. Then for every section f of L at y, there exists $F \in H^0(X \backslash N, L)$ such that $F(y) = f(y)$ and*

$$\int_X |F|_h^2 |\Lambda^n ds(y)|_\omega^2 \omega^n \leq C_n |f|_h^2,$$

where C_n is a numerical constant depending only on n.

Proof. Applying Theorem 5.4 to $f_1 := f \Lambda^n ds$, we find a $\bar{\partial}$-closed $(n,0)$-form F_1 with values in L such that

$$\int_X \frac{|F|_{h,\omega}^2}{|s|^{2n}(-\log|s|)^2} \omega^n \leq C_n |f|_h^2.$$

Put $F := F_1 / \Lambda^n ds$. By the hypothesis on s we see that F is holomorphic on $X \backslash N$ and satisfies the desired inequality. $\qquad\square$

5.2. **Analytic regularisation of psh functions.** Let (X, ω) be a compact Kähler manifold. Let L be a big and semi-ample line bundle on X (hence X is forced to be projective by Moishezon's theorem). Let ρ, Φ_{k_L}, X' be as in Introduction. From now on we assume the following hypothesis:

(H)   The Chern class of K_X^* contains a closed positive $(1,1)$-current of bounded potentials, *i.e*, there exists a bounded η_ω-psh function ϑ on X, where η_ω is the Chern form of the Hermitian metric on K_X^* induced by ω. Furthermore, X' has only isolated singularities.

In particular, this assumption is fulfilled if K_X^* is semi-positive and $n = 2$. Let

$$\theta := \frac{1}{2k_L} dd^c \log \sum_{j=1}^{d_{k_L}} |s_j|^2$$

which is smooth closed form in $c_1(L)$. Hence θ is the pull-back of the Fubini-Study form in $\mathbb{CP}^{d_{k_L}-1}$ under Φ_{k_L}. Let h_0 be a smooth Hermitian metric on L with $c_1(L, h_0) = \theta$.

Fix a smooth Riemannian metric on X and let $\mathbb{B}(x, r)$ be the ball of radius r with respect to this metric. Let $r_X > 0$ be a constant so that for every $x \in X$ the closure of the ball $\mathbb{B}(x, r_X)$ is contained in a local chart of X which is biholomorphic to a ball in $\mathbb{C}^n$.

Lemma 5.6. *There exist constants $C_0 > 0$, $r_0 > 0$ small enough such that for every $y \in X$, there exist global negative θ-psh functions u_y on X so that*

$$u_y(x) \leq \log|y - x| + C_0$$

on $\Phi_{k_L}^{-1}\big(\mathbb{B}(y', r_0)\big)$ where $y' := \Phi_{k_L}(y)$ and by abuse of notation, for every $r > 0$, we denote by $\mathbb{B}(y', r)$ the ball of radius r centered at $y' \in \mathbb{CP}^{d_{k_L}-1}$ with respect to a fixed smooth metric on $\mathbb{CP}^{d_{k_L}-1}$. Furthermore, for every constant $\epsilon > 0$, we have

(5.1) $u_y \geq \log \epsilon - C$

outside $\Phi_{k_L}^{-1}\big(\mathbb{B}(y', \epsilon)\big)$ *for some constant C independent of y, ϵ.*

Proof. Let $y' := \Phi_{k_L}(y)$ and let $v_y(z)$ be a ω_{FS}-psh function on $\mathbb{CP}^{d_{k_L}-1}$ given by

$$v_y(z) := \log|z - y'|$$

where we use homogeneous coordinates for z, y', and ω_{FS} is the Fubini-Study form on $\mathbb{CP}^{d_{k_L}-1}$. Thus, for every $\epsilon > 0$, there holds $v_y \geq C \log \epsilon$ outside $\mathbb{B}(y', \epsilon)$ for some constant C independent of y, ϵ.

Since $\Phi_{k_L}^* \omega_{FS} = \theta$, we infer $u_y := \Phi_{k_L}^* v_y$ and $\tilde{u}_y := \Phi_{k_L}^* \tilde{v}_y$ are θ-psh and satisfies that

$$u_y(x) = \log|\Phi_{k_L}(y) - \Phi_{k_L}(x)| \leq \log|y - x| + C_0$$

on $\Phi_{k_L}^{-1}\big(\mathbb{B}(y', r_0)\big)$. Moreover one also has (5.1) because of the smoothness of v_y outside y'. $\qquad\square$

We denote by $Sing(X')$ the singular locus of X'. By the hypothesis, the set $Sing(X')$ is finite. Let $\mathbb{B}_r(y)$ be the ball of radius r centered at y in $\mathbb{C}^{k_L-1}$. If $y = 0$, then we write $\mathbb{B}_r$ for $\mathbb{B}_r(y)$. Put $N' := \Phi_{k_L}(N)$ which is an analytic subset in X'. Since $Sing(X')$ is finite, we can find open subsets $U_1', \ldots, U_l'$ in $\mathbb{CP}^{d_{k_L}-1}$ such that the following properties hold:

(i) $U_j' \Subset U_j''$ which is biholomorphic to the ball $\mathbb{B}_3$ in $\mathbb{C}^{d_{k_L}-1}$ under a map Ψ_j for every $1 \leq j \leq l$ and U_j is biholomorphic to $\mathbb{B}_2$ under Ψ_j. Furthermore, if $U_j' \cap Sing(X')$ is non empty, then $U_j' \cap Sing(X')$ is contained in $\mathbb{B}_1 \Subset U_j'$,

(ii) $X' \subset \cup_{j=1}^{l} U_j'$,

(iii) There is a hyperplane H_j on $\mathbb{CP}^{d_{k_L}-1}$ such that H_j does not intersect U_j'' for every $1 \leq l \leq j$.

By our choice of U_j', we see that U_j' is hyperconvex (hence weakly pseudoconvex), i.e, there is a smooth psh function w_j on U_j' such that $\{w_j < c\}$ is relatively compact in U_j' for every constant $c < 0$ and every j. Let

$$U_j := \Phi_{k_L}^{-1}(U_j').$$

Note that U_j is also hyperconvex and L is trivial over U_j because $L = \Phi_{k_L}^* \mathcal{O}(1)$ and $\mathcal{O}(1)$ is trivial over $X' \backslash H_j$.

Lemma 5.7. *Let $h := h_0 e^{-2\phi}$ be a singular positively curved metric on L such that ϕ is smooth outside N. Fix $1 \leq j \leq l$. Let $y \in U_j \backslash N$. Let e be a local holomorphic frame of L over U_j. Then for every $a \in \mathbb{C}$, there exists a section $f \in H^0(U_j, L)$ such that $f(y) = ae(y)$ and*

$$\int_{U_j} |f|_{h_0}^2 e^{-2\phi} \theta^n \leq C|a|^2 |e(y)|_{h_0}^2 e^{-2\phi(y)}.$$

where $C > 0$ is a constant independent of y and a.

Proof. We treat first the case where ϕ is smooth. Let $L|_{U_j}$ be the restriction of L to U_j, and $E := L|_{U_j} \oplus \cdots \oplus L|_{U_j}$ (n times). Since L is trivial on U_j, so is E. Equip E with the trivial Hermitian metric. Hence a section of E is simply a collection of n holomnorphic functions on U_j. Let $z = (z_1, \ldots, z_{k_L-1})$ be the local coordinates on $U_j'' \approx \mathbb{B}_3$. We can assume that y' is the origin in these local coordinates. Let $s_E'(z) := z$. Observe that $s_E := s_E' \circ \Phi_{k_L}^{-1}$ is a section of E on U_j and vanishes only at y. Recall that $\theta = \Phi_{k_L}^* \omega_{FS}$, where ω_{FS} is the Fubini-Study form on $\mathbb{CP}^{k_L-1}$.

Let $N' := \Phi_{k_L}(N)$. Then $X'\backslash N'$ is smooth and Φ_{k_L} is a biholomoprhism from $U_j\backslash N$ to $U_j'\backslash N'$. Let $X'' := X' \cap \mathbb{B}_2$, we have a natural inclusion $\xi : X'' \to U_j' \approx \mathbb{B}_2$. Let Ψ be an orthogonal change of coordinates on $\mathbb{C}^{k_L-1}$ so that $\Psi_* \xi_* T_y X''$ is given by the subspace $\{z_1, \ldots, z_n, 0, \ldots, 0\}$ at 0 in $\mathbb{C}^{k_L-1}$. Write $\Psi = (\Psi_1, \ldots, \Psi_{k_L})$. Let

$$s_E' := (\Psi_1, \ldots, \Psi_n) \circ \xi$$

regarded as a section of $\Phi_{k_L-1}^* E$. Let $Y := X'' \cap \{s_E' = 0, \det J_{s_E'} \neq 0 : 1 \leq k \leq n\}$ contains 0 as an isolated point, where $\xi_y(z) := (z_{j_1}, \ldots, z_{j_n})$ for $z \in X''$ (note that Y may not be connected). Note that ω_0 is preserved under Ψ. By the choice of s_E', there is a constant $\epsilon_0 > 0$ independent of y such that

$$(5.2) \qquad |\Lambda^n ds_E'(y')|_{\xi^* \omega_{FS}} \geq \epsilon_0.$$

Indeed, by the choice of Ψ, the norm $|\Lambda^n ds_E'(y')|_{\xi^* \omega_0}$ (which is the norm of $\det J_{s_E'}$ with respect to $\xi^* \omega_0$) is equal to the absolute value of the determinant of the (n,n)-submatrix of the Jacobian of $(\Psi_1, \ldots, \Psi_n)$ given by the first n rows. Hence $|\Lambda^n ds_E'(y)|_{\xi^* \omega_0} = 1$. Since ω_{FS} and ω_0 are equivalent on U', we get (5.2).

Let $s_E := s_E' \circ \Phi_{k_L}$. Observe that the expression $|\Lambda^n ds_E|_\omega^2 \omega^n$ is independent of ω and we have

$$|\Lambda^n ds_E|_\omega^2 \omega^n = |\Lambda^n ds_E|_\theta^2 \theta^n = |\Lambda^n ds_E'|_{\xi^* \omega_{FS}}^2 \theta^n \geq \epsilon_0 \theta^n$$

by (5.2).

Applying Corollary 5.5 to U_j, s_E, N implies that there exists a section $f \in H^0(U_j\backslash N, L)$ such that $f(y) = ae(y)$ and

$$\int_{U_j} |f|_{h_0}^2 e^{-2\phi} \theta^n \leq C|a|^2 |e(y)|_{h_0}^2 e^{-2\phi(y)}$$

where $C > 0$ is a constant independent of y and a. Since Φ_{k_L} is biholomorphic on $U_j\backslash N$, we see that $f \circ \Phi_{k_L}^{-1}$ is holomorphic on $U_j' \cap (X'\backslash Sing(X'))$ (which is a normal variety). Hence $f \circ \Phi_{k_L}^{-1}$ extends to be a holomorphic function on $U_j' \cap X'$ (see [23]). We infer that $f \in H^0(U_j, L)$.

It remains to treat the case where ϕ is not necessarily smooth. Put $\phi' := \phi \circ \Phi_{k_L}^{-1}$ which is well-defined on $X'\backslash Sing(X')$. Since X' is normal, ϕ' extends to a ω_{FS}-psh function on X'. By [9] we can extends ϕ' to an ω_{FS}-function on $\mathbb{CP}^{d_{k_L}-1}$. Standard regularisation (see [4]) gives us a decreasing sequence of smooth ω_{FS}-functions to ϕ'. Hence we obtain a decreasing sequence of smooth θ-psh functions $(\phi_l)_l$ to ϕ. Applying the first part to each ϕ_l we get $f_l \in H^0(U_j, L)$ with $f_j(y) = ae(y)$ and

$$\int_{U_j} |f_l|_{h_0}^2 e^{-2\phi_l} \theta^n \leq C|a|^2 |e(y)|_{h_0}^2 e^{-2\phi_l(y)}.$$

Let $V \Subset U_j\backslash N$ be an open set. We infer that $f_l \in L^2(V, \theta^n)$. Extracting a subsequence if necessary, we can assume that f_l converges uniformly to f in V. This combined with the fact that ϕ_l is bounded uniformly on V (by the hypothesis on ϕ) gives

$$\int_V |f|_{h_0}^2 e^{-2\phi} \theta^n \leq \liminf_{l \to \infty} \int_V |f_l|_{h_0}^2 e^{-2\phi_l} \theta^n \leq C|a|^2 |e(y)|_{h_0}^2 e^{-2\phi(y)}.$$

Letting V increasing to $U_j\backslash N$ gives to the desired estimate for f (and again $f \in H^0(U_j, L)$ as in the end of the first part of the proof). The proof is complete. $\qquad\square$

Since L is big, by Demailly [16], there exists a negative θ-psh function ρ such that locally

$$\rho = \log\Big(\sum_{j=1}^{r} |f_j|\Big) + O(1),$$

for some local holomorphic functions $f_1, \ldots, f_r$, and

$$dd^c\rho + \theta \geq \delta_0\omega,$$

where $\delta_0 > 0$ is a constant. We can choose ρ so that $N := \{\rho = -\infty\}$ is equal to the non-Kähler locus of $c_1(L)$, see [6]. Recall that the non-Kähler locus of $c_1(L)$ is equal to the augmented base locus of L (see [40, Theorem 2.3] or [5]).

Let h be a positive Hermitian metric on L. Let e_L is a local holomorphic frame for L (i.e., e_L is a local holomorphic section of L and $e_L \neq 0$ everywhere). Write $h = h_0 e^{-2\varphi}$. Thus by hypothesis one gets

$$0 \leq c_1(L, h) = -dd^c \log |e_L|_h = dd^c\varphi + \theta.$$

In other words, φ is θ-psh function. By multiplying a large constant with h_0, without loss of generality we can assume that $\varphi \leq 0$. We assume from now on that φ is bounded.

For every constant $\delta \in (0, 1)$, define

$$\varphi_\delta := (1 - \delta)\varphi + \delta\rho, \quad h_\delta := h_0 e^{-2\varphi_\delta}.$$

We have $dd^c\varphi_\delta + \theta \geq \delta\delta_0\omega$. Let $m \in \mathbb{N}$ and $d_m := \dim H^0(X, L^m)$ which is $\approx m^n$ as $m \to \infty$. Let $m_0 := 2n + 3$. Let $a_0 \in (0, 1/2)$ be a constant such that

$$\int_X e^{-2a_0\rho}\omega^n < \infty.$$

For $m > m_0$, $\delta \in (0, a_0/m)$ and $s, s' \in H^0(X, L^m)$, we put

$$\langle s, s'\rangle_{L^2} = \langle s, s'\rangle_{L^2, m, \delta} := \int_X \langle s, s'\rangle_{h_0^m} e^{-2(m-m_0)\varphi_\delta}\omega^n$$

which is finite because the boundedness of φ and the choice of a_0. To give readers a hint what we do with δ, we remark that we will choose later $\delta := m^{-2D}$ for some constant $D \geq 1$ (see the proof of Lemma 6.2 below), thus the condition $\delta < a_0/m$ is automatically satisfied for $m \geq a_0^{-1}$.

Let $\{\sigma_1, \ldots, \sigma_{d_m}\}$ be an orthonormal basis of $H^0(X, L^m)$ with respect to L^2-product, and let

$$\psi_{m,\delta} := \frac{1}{2m} \log\Big(\sum_{j=1}^{d_m} |\sigma_j|_{h_0^m}^2\Big) = \frac{1}{2m} \sup_{s \in H^0(X, L^m): \|s\|_{L^2} = 1} \log |s|_{h_0^m}^2.$$

Since $\log |\sigma_j|_{h_0^m}$ is $m\theta$-psh, we infer that ψ_m is θ-psh.

The following result is a variant from [16, Theorem 14.21]. Recall $N = \{\rho = -\infty\}$.

Theorem 5.8. *There exists a constant $C > 0$ such that for every $\delta \in (0, a_0/m)$ and every $m \geq m_0 + 1$ there holds:*

(i)

$$\frac{m - m_0}{m}\varphi_\delta(x) - \frac{C + |\log\delta|}{2m} \leq \psi_{m,\delta} \leq \frac{m - m_0}{m} \sup_{x' \in \mathbb{B}(x,r)} \varphi_\delta(x') + Cr + C\frac{|\log r|}{m},$$

for every $x \in X$ and $r > 0$.

(ii)

$$|\nabla \psi_{m,\delta}(x)| \le C + \frac{C}{m\delta^{1/2}r^{n+1}}e^{(m-m_0)(\sup_{\mathbb{B}(x,r)}\varphi_\delta - \varphi_\delta(x)) + C(m-m_0)r},$$

for every $x \in X \backslash N$ and $r > 0$.

Proof. We check the second inequality in (i). Let U be a small local chart around x. We trivialize L over U and let $e_{L,U}$ be a nowhere vanishing holomorhic section of L over U. Hence we can identify $h_0 = e^{-2\phi_0}$ for some smooth function ϕ_0, and sections of L^m are identified with holomorphic functions on U.

Let $s \in L^m$ with $\|s\|_{L^2} = 1$. By abuse of notation we still denote by s the holomorphic function corresponding to a section s of L^m. Thus $|s|^2_{h_0^m} = |s|^2 e^{-2m\phi_0}$. By the submean inequality, one gets

$$|s(x)|^2 \lesssim r^{-2n} \int_{\mathbb{B}(x,r)} |s|^2 d\,\mathrm{Leb}_{\mathbb{C}^n} \lesssim r^{-2n}e^{\sup_{\mathbb{B}(x,r)} q} \int_{\mathbb{B}(x,r)} |s|^2 e^{-q}\omega^n \lesssim r^{-2n}e^{\sup_{\mathbb{B}(x,r)} q}\|s\|^2_{L^2},$$

where $q = 2(m-m_0)\varphi_\delta + 2m\phi_0$. Then

$$|s(x)|^2_{h_0^m} = |s(x)|^2 e^{-2m\phi_0(x)} \lesssim r^{-2n}e^{2(m-m_0)(\sup_{\mathbb{B}(x,r)}(\varphi_\delta + \phi_0) - \phi_0(x))}.$$

By this and the fact that $\phi_0 \in \mathcal{C}^1$ we infer that

$$|s(x)|^2_{h_0^m} \lesssim r^{-2n}e^{2(m-m_0)\sup_{\mathbb{B}(x,r)}\varphi_\delta + 2(m-m_0)C_1 r},$$

for some constant $C_1 > 0$ independent of δ, m, φ, s, for $s \in H^0(X, L^m)$ with $|s|_{L^2} = 1$. It follows that

$$(5.3) \quad e^{2m\psi_{m,\delta}} = \sup_{s \in H^0(X,L^m):\|s\|_{L^2}=1} |s(x)|^2_{h_0^m} \le e^{C_2}r^{-2n}e^{2(m-m_0)\sup_{\mathbb{B}(x,r)}\varphi_\delta + 2(m-m_0)C_1 r},$$

where $C_1, C_2 > 0$ are constants independent of δ, m, φ, s. Hence we obtain

$$\psi_{m,\delta} \le \frac{m-c_0}{m}\sup_{x' \in \mathbb{B}(x,r)}\varphi_\delta(x') + C_1 r + \frac{2n|\log r| + C_2}{2m} \le \frac{m-m_0}{m}\sup_{x' \in \mathbb{B}(x,r)}\varphi_\delta(x') + C_3 r + \frac{C_3|\log r|}{m},$$

for every $x \in X$ and $r > 0$, where $C_3 = C_1 + C_2 e + n$.

The remaining inequality of (i) requires the L^2-estimate. It suffices to consider $x \notin N$. Let $U_1, \ldots, U_l$ be the open cover of X defined above. Without loss of generality we can assume that $x \in U_1$. Choose $U := U_1$. We can modify the coordinates on $U_1' \approx \mathbb{B}_2$ so that $\Phi_{k_L}(x)$ is the center of U_1'. By Lemma 5.7, there are a constant $B_1 > 0$ independent of x such that for every $a \in \mathbb{C}$, there is a $f \in H^0(U_1, L^m)$ so that $f(x) = ae_{L,U}^m$ and

$$\int_{U_1} |f|^2_{h_0^m}e^{-2(m-m_0)\varphi_\delta}\theta^n \le B_1|a|^2 |e_{L,U}(x)|^{2m}_{h_0}e^{-2(m-m_0)\varphi_\delta(x)}.$$

Fix a cut-off function χ' supported on U_1' and equal to 1 on $\mathbb{B}_1$ (recall that $U_1' \approx \mathbb{B}_2$ and $\Phi_{k_L}(x) = 0$ is the origin). By our choice of U_1', we see that $Sing(X') \cap U_j'$ is contained in the interior of $\{\chi' = 1\}$.

Put $\chi := \chi' \circ \Phi_{k_L}^{-1}$. By Lemma 5.6, there exist a constant $C_4 > 0$ independent of x and a negative θ-psh function u_x satisfying $u_x(z) \le \log|z - x| + C_4$, and $u_x(z) \ge -C_4$ for $z \notin \Phi_{k_L}^{-1}(\mathbb{B}_{1/2}(x'))$ ($x' := \Phi_{k_L}(x)$). Let

$$w := (m - m_0)\varphi_\delta + m_0 u_x.$$

Observe that

$$dd^c w + m\theta \geq (m - m_0)\delta\delta_0\omega.$$

Let

$$h_{u_x} := h_0 e^{-2u_x}, \quad \tilde{h} := h_0^m e^{-2w}$$

which is a singular Hermitian metric on L^m. Thanks to the hypothesis about the semi-positivity of K_X^*, we can apply Corollary 5.3 to $\tilde{h}$ and $g := \bar{\partial}(\chi f)$ which is smooth. Hence we find a smooth section v of L over X so that $\bar{\partial}v = g$ and

$$(5.4) \qquad \int_X |v|_{h_0^m}^2 e^{-2w}\omega^n \leq \frac{1}{(m - m_0)\delta\delta_0} \int_{U_1 \backslash \Phi_{k_L}^{-1}(\mathbb{B}_1(x'))} |f|_{h_0^m}^2 e^{-2w}\omega^n$$

because $\partial\chi'$ vanishes on $B_1(x')$. This combined with the fact that $u_x(x') \geq C_4$ outside $\Phi_{k_L}^{-1}(\mathbb{B}_1(x'))$ yields

$$(5.5) \qquad \int_X |v|_{h_0^m}^2 e^{-2w}\omega^n \leq \frac{1}{(m - m_0)\delta\delta_0} \int_{U_1 \backslash \Phi_{k_L}^{-1}(\mathbb{B}_1(x'))} |f|_{h_0^m}^2 e^{-2(m-m_0)\varphi_\delta}\omega^n$$

$$\leq \frac{1}{(m - m_0)\delta\delta_0} \int_{U_1 \backslash \Phi_{k_L}^{-1}(\mathbb{B}_1(x'))} |f|_{h_0^m}^2 e^{-2(m-m_0)\varphi_\delta}\theta^n$$

(because θ^n and ω^n are comparable on $U_1 \backslash \Phi_{k_L}^{-1}(\mathbb{B}_1(x'))$)

$$\leq |a|^2 |e_{L,U}(x)|_{h_0}^{2m} e^{-2(m-m_0)\varphi_\delta(x)}.$$

Note that since g vanishes near x, one gets that $\bar{\partial}v = 0$ near x. Thus v is holomorphic near x. By properties of u_x, observe that

$$e^{-2w(x')} \gtrsim \frac{1}{|x' - x|^{2n+2}}$$

which in turn implies that $v(x) = 0$ because $\int_X |v|_{h_0^m}^2 e^{-2w}\theta^n$ is finite and θ is Kähler near x (recall that $x \notin N$). This together with (5.5) gives

$$(5.6) \qquad \int_X |v|_{h_0^m}^2 e^{-2w}\omega^n \lesssim \frac{|a|^2}{(m - m_0)\delta} e^{-2(m-m_0)\varphi_\delta(x) - 2m\phi_0(x)}.$$

Let $\tilde{v} := \chi f - v \in H^0(X, L^m)$. Since $u_x \leq 0$, using (5.6) and the choice of f, we obtain

$$\int_X |\tilde{v}|_{h_0^m}^2 e^{-2(m-m_0)\varphi_\delta}\omega^n \leq \frac{B_2 |a|^2 |e_{L,U}(x)|_{h_0}^{2m}}{(m - m_0)\delta} e^{-2(m-m_0)\varphi_\delta(x)},$$

for some constant $B_2 > 0$ independent of x, a, m, δ. Choose

$$a := B_2^{-1/2} |e_{L,U}(x)|_{h_0}^{-m} \delta^{1/2} (m - m_0)^{1/2} e^{(m-m_0)\varphi_\delta(x)}.$$

We see that

$$\int_X |\tilde{v}|_{h_0^m}^2 e^{-2(m-m_0)\varphi_\delta}\omega^n \leq 1.$$

and

$$\tilde{v}(x) = f(x) - v(x) = f(x) = a\, e_{L,U}^m(x).$$

It follows that

$$(5.7) \qquad e^{2m\psi_{m,\delta}(x)} \geq |\tilde{v}(x)|_{h_0^m}^2 \geq \frac{\delta(m - m_0)}{B_2} e^{2(m-m_0)\varphi_\delta(x)}.$$

Thus

$$\psi_{m,\delta}(x) \geq \frac{m - m_0}{m} \varphi_\delta(x) + \frac{1}{2m} \log \frac{\delta(m - m_0)}{B_2}$$
$$\geq \frac{m - m_0}{m} \varphi_\delta(x) - \frac{|\log \delta| + \log B_2}{2m}.$$

This finishes the proof for (i).

We now check (ii). We work in a small local chart (U, x) and write $h_0 = e^{-2\phi_0}$ as above. We identify sections with holomorphic functions on a trivialization of L over this local chart. We have

$$\psi_{m,\delta} = \frac{1}{2m} \log \sum_{j=1}^{d_m} |\sigma_j|^2 - \phi_0.$$

Direct computations give

$$\partial \psi_{m,\delta} = \frac{1}{2m} \frac{\sum_{j=1}^{d_k} \bar{\sigma}_j \partial \sigma_j}{\sum_{j=1}^{d_k} |\sigma_j|^2} - \partial \phi_0.$$

Hence

(5.8) $$|\partial \psi_{m,\delta}| \leq \frac{1}{2m} \frac{\left(\sum_{j=1}^{d_k} |\partial \sigma_j|^2 \right)^{1/2}}{\left(\sum_{j=1}^{d_k} |\sigma_j|^2 \right)^{1/2}} + |\partial \phi_0|.$$

By (5.7), we have

(5.9) $$\sum_{j=1}^{d_k} |\sigma_j(x)|^2 = e^{2m(\psi_{m,\delta} + \phi_0)} \geq \frac{\delta(m - m_0)}{B_2} e^{2(m - m_0)\varphi_\delta(x) + 2m\phi_0(x)}.$$

On the other hand, since σ_j is homomorphic, it follows from Cauchy's integral formula that

$$\sum_{j=1}^{d_k} |\partial \sigma_j(x)|^2 \lesssim r^{-n-2} \sum_{j=1}^{d_k} \int_{x + \partial_0 \Delta_r^n} |\sigma_j|^2 d\xi_1 ... d\xi_n \lesssim r^{-2} \sup_{x + \Delta_r^n} \sum_{j=1}^{d_k} |\sigma_j|^2 = r^{-2} \sup_{x + \Delta_r^n} e^{2m(\psi_{m,\delta} + \phi_0)},$$

for every $0 < r < \text{dist}(x, \partial U)$, where Δ_r denotes the disk of radius r with center at 0 in $\mathbb{C}$, and $\partial_0 \Delta_r^n := (\partial \Delta_r)^n$. Therefore, by (5.3) and the fact $\phi_0 \in \mathcal{C}^1$, we get

(5.10) $$\sum_{j=1}^{d_k} |\partial \sigma_j(x)|^2 \lesssim r^{-2n-2} e^{2(m - m_0)(\sup_{\mathbb{B}(x,r)} \varphi_\delta + \phi_0) + C_5(m - m_0)r}.$$

Combining (5.8), (5.9) and (5.10), we get

$$|\partial \psi_{m,\delta}| \lesssim 1 + \frac{1}{m\delta^{1/2} r^{n+1}} e^{(m - m_0)(\sup_{\mathbb{B}(x,r)} \varphi_\delta - \varphi_\delta(x)) + C_5(m - m_0)r}.$$

$\square$

The new point here is that we approximate φ through the analytic approximation sequence for φ_δ with δ depending on m. We will choose δ to be very small compared to m.

Lemma 5.9. *Let u be a bounded negative psh function on the open unit ball $\mathbb{B}$ of $\mathbb{C}^n$ and let $K \Subset \mathbb{B}$. Let v be a Hölder continuous plurisubharmonic function on $\mathbb{B}$ and denote $\mu = (dd^c v)^n$. Then there exist constants $\alpha = \alpha(\mu, K)$ and $C = C(n, K, \mu, \|u\|_{L^\infty}) > 0$ such that*

$$\int_K \big|\sup_{x' \in \mathbb{B}(x,s)} u(x') - u(x)\big| d\mu \leq Cs^\alpha,$$

for every $0 < s < r_0^3$, where $r_0 := \frac{1}{4} \inf_{w \in K} \mathrm{dist}(w, \partial\mathbb{B})$.

Proof. Denote $M = \|u\|_{L^\infty}$, $U = (1 - 3r_0)\mathbb{B}$ and $V = (1 - 2r_0)\mathbb{B}$. First, we prove that

$$(5.11) \qquad \int_V \big|\sup_{y \in \mathbb{B}(x,s)} u(y) - u(x)\big| d\,\mathrm{Leb} \leq C_0 M s^{2/3},$$

for every $0 < s < r_0^3$, where $C_0 > 0$ is a constant depending only on n and r_0.

For every $0 < r < r_0$ and $z \in U$, we denote

$$\hat{u}_r(z) = \frac{1}{\mathrm{vol}(\mathbb{B}(z,r))} \int_{\mathbb{B}(z,r)} u(\xi) dV(\xi),$$

and

$$\bar{u}_r(z) = \sup_{\xi \in \mathbb{B}(z,r)} u(\xi).$$

Let $z_0 \in V$ and $v_M := \frac{M}{r_0}(|z - z_0|^2 - 1)$. We have $v_M < u$ on $\mathbb{B}_{\sqrt{1/2 - r_0}}(z_0)$ (which contains $V + r_0\mathbb{B}$ because $r_0 < 1$). By the comparison principle for Laplace operator, one has

$$\int_{\{v_M < u\}} \Delta u \leq \int_{\{v_M < u\}} \Delta v_M \lesssim M/r_0.$$

It follows that there exists $C_1 > 0$ depending only on n such that

$$\int_{V + r_0\mathbb{B}} \Delta u \leq \frac{C_1 M}{r_0}.$$

Then, by Jensen formula (see, for example, [1, 17]), one has

$$(5.12) \qquad \int_V |\hat{u}_r(z) - u(z)| d\,\mathrm{Leb} \leq C_2 M r^2,$$

for every $0 < r < r_0$, where $C_2 > 0$ depends only on n and r_0.

For every $z \in V$ and for every $0 < s < r$, there exists $\hat{z} \in \overline{\mathbb{B}(z,s)}$ such that

$$\bar{u}_s(z) = u(\hat{z}) \leq \hat{u}_r(\hat{z}).$$

Since u is negative, it follows that

$$(5.13) \qquad \bar{u}_s(z) \leq \left(\frac{r - s}{r}\right)^{2n} \hat{u}_{r-s}(z).$$

Combining (5.12) and (5.13), we get

$$\int_V |\bar{u}_s(z) - u(z)| d\,\mathrm{Leb} \leq \left(\frac{r - s}{r}\right)^{2n} \int_V |\hat{u}_{r-s}(z) - u(z)| d\,\mathrm{Leb} + \frac{r^{2n} - (r - s)^{2n}}{r^{2n}} \int_V |u(z)| d\,\mathrm{Leb}$$

$$\leq C_3 \left(M(r - s)^2 + \frac{Ms}{r}\right),$$

for every $0 < s < r < r_0$, where $C_3 > 0$ is a constant depending only on n and r_0. Choosing $s = r^3$, we obtain (5.11) (with $C_0 = 2C_3$).

Fix $s \in (0, r_0^3)$. For every $z \in V$, we denote $\psi(z) = \frac{M}{r_0}(|z|^2 - (1 - 2r_0)^2)$, $u' := \max\{u, \psi\}$ and $v' := \max\{\bar{u}_s, \psi\}$. We have $u = u'$ on K, $\bar{u}_s = v'$ on K and $u' = v' = \psi$ on $V \setminus U$. Let $\phi \in \mathscr{C}_0^\infty(V)$ such that $0 \le \phi \le 1$ and $\phi \equiv 1$ on U. Put $\tilde{u} = \phi u'$ and $\tilde{v} = \phi v'$. By using the standard embedding $\mathbb{C}^n \hookrightarrow \mathbb{CP}^n$, one can extend $\tilde{u}$ and $\tilde{v}$ to $A\omega_{FS}$-plurisubharmonic functions on $\mathbb{CP}^n$, where $A \ge 1$ is a constant depending only on n, M and r_0. Since $\mu = (dd^c v)^n$, we have $\tilde{\mu} := \mathbf{1}_{\mathbf{V}}\mu$ is a Hölder continuous measure on $\mathbb{CP}^n$. Therefore, there exist constants $\beta = \beta(\tilde{\mu}) > 0$ and $C_4 = C_4(\tilde{\mu}, M, A) > 0$ such that

$$(5.14) \qquad \int_K |\bar{u}_s - u|d\mu \le \|\tilde{u} - \tilde{v}\|_{L^1(\tilde{\mu})} \le C_4\|\tilde{u} - \tilde{v}\|_{L^1(\mathbb{CP}^n)}^\beta \le C_4\|\bar{u}_s - u\|_{L^1(V)}^\beta.$$

Combining (5.11) and (5.14), we get

$$\int_K |\bar{u}_s - u|d\mu \le C_5 s^{2\beta/3},$$

where $C_5 > 0$ is a constant depending on $n, r_0, \tilde{\mu}$ and M. The proof is completed. $\qquad\square$

Recall that $N = \{\rho = -\infty\}$. By the choice of ρ, and Lojasiewicz's inequality (e.g., see [3]), there exist constants $A_0, A_1 > 1$ such that

$$(5.15) \qquad A_0 \log \mathrm{dist}(x, N) - A_1 \le \rho(x) \le \frac{1}{A_0} \log \mathrm{dist}(x, N) + A_1,$$

for every $x \in X$.

Theorem 5.10. *Let μ be a Hölder continuous measure on X and $p \ge 1$ be a constant. Assume that φ is bounded on X and $B := \|\varphi\|_{L^\infty}$. Then there exist a constant $C > 0$ and a family of θ-psh functions $\psi_{m,\delta}$ with $\delta \in (0, a_0/m), m \in \mathbb{Z}^+$ satisfying the following three properties:*
  (i)

$$\|\psi_{m,\delta} - \varphi\|_{L^p(\mu)} \le C\frac{|\log \delta| + \log m}{m} + C\delta,$$

  (ii)

$$\psi_{m,\delta}(x) \ge \varphi(x) - \frac{Bm_0}{m} + A_0(\delta + m^{-1}) \log \mathrm{dist}(x, N) - C\left(\delta + \frac{|\log \delta|}{m}\right)$$

for every $x \in X$,
  (iii)

$$|\nabla \psi_{m,\delta}(x)| \le C\delta^{-1/2} e^{(B+1)m} e^{-A_0 m\delta \log \mathrm{dist}(x, N)}$$

for every $x \in X$.

Proof. We note that the assumption that φ is bounded implies that the Chern class of L is nef by Demailly's regularisation theorems. The property (ii) follows from Theorem 5.8 (i) and from (5.15). The property (iii) follows from Theorem 5.8 (ii) applied to $r = 1$ and from (5.15). It remains to prove (i).

Since $\varphi_\delta = (1 - \delta)\varphi + \delta\rho$, we have

$$(5.16) \quad \sup_{\mathbb{B}(x,r)} \varphi_\delta - \varphi_\delta(x) \le (1 - \delta)(\sup_{\mathbb{B}(x,r)} \varphi - \varphi(x)) + \delta|\rho(x)| \le \sup_{\mathbb{B}(x,r)} \varphi - \varphi(x) + \delta|\rho(x)|,$$

and

$$(5.17) \qquad \left| \frac{m - m_0}{m} \varphi_\delta(x) - \varphi(x) \right| \leq \left(\frac{m_0}{m} + \delta \right) |\varphi(x)| + \delta|\rho(x)|,$$

for every $x \in X$, $m > m_0$ and $0 < \delta < 1$.

Using (5.16), (5.17) and Theorem 5.8 (i), we get

$$|\psi_{m,\delta} - \varphi| \leq \left| \psi_{m,\delta} - \frac{m - m_0}{m} \varphi_\delta \right| + \left| \frac{m - m_0}{m} \varphi_\delta - \varphi \right|$$

$$\leq \sup_{B(x,r)} \varphi_\delta - \varphi_\delta(x) + C_1 r + C_1 \frac{|\log r| + |\log \delta| + 1}{m} + \left(\frac{m_0}{m} + \delta \right) |\varphi(x)| + \delta|\rho(x)|$$

$$\leq \sup_{\mathbb{B}(x,r)} \varphi - \varphi(x) + 2\delta|\rho(x)| + C_1 r + B\delta + C_2 \frac{|\log r| + |\log \delta|}{m},$$

for every $m > m_0, r > 0$ and $0 < \delta < 1/2$, where $C_1, C_2 > 0$ are constants. Then we have
(5.18)

$$\|\psi_{m,\delta} - \varphi\|_{L^p(\mu)} \leq \| \sup_{\mathbb{B}(x,r)} \varphi - \varphi(x)\|_{L^p(\mu)} + C_3 \left(\frac{|\log \delta| + |\log r|}{m} + \delta + r + \delta\|\rho\|_{L^p(\mu)} \right).$$

It follows from [20, Proposition 4.4] that there exist constants $\epsilon, M > 0$ depending only on X, ω, θ and μ satisfying

$$\int_X e^{-\epsilon w} d\mu \leq M,$$

for every $w \in \mathrm{PSH}(X, \theta)$ with $\sup_X w = 0$. Then, by Hölder inequality, we have

$$\| \sup_{\mathbb{B}(x,r)} \varphi - \varphi(x)\|_{L^p(\mu)} \leq \| \sup_{\mathbb{B}(x,r)} \varphi - \varphi(x)\|_{L^1(\mu)}^{\frac{1}{2p}} \| \sup_{\mathbb{B}(x,r)} \varphi - \varphi(x)\|_{L^{2p-1}(\mu)}^{\frac{2p-1}{2p}} \leq C_4 \| \sup_{\mathbb{B}(x,r)} \varphi - \varphi(x)\|_{L^1(\mu)}^{\frac{1}{2p}},$$

where $C_4 > 0$ is a constant depending only on M, ϵ, μ and p. This combined with Lemma 5.9 gives

$$(5.19) \qquad \| \sup_{\mathbb{B}(x,r)} \varphi - \varphi(x)\|_{L^p(\mu)} \leq C_5 r^{\alpha/p},$$

for every $0 < r < r_0$, where $r_0 = r_0(X, \omega), \alpha = \alpha(X, \omega, \mu)$ and $C_5 = C_5(n, X, \omega, \theta, \mu, B, p)$ are positive constants.

Combining (5.18) and (5.19), we get

$$\|\psi_{m,\delta} - \varphi\|_{L^p(\mu)} \leq C_6 \left(r^{\alpha/p} + \frac{|\log \delta| + |\log r|}{m} + \delta + r \right),$$

for every $m > m_0$, $0 < r < r_0$ and $0 < \delta < 1/2$. Choosing $r = \frac{r_0}{m^{p/\alpha}}$, we obtain (i). The proof is completed. $\qquad \square$

6. Going up to the desingularisation of N

In this section we will prove Theorem 1.1. Let $\pi : \widehat{X} \to X$ be the composition of sequence of blowups along smooth centers over N such that $\widehat{N} := \pi^{-1}(N)$ is a simple normal crossing hypersurface in X'. By Lojasiewicz's inequality, one has

$$(6.1) \qquad \mathrm{dist}(\pi(x), N) \lesssim \mathrm{dist}(x, \widehat{N}) \lesssim \mathrm{dist}^\beta(\pi(x), N),$$

for some constant $\beta > 0$ independent of $x \in \widehat{X}$. Let $\widehat{\varphi} := \pi^* \varphi$ which is $\widehat{\theta}$-psh, where $\widehat{\theta} := \pi^* \theta$.

Theorem 6.1. *Let μ be a Hölder continuous measure on $\widehat{X}$ and $p \geq 1$ be a constant. Assume that φ is bounded and let $B := \|\varphi\|_{L^\infty}$. Then there exist constants $A, C > 0$ and a family of $\widehat{\theta}$-psh functions $\widehat{\psi}_{m,\delta}$ with $\delta \in (0, a_0/m), m \in \mathbb{Z}^+$ satisfying the following three properties:*

(i)
$$\|\widehat{\psi}_{m,\delta} - \widehat{\varphi}\|_{L^p(\mu)} \leq C \frac{|\log \delta| + \log m}{m} + C\delta,$$

(ii)
$$\widehat{\psi}_{m,\delta}(x) \geq \widehat{\varphi}(x) - \frac{Bm_0}{m} + A\delta \log \operatorname{dist}(x, \widehat{N}) - C\left(\delta + \frac{|\log \delta|}{m}\right),$$

for every $x \in \widehat{X}$,

(iii)
$$|\nabla \widehat{\psi}_{m,\delta}(x)| \leq C\delta^{-1/2} e^{(B+1)m} e^{-Am\delta \log \operatorname{dist}(x, \widehat{N})},$$

for every $x \in \widehat{X}$.

Proof. Let $\psi_{m,\delta}$ be functions in Theorem 5.10. Let $\widehat{\psi}_{m,\delta} := \pi^* \psi_{m,\delta}$. The desired assertions (ii) and (iii) follow directly from (6.1) and Theorem 5.10. To see why (i) holds, we recall that π is a composition of successive blowups along smooth centers. Thus the desired inequality (i) is deduced by Theorem 5.10 and Corollary 4.3 applied to μ. The proof is complete. $\qquad \square$

Lemma 6.2. *Assume that φ is bounded on X and $(dd^c \varphi + \theta)^n = \mu$ is a Hölder continuous measure. Let γ be an arbitrary constant in $(0, 1)$. Then for every constant $D > 1$, there is a constant $c_{D,\gamma} > 0$ so that*
$$|\widehat{\varphi}(x) - \widehat{\varphi}(y)| \leq \frac{c_{D,\gamma}}{|\log \operatorname{dist}(x, y)|^\gamma},$$

for every $x, y \in \widehat{X} \backslash \widehat{N}$ with
$$(\operatorname{dist}(x, y))^D \leq \min\{\operatorname{dist}(x, \widehat{N}), \operatorname{dist}(y, \widehat{N})\}.$$

Proof. Without loss of generality, we can assume that $0 < \operatorname{dist}(x, y) < 1/2$. Let $p > 1$ be a constant. Denote $\gamma_0 := p/(p + 2n + 1)$ and $\gamma = p/(p + 2n + 2)$. Note that if $p \to \infty$, then $\gamma \to 1$. Let $\delta := m^{-2D}$. By Lemma 4.1 (we choose the constant $\gamma = 1$ in Lemma 4.1) and Theorem 6.1(i), one get
$$\widehat{\psi}_{m,\delta}(x) - \widehat{\varphi}(x) \lesssim_{\gamma_0} \|\widehat{\psi}_{m,\delta} - \widehat{\varphi}\|_{L^p(\mu)}^{\gamma_0} \lesssim_{\gamma_0} \left(\frac{\log m}{m}\right)^{\gamma_0},$$

for every $x \in \widehat{X}$, $m > m_0$. This combined with Theorem 6.1 (ii) yields

(6.2)
$$|\widehat{\psi}_{m,\delta}(x) - \widehat{\varphi}(x)| \lesssim \left(\frac{\log m}{m}\right)^{\gamma_0} + m^{-2D}(-\log \operatorname{dist}(x, \widehat{N}))_+$$

for every $x \in \widehat{X}$, $m > m_0$. Here $(-\log \operatorname{dist}(x, \widehat{N}))_+ = \max\{-\log \operatorname{dist}(x, \widehat{N}), 0\}$.

Let $l_{x,y}$ be the curve chosen as in Lemma 2.4 (for $\widehat{N}$ in place of N). Now using (6.2) and Theorem 6.1 (iii), and Lemma 2.4, we estimate

$$|\widehat{\varphi}(x) - \widehat{\varphi}(y)| \leq |\widehat{\varphi}(x) - \widehat{\psi}_{m,\delta}(x)| + |\widehat{\varphi}(y) - \widehat{\psi}_{m,\delta}(y)| + |\widehat{\psi}_{m,\delta}(x) - \widehat{\psi}_{m,\delta}(y)|$$

$$\lesssim \left(\frac{\log m}{m}\right)^{\gamma_0} + m^{-2D}(-\log \operatorname{dist}(x, \widehat{N}))_+ + m^{-2D}(-\log \operatorname{dist}(y, \widehat{N}))_+$$

$$+ \operatorname{dist}(x,y) m^D e^{m(B+1)} e^{-Am^{-2D+1}\log \operatorname{dist}(l_{x,y}(t),\widehat{N})},$$

for some point $t \in [0,1]$. Since $\operatorname{dist}(l_{x,y}(t), \widehat{N}) \geq C^{-1}\min\{\operatorname{dist}(x,\widehat{N}), \operatorname{dist}(y,\widehat{N})\}$, we obtain

$$|\widehat{\varphi}(x) - \widehat{\varphi}(y)| \lesssim \left(\frac{|\log \delta|}{m}\right)^{\gamma_0} + \delta^{\gamma_0} + \delta(-\log\min\{\operatorname{dist}(x,\widehat{N}), \operatorname{dist}(y,\widehat{N})\})_+$$

$$+ \operatorname{dist}(x,y)\delta^{-1/2} e^{m(B+1)} e^{-Am^{-2D+1}\log\min\{\operatorname{dist}(x,\widehat{N}),\operatorname{dist}(y,\widehat{N})\}}.$$

Hence, if $(\operatorname{dist}(x,y))^D \leq \min\{\operatorname{dist}(x, \widehat{N}), \operatorname{dist}(y, \widehat{N})\}$ then we have

$$|\widehat{\varphi}(x) - \widehat{\varphi}(y)| \lesssim m^{-\gamma} - Dm^{-2D}\log \operatorname{dist}(x,y)$$

$$+ \operatorname{dist}(x,y) m^D e^{m(B+1)} e^{-ADm^{-2D+1}\log \operatorname{dist}(x,y)}.$$

By choosing

$$m := \max\left\{m_0 + 1, \frac{\gamma|\log \operatorname{dist}(x,y)|}{3(B+1)}\right\},$$

we get

$$|\widehat{\varphi}(x) - \widehat{\varphi}(y)| \leq \frac{c_D}{|\log \operatorname{dist}(x,y)|^{\gamma}},$$

for every $x, y \in \widehat{X} \backslash \widehat{N}$ with

$$(\operatorname{dist}(x,y))^D \leq \min\{\operatorname{dist}(x, \widehat{N}), \operatorname{dist}(y, \widehat{N})\}.$$

This finishes the proof. $\square$

Proposition 6.3. *Assume that φ is bounded on X and θ_φ^n is a Hölder continuous Monge-Ampère measure. Then for every constant $\gamma \in (0,1)$, there exists a constant $C_\gamma > 0$ such that*

$$|\varphi(x) - \varphi(y)| \leq \frac{C_\gamma}{|\log \operatorname{dist}(x,y)|^{\gamma}},$$

for every $x, y \in X \backslash N$.

Proof. By Lemma 6.2, we can apply Proposition 2.3 to $\widehat{\varphi}$, and we see that $\widehat{\varphi}$ is $\log^\gamma$-continuous on $\widehat{X}$. This combined with Lemma 3.1 yields that φ is $\log^\gamma$-continuous. $\square$

Proof of Theorem 1.1. Throughout this proof, C_j ($j = 1, 2, 3...$) is a constant independent of m, δ, x, r.

Let $\gamma \in (0,1)$. By Proposition 6.3, we have

$$\sup_{x' \in \mathbb{B}(x,r)} \varphi_\delta(x') - \varphi_\delta(x) \leq (1-\delta)(\sup_{x' \in \mathbb{B}(x,r)} \varphi(x') - \varphi(x)) - \delta\rho(x)$$

$$\leq C_1\left(|\log r|^{-\gamma} + \delta|\log \operatorname{dist}(x,N)|\right),$$

for every $x \in X$ and $0 < r, \delta < 1/2$. This combined with Theorem 5.8 (ii) yields

$$|\nabla\psi_{m,\delta}(x)| \leq C_2\delta^{-1/2} r^{-n-1} e^{C_2 m(|\log r|^{-\gamma} + \delta|\log \operatorname{dist}(x,N)| + r)},$$

for every $m > m_0$. Now choose

$$r := e^{-m^{\frac{1}{1+\gamma}}}.$$

We obtain that

(6.3) $|\nabla \psi_{m,\delta}(x)| \leq C_3 \delta^{-1/2} e^{C_3 m^{\frac{1}{1+\gamma}} + C_3 m \delta |\log \text{dist}(x,N)|},$

for every $x \in X$, $0 < \delta < 1/2$ and $m > m_0$.

Let $\pi : \widehat{X} \to X$ and $\widehat{N}$ be as above. Let $\widehat{\psi}_{m,\delta} := \pi^* \psi_{m,\delta}$. Thanks to (6.3) one gets immediately the following property (which is a stronger version of Theorem 6.1 (iii)):

(6.4) $|\nabla \widehat{\psi}_{m,\delta}(x)| \leq C_4 \delta^{-1/2} e^{C_4 m^{\frac{1}{1+\gamma}} + C_4 m \delta |\log \text{dist}(x,\widehat{N})|},$

for every $x \in \widehat{X}$.

Now arguing exactly as in the proofs of Lemma 6.2 (use (6.4) in place of Theorem 6.1 (iii)) with $\delta := m^{-2D}$, we get

$$|\widehat{\varphi}(x) - \widehat{\varphi}(y)| \lesssim m^{-\gamma} - Dm^{-2D} \log \text{dist}(x,y)$$
$$+ \text{dist}(x,y) m^D e^{C_4 m^{\frac{1}{1+\gamma}}} e^{-C_5 m^{-2D+1} \log \text{dist}(x,y)},$$

for every $x, y \in \widehat{X} \backslash \widehat{N}$ with $(\text{dist}(x,y))^D \leq \min\{\text{dist}(x,\widehat{N}), \text{dist}(y,\widehat{N})\}$. Now letting

$$m := \max \left\{ m_0 + 1, \left(\frac{\gamma |\log \text{dist}(x,y)|}{3C_4} \right)^{1+\gamma} \right\},$$

we obtain

$$|\widehat{\varphi}(x) - \widehat{\varphi}(y)| \lesssim |\log |x - y||^{-\gamma(1+\gamma)},$$

for every $x, y \in \widehat{X} \backslash \widehat{N}$ with $(\text{dist}(x,y))^D \leq \min\{\text{dist}(x,\widehat{N}), \text{dist}(y,\widehat{N})\}$. We note that if $\gamma \to 1$, then $\gamma(1+\gamma) \to 2$. Using again arguments from the proof of Proposition 6.3 we infer that Proposition 6.3 holds for $\gamma(1+\gamma)$ in place of γ. Applying now Proposition 2.3 to $\widehat{\varphi}$, we see that $\widehat{\varphi}$ is $\log^{\gamma(1+\gamma)}$-continuous on $\widehat{X}$. This combined with Lemma 3.1 yields that φ is $\log^{\gamma'}$-continuous for every $\gamma' \in (0,2)$. Repeating this procedure gives the desired assertion.   □

7. Log continuity of Monge-Ampère metrics

In this section we prove Corollary 1.2. We start with some auxiliary results. We fix a smooth Kähler form ω on X which induces a distance on X.

Lemma 7.1. *Let v be a bounded ω-psh function. Then there exists a constant $C > 0$ such that for every $x \in X$ and $\epsilon \in (0,1]$ one has*

$$\lambda(v,x,\epsilon) := \epsilon^{-2n+2} \int_{\mathbb{B}(x,\epsilon)} (dd^c v + \omega) \wedge \omega^{n-1} \leq C/|\log \epsilon|.$$

Proof. Fix $x_0 \in X$. Let ψ be a negative ω-psh function on X such that $\psi = c \log |x - x_0|$ on an open neighborhood of x_0 in X for some constant $c > 0$, and ψ is smooth outside x_0. For

ϵ small enough, we see that $\psi(x) = c \log |x - x_0|$ on $\mathbb{B}(x, \epsilon)$. Hence $\psi \leq c \log \epsilon$ on $\mathbb{B}(x_0, \epsilon)$. Recall also that

$$\epsilon^{-2n+2} \int_{\mathbb{B}(x,\epsilon)} (dd^c v + \omega) \wedge \omega^{n-1} \lesssim \int_{\mathbb{B}(x,\epsilon)} (dd^c v + \omega) \wedge (dd^c \psi + \omega)^{n-1},$$

see [13, Page 159]. Hence we get

$$\lambda(v, x, \epsilon) \lesssim |\log \epsilon|^{-1} \int_{\mathbb{B}(x,\epsilon)} -\psi(dd^c v + \omega) \wedge (dd^c \psi + \omega)^{n-1}$$

$$\leq |\log \epsilon|^{-1} \int_X -\psi(dd^c v + \omega) \wedge (dd^c \psi + \omega)^{n-1}$$

$$= |\log \epsilon|^{-1} \int_X -\psi \omega \wedge (dd^c \psi + \omega)^{n-1}$$

$$+ |\log \epsilon|^{-1} \int_X -v dd^c \psi \wedge \omega \wedge (dd^c \psi + \omega)^{n-1}$$

$$\lesssim |\log \epsilon|^{-1} (\|v\|_{L^\infty} + 1).$$

This finishes the proof. □

Lemma 7.2. *Let u be a $\log^M$-continuous θ-psh function on X such that u is smooth outside N. Let $\delta \in (0, 1]$ be a constant. Then there exist a constant $C > 0$ independent of δ and a sequence of smooth $(\theta + \delta\omega)$-psh function $(u_\epsilon)_\epsilon$ so that u_ϵ converges uniformly to u and u_ϵ converges to u locally in the C^∞-topology on $X \backslash N$.*

Proof. This follows essentially from Demailly's regularisation of psh functions ([14, Section 8]). Let $\exp_z$ be the exponential map at $z \in X$ of (X, ω). Let χ be a cut-off function as in [14, Page 492]. We define

$$u_\epsilon(z) := \frac{1}{C\epsilon^{2n}} \int_{\zeta \in T_z X} u(\exp_z(\zeta)) \chi'(|\zeta|^2/\epsilon^2) d\lambda(\zeta),$$

where $d\lambda$ denotes the Lebesgue measure on the Hermitian space $T_z X$ and

$$C := \int_{\zeta \in T_z X} \chi'(|\zeta|^2) d\lambda(\zeta).$$

One sees immediately that u_ϵ is $\log^M$-continuous uniformly in ϵ because u is already $\log^M$-continuous.

By [14, Proposition 8.5 and Lemma 8.6], we know that there is a constant $A_1 > 0$ such that

$$dd^c u_\epsilon(x) + \theta(x) \geq \left(-A_1 \lambda(u, x, \epsilon) - A_1 \epsilon / |\log \epsilon| \right) \omega,$$

where

$$\lambda(u, x, \epsilon) := \epsilon^{-2n+2} \int_{\mathbb{B}(x,\epsilon)} (dd^c u + \omega) \wedge \omega^{n-1} \leq A_2 / |\log \epsilon|$$

for some constant A_2 independent of x by Lemma 7.1. Hence we infer that

$$dd^c u_\epsilon + \omega \geq -A_3 |\log \epsilon|^{-1} \omega$$

for some constant $A_3 > 0$ independent of ϵ. This finishes the proof. □

Lemma 7.3. *Let $\delta \in (0, 1]$, $M > 1$ and $C_0 > 0$ be constants. Let u be a smooth $(\theta + \delta\omega)$-psh functions such that*

$$|u(x) - u(y)| \leq C_0 |\log \mathrm{dist}(x, y)|^{-2M}$$

for every $x, y \in X$. Denote by $\tilde{d}$ the distance induced by $dd^c u + \theta + \delta\omega$. Then

$$\tilde{d}(x, y) \leq C |\log \mathrm{dist}(x, y)|^{-M+1}$$

for every $x, y \in X$, where $C > 0$ is a constant independent of u and δ.

Proof. Let $\Omega(r)$ be the modulus of continuity of u. By hypothesis, one has

$$(7.1) \qquad\qquad\qquad \Omega(r) \leq C_0 |\log r|^{-2M}$$

for every $0 < r < 1$. We cover X by finitely many local charts (which are relatively compact in bigger local charts) and since the Kähler form ω is equivalent to the standard Kähler form on $\mathbb{C}^n$ in these local charts, we can assume that ω is equal to the standard form on $\mathbb{C}^n$ on these local charts.

Let $\mathbb{B}(x, r)$ denotes the ball of radius r with center at $x \in \mathbb{C}^n$. Fix $x^* \in X$ and a local chart U around x^* biholomorphic to $\mathbb{B}(0, 2)$ such that $x^* = 0$ in these local coordinates. Define $d(x) := \tilde{d}(x, 0)$. Recall that $\tilde{d}$ is the Riemannian metric induced by $dd^c u + \theta + \delta\omega$. For $x \in \mathbb{B}(0, 1)$, let

$$d_r(x) := \mathrm{vol}(\mathbb{B}(x, r))^{-1} \int_{x' \in \mathbb{B}(x, r)} d(x') \, \omega^n.$$

Arguing as in the proof of [28, Lemma 5] (see also [34] or [27]) and using (7.1), for every $x_0 \in \mathbb{B}(0, 1)$, one obtains

$$\int_{\mathbb{B}(x_0, r)} |\nabla d|_\omega^2 \omega^n \leq C_1 r^{2n} + C_1 \int_{\mathbb{B}(x_0, 3r/2)} |u(x) - u(x_0)| \omega^n \leq C_2 r^{2n-2} |\log r|^{-2M}.$$

Therefore, by Poincaré inequality, we infer

$$r^{-2n} \int_{\mathbb{B}(x_0, r)} |d(x) - d_r(x_0)|^2 \omega^n \leq C_3 |\log r|^{-2M},$$

where $C_3 > 0$ is a uniform constant independent of x_0, δ and r. This combined with Hölder inequality gives

$$(7.2) \qquad\qquad r^{-2n} \int_{\mathbb{B}_\omega(x_0, r)} |d(x) - d_r(x_0)| \omega^n \leq C_3 |\log r|^{-M}.$$

We now use some arguments similar to the proof of Campanato's lemma. We follow the presentation in [29, Chapter 3]. Assume $0 < r_1 < r_2 < 1$ and $x_1, x_2 \in \mathbb{B}(0, 1)$ with $\mathbb{B}(x_1, r_1) \subset \mathbb{B}(x_2, r_2)$. Observe that

$$|d_{r_1}(x_0) - d_{r_2}(x_0)| \leq |d_{r_1}(x_0) - d(x)| + |d_{r_2}(x_0) - d(x)|$$

for every $x \in \mathbb{B}(x_1, r_1) \subset \mathbb{B}(x_2, r_2)$. It follows that

$$|d_{r_1}(x_1) - d_{r_2}(x_2)| \leq$$
$$\mathrm{vol}(\mathbb{B}_\omega(x_1, r_1))^{-1} \left(\int_{\mathbb{B}_\omega(x_1, r_1)} |d_{r_1}(x_1) - d(x)| \omega^n + \int_{\mathbb{B}_\omega(x_2, r_2)} |d_{r_2}(x_2) - d(x)| \omega^n \right)$$

By (7.2), it follows that

$$(7.3) \qquad |d_{r_1}(x_1) - d_{r_2}(x_2)| \lesssim r_1^{-2n} \left(r_1^{2n} |\log r_1|^{-M} + r_2^{2n} |\log r_2|^{-M} \right).$$

Applying the last inequality to $r_1 = r/2^{k+1}, r_2 = r/2^k$ and $x_1 = x_2 = x_0$ yields

$$|d_{2^{-k}r}(x_0) - d_{2^{-k-1}r}(x_0)| \leq C_4 \big(|\log r| + (k+1)\log 2\big)^{-M} \leq \int_k^{k+1} \frac{C_4 dt}{(|\log r| + t\log 2)^{-M}},$$

for some uniform constant $C_4 > 0$ independent of $x_0, r, k, \epsilon, \delta$. Summing over $k = 0, 1, 2, \dots$ yields

$$\sum_{k=0}^{\infty} |d_{2^{-k}r}(x_0) - d_{2^{-k-1}r}(x_0)| \leq \int_0^{\infty} \frac{C_4 dt}{(|\log r| + t\log 2)^{-M}} \leq C_5 |\log r|^{-M+1}.$$

Since d_r converges uniformly to d on $\mathbb{B}(0,1)$ as $r \searrow 0$, it follows that

$$(7.4) \qquad\qquad |d(x_0) - d_r(x_0)| \leq C_5 |\log r|^{-M+1}$$

for every $x_0 \in \mathbb{B}(x^*, 1)$ (recall $x^* = 0$), and $0 < r < 1$. Let $x \in \mathbb{B}(x^*, 1/8)$ and $r := \mathrm{dist}(x, x^*) \leq 1/8$. Using (7.4) and then applying (7.3) for $x_1 = x$, $x_2 = x^*$ and $r_2 = 3r_1 = 3r$, we get

$$\begin{aligned}
d(x) &= d(x) - d(x^*) \\
&\leq |d(x) - d_r(x)| + |d(x^*) - d_{3r}(x^*)| + |d_r(x) - d_{3r}(x^*)| \\
&\lesssim |\log r|^{-M+1} + |d_r(x) - d_{3r}(x^*)| \\
&\lesssim |\log r|^{-M+1} + |\log r|^{-M} \\
&\lesssim |\log r|^{-M+1}.
\end{aligned}$$

This finishes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Proof of Corollary 1.2. Write $\omega_F = dd^c u + \theta$, where u is $\log^M$-continuous θ-psh function for every constant $M > 1$ by Theorem 1.1. Fix $\delta \in (0, 1]$. Let u_ϵ be as in Lemma 7.2 for u. Since $dd^c u_\epsilon + \theta + \delta\omega$ converges to $dd^c u + \theta + \delta\omega \geq dd^c u + \theta$ locally in the $\mathcal{C}^\infty$-topology in $X \setminus N$, one sees that the desired assertion follows from Lemma 7.3 by letting $\epsilon \to 0$. $\qquad \square$

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