MODULUS OF CONTINUITY OF MONGE–AMPÈRE POTENTIALS IN BIG COHOMOLOGY CLASSES

QUANG-TUAN DANG, HOANG-SON DO, HOANG HIEP PHAM

ABSTRACT. In this paper, we prove a uniform estimate for the modulus of continuity of solutions to degenerate complex Monge–Ampère equation in big cohomology classes. This improves the previous results of Di Nezza–Lu and of the first author.

1. INTRODUCTION

The study of complex Monge-Ampère equations on compact Kähler manifolds has attracted considerable interest since Yau's resolution of the Calabi conjecture. In connection with the Minimal Model Program, the search for singular Kähler–Einstein metrics leads to the study of degenerate complex Monge–Ampère equations; see [EGZ09, BBE+19, GZ17] and references therein.

Guedj and Zeriahi [GZ07] developed the first step of the study of the non-pluripolar Monge–Ampère measure and solved degenerate complex Monge–Ampère equations with rather general measures on the right-hand side. Their approach was later extended to the setting of big cohomology classes by Boucksom, Eyssidieux, Guedj, and Zeriahi [BEGZ10]. Since then, when the right-hand side belongs to L^p for some $p > 1$, the Hölder continuity of solutions to degenerate complex Monge–Ampère equations has been intensively studied by many authors [Koł98, Koł08, Hie10, EGZ11, BEGZ10, DDG+14, DN14, Dan22, DKN22]. The modulus of continuity of the solution plays a crucial role since it is closely related to the geometric properties of the corresponding Kähler metrics, such as the uniform bounds for diameter of metrics and Gromov–Hausdorff convergence; cf. [FGS20, GS22, GPTW21, GPSS24a, GPSS24b, GGZ25].

The primary goal of this paper is to study the modulus of continuity of solutions to complex Monge–Ampère equations when the right-hand side is not integrable. To state our result, we first introduce some notation and terminology. Let X be a compact Kähler manifold of dimension n equipped with a Kähler form ω_X. We let d, d^c denote the real differential operators on X defined by $d := \partial + \bar\partial$, $d^c := \frac{i}{2}(\bar\partial - \partial)$ so that $dd^c = i\partial\bar\partial$. Fix a closed real smooth (1,1)-form θ representing a big cohomology class. Let $\mathrm{PSH}(X,\theta)$ denote the set of all θ-psh functions. We say that the cohomology class $\{\theta\}$ is *big* if the set $\mathrm{PSH}(X,\theta-\varepsilon\omega_X) \neq \varnothing$ for some $\varepsilon > 0$. Let $\mathrm{Amp}(\theta)$ denote the *ample locus* of θ which is roughly speaking the largest Zariski open subset where $\{\theta\}$ locally behaves like a Kähler class.

We are interested in studying the regularity of solutions to the complex Monge–Ampère equation type $u \in \mathrm{PSH}(X,\theta)$ satisfying

$$(\theta + dd^c u)^n = \nu, \quad \sup_X u = 0, \tag{1.1}$$

Date: April 20, 2025.
2020 *Mathematics Subject Classification.* 32U15, 32W20, 32Q15.
Key words and phrases. Complex Monge-Ampère equations; Hölder measure; Capacities.

where ν is a positive measure on X that puts no mass on pluripolar subsets and satisfies the compatibility condition $\nu(X) = \mathrm{Vol}(\theta)$, u is the unknown θ-function, and the left-hand side of (1.1) denotes the non-pluripolar Monge–Ampère product [BT87, BEGZ10].

Let $\mathcal{C}$ denote the set of ω_X-psh functions u normalized by $\int_X u\omega_X^n = 0$. This is a convex compact set in the $L^1(X)$ topology. Define the following distance on $\mathcal{C}$

$$\mathrm{dist}(u, v) := \|u - v\|_{L^1}$$

where the L^1-norm is with respect to the measure ω_X^n. We say that a measure μ is *Hölder continuous* with Hölder constant B and Hölder exponent $0 < \beta \leq 1$ with respect to dist_{L^1} on $\mathcal{C}$ if for any $u, v \in \mathcal{C}$

$$\int_X |u - v|\omega_X^n \leq B\|u - v\|_{L^1}^\beta.$$

Our main result is the following:

Theorem 1.1. *Let (X, ω_X) be a compact Kähler manifold of dimension n and θ a smooth closed (1,1) form whose cohomology class is big. Let μ be a Hölder continuous measure with Hölder constant B and Hölder exponent $0 < \beta \leq 1$ with respect to dist_{L^1} on $\mathcal{C}$. Assume that ψ is a quasi-psh function on X satisfying $\int_X e^{-\psi} d\mu = \mathrm{Vol}(\theta)$ and $\omega_X + a_0 dd^c\psi \geq 0$, with $a_0 > 0$. Suppose that $u \in \mathcal{E}(X, \theta)$ is a solution to the following equation*

$$(\theta + dd^c u)^n = e^{-\psi}\mu, \quad \sup_X u = 0. \tag{1.2}$$

Then for each $U \Subset \mathrm{Amp}(\theta)\backslash\{\psi = -\infty\}$, there exists a continuous function $F_U : \mathbb{R}_+ \to \mathbb{R}_+$ with $F(0) = 0$ such that

$$|u(z_1) - u(z_2)| \leq F_U(\mathrm{dist}(z_1, z_2)),$$

for every $z_1, z_2 \in U$. Moreover, the choice of F_U depends only on X, U, ω_X, n, θ, a_0, B, β, $\sup_U(-\psi)$ and an upper bound function for $H(a) = \int_X e^{2(V_\theta - u)/a} d\mu$.

Here we notice that $e^{2(V_\theta - u)/a} \in L^1(\mu)$ for every $a > 0$ by Lemma 2.11. In the case $\theta = \omega_X$, we have the following corollary:

Corollary 1.2. *Let (X, ω_X) be a compact Kähler manifold of dimension n. Let μ be a Hölder continuous measure with Hölder constant B and Hölder exponent $0 < \beta \leq 1$ with respect to $dist_{L^1}$ on $\mathcal{C}$. Assume that ψ is a quasi-psh function on X satisfying $\int_X e^{-\psi} d\mu = \mathrm{Vol}(\theta)$, $\omega_X + a_0 dd^c\psi \geq 0$ and*

$$\int_X h(-\psi)e^{-\psi}\mu \leq C_0 \tag{1.3}$$

where $a_0, C_0 > 0$ are constants and $h : (0, \infty) \to (0, \infty)$ is an increasing, concave function with $h(\infty) = \infty$. Suppose that $u \in \mathcal{E}(X, \omega_X)$ is a solution to the following equation

$$(\omega_X + dd^c u)^n = e^{-\psi}\mu, \quad \sup_X u = 0. \tag{1.4}$$

Then for each $U \Subset X\backslash\{\psi = -\infty\}$, there exists a continuous function $F_U : \mathbb{R}_+ \to \mathbb{R}_+$ with $F(0) = 0$ such that

$$|u(z_1) - u(z_2)| \leq F_U(\mathrm{dist}(z_1, z_2)),$$

for every $z_1, z_2 \in U$. Moreover, the choice of F_U depends only on X, U, ω_X, n, θ, a_0, B, β, $\sup_U(-\psi)$, C_0, and h.

Let us emphasize that the continuity of the solution u to equation (1.4) was established by Di Nezza and Lu [DL17, Theorem 3.1]. The contribution of the corollary above is to establish the equicontinuity of the family of solutions corresponding to pairs (μ, ψ) which satisfy the condition (1.3), given h and C_0. We also note that for each pair (μ, ψ) with $e^{-\psi} \in L^1(X, \mu)$, there always exists a pair (h, C) such that the condition (1.3) holds; cf. Remark 2.14.

Acknowledgment. This work was done while the authors were visiting the Vietnam Institute for Advanced Study in Mathematics (VIASM), and they would like to thank VIASM for its hospitality and support.

Ethics declarations. The authors declare no conflict of interest.

2. Preliminaries

Throughout the paper, we let X denote a compact Kähler manifold of dimension n, equipped with a Kähler form ω_X.

2.1. Quasi-plurisubharmonic functions.

Recall that an upper semi-continuous function $\varphi : X \to \mathbb{R} \cup \{-\infty\}$ is called *quasi-plurisubharmonic* (*quasi-psh* for short) if it is locally the sum of a smooth and a plurisubharmonic (psh for short) function.

We say that φ is *θ-plurisubharmonic* (*θ-psh* for short) if it is quasi-psh, and $\theta + dd^c\varphi \geq 0$ in the sense of currents, where $\mathrm{d} = \partial + \bar\partial$ and $\mathrm{d}^c = \frac{i}{2\pi}(\bar\partial - \partial)$ so that $dd^c = \frac{i}{\pi}\partial\bar\partial$. We let $\mathrm{PSH}(X, \theta)$ denote the set of all θ-psh functions, which are not identically $-\infty$. This set is endowed with the weak topology, which coincides with the $L^1(X)$-topology. By Hartogs' lemma, $\varphi \mapsto \sup_X \varphi$ is continuous in this weak topology, it follows that the set of $\varphi \in \mathrm{PSH}(X, \theta)$, with $\sup_X \varphi = 0$ is compact. We refer the reader to [Dem12, GZ17] for more properties of θ-psh functions.

The cohomology class $\{\theta\}$ is said to be *big* if the set $\mathrm{PSH}(X, \theta - \varepsilon\omega_X)$ is not empty for some $\varepsilon > 0$. We now assume that $\{\theta\}$ is big unless otherwise specified. By Demailly's regularization theorem [Dem92], we can find a closed positive $(1,1)$-current $T_0 \in \{\theta\}$ such that

$$T_0 = \theta + dd^c\Psi_0 \geq \varepsilon_0\omega_X$$

for some $\varepsilon_0 > 0$, where Ψ_0 is a quasi-psh function with *analytic singularities*, i.e., locally

$$\Psi_0 = c\log\left[\sum_{j=1}^{N} |f_j|^2\right] + O(1),$$

where the f_j's are holomorphic functions. Such a current T_0 is then smooth on a Zariski open subset $X \setminus \{\Psi_0 = -\infty\}$. We thus define the *ample locus* $\mathrm{Amp}(\theta)$ of θ as the largest Zariski open subset (which exists by the Noetherian property of closed analytic subsets; cf. [Bou04]).

Given $\varphi, \psi \in \mathrm{PSH}(X, \theta)$, we say that φ is *less singular* than ψ, and denote by $\psi \preceq \varphi$, if there exists a constant C such that $\psi \leq \varphi + C$ on X. We say that φ, ψ have the *same singularity type*, and denote by $\varphi \simeq \psi$ if $\varphi \preceq \psi$ and $\psi \preceq \varphi$. There is a natural least singular potential in $\mathrm{PSH}(X, \theta)$ given by

$$V_\theta := \sup\{\varphi \in \mathrm{PSH}(X, \theta) : \varphi \leq 0\}.$$

A function φ is said to have minimal singularities if it has the same singularity type as V_θ. In particular, $V_\theta = 0$ if θ is semi-positive. We see that V_θ is locally bounded in the ample locus.

2.2. **Demailly's regularization.** We consider the exponential mapping $\exp_x : T_xX \ni \zeta \to X$ as follows: $\exp_x(\zeta) = \gamma(1)$ where $\gamma : [0,1] \to X$ is the geodesics starting from $x = \gamma(0)$ with the initial velocity $\gamma'(0) = \zeta$. In the Euclidean space $\mathbb{C}^n$ the exponential map $\exp_x(\zeta) = x + \zeta$.

Following [Dem82, Dem94], for each function $u \in L^1(X)$, we define its δ-*regularization* $\rho_\delta u = \Psi(u)(z,\delta)$ where

$$\Psi(u)(z,w) = \int_{\zeta\in T_zX} u(\exp_z(w\zeta))\chi\Big(|\zeta|^2_{\omega_X}\Big)\, dV_{\omega_X}(\zeta),\ \delta > 0, \tag{2.1}$$

where $\chi : \mathbb{R}_+ \to \mathbb{R}_+$ is defined by

$$\chi(t) = \begin{cases} \frac{\eta}{(1-t)^2}\exp(\frac{1}{t-1}) & \text{if } 0 \le t \le 1, \\ 0 & \text{if } t > 1 \end{cases},$$

with a suitable constant η, such that $\int_{\mathbb{C}^n} \chi(\|z\|^2)\, dV(z) = 1$. The δ-regularization $\rho_\delta u$ can be written by

$$\rho_\delta u(z) = \frac{1}{\delta^{2n}}\int_{\zeta\in T_zX} u(\exp_z(\zeta))\chi\Big(\frac{|\zeta|^2_{\omega_X}}{\delta^2}\Big)\, dV_{\omega_X}(\zeta).$$

Intuitively, this corresponds to the familiar convolution with smoothing kernels. Actually, in the case of $\mathbb{C}^n$ endowed with the Euclidean metric, this is exactly the smoothing convolution; see [Dem94, Remark 4.6].

The following lemma is based on [BD12, Lemma 1.12] and [Dem94, Theorem 4.1], which will be crucial in the sequel.

Lemma 2.1. *Let u be a θ-psh function and define the Kiselman-Legendre transform with level c by*

$$\Phi_{c,\delta} := \inf_{0<t\le\delta}\Big[\rho_t u(z) + K(t^2-\delta^2) + K(t-\delta) - c\log\Big(\frac{t}{\delta}\Big)\Big]. \tag{2.2}$$

Then for some positive constants $0 < \delta_0 < 1$ and $K > 1$ depending on the curvature, the function $\rho_t u(z) + Kt^2$ is increasing in $t \in (0,\delta_0]$ and one has the following estimate for the complex Hessian:

$$\theta + dd^c\Phi_{c,\delta} \ge -(Ac + 2K\delta)\,\omega_X, \tag{2.3}$$

where A is a lower bound of the negative part of the bisectional curvature of ω_X.

Proof. The proof is identical to that of [KN19, Lemma 4.1], which is still valid without the boundedness of u. □

Remark 2.2. The above lemma is extremely crucial for our goal in studying the modulus of continuity of solutions to complex Monge–Ampère equations. We proceed the same way as in the proof of [DDG+14, Theorem D]. The function $\Phi_{c,\delta}$ in our lemma differs from the latter by adding term $K(t-\delta)$ (as shown in [Dem94, Remark 4.7]); this term is applied to take care of the mixed term $|dz||dw|$.

Lemma 2.3. *Let $u \in \mathrm{PSH}(X,\theta)$. If $\rho_\delta u$ is the regularization of u defined as in (2.1) then for δ small enough we have*

$$\int_X \frac{\rho_\delta u - u}{\delta^2}\omega_X^n \le C\|u\|_{L^1(X,\omega_X)},$$

where C only depends on n, X, ω_X.

Proof. The proof follows verbatim from that of [DDG+14, Lemma 2.3]. □

Since the cohomology class $\{\theta\}$ is big, there exists a negative θ-psh function Ψ_0 with analytic singularities such that $\theta + dd^c\Psi_0 \geq \varepsilon_0\omega_X$ for some $\varepsilon_0 > 0$. It follows from [Bou04] that we can choose ψ so that $\mathrm{Amp}(\theta) = X \setminus \mathrm{Sing}(\Psi_0)$. By subtracting a positive constant, we can assume $\Psi_0 \leq V_\theta$.

Fix $0 < c_1 \leq \frac{\varepsilon_0}{4A}$ and $0 < \delta_1 < \min\{\delta_0, \sqrt{\frac{\varepsilon_0}{4K}}\}$, where δ_0 is defined as in Lemma 2.1. For every $0 < c < c_1$ and $0 < \delta < \delta_1$, we define

$$B_0 = B_0(c,\delta) = \frac{Ac + 2K\delta}{\varepsilon_0}. \tag{2.4}$$

Then, we have $0 < B_0 < 1/2$. We set

$$u_{c,\delta} = B_0\Psi_0 + (1 - B_0)\Phi_{c,\delta}. \tag{2.5}$$

where $\Phi_{c,\delta}$ is defined by (2.2). It follows from Lemma 2.1 that $\theta + d^c u_{c,\delta} \geq \varepsilon_0 B_0^2\omega_X$ hence $u_{c,\delta}$ is θ-psh. From now on, we set $\Omega := \mathrm{Amp}(\theta) \setminus \{\psi = -\infty\}$.

Lemma 2.4. *Let $c : (0,\delta_1) \to (0,c_1)$ and $B_1 : \mathbb{R}^+ \to \mathbb{R}^+$ be functions. Assume $\lim_{\delta\to 0^+} c(\delta)/\delta = +\infty$, so that $2K\delta < Ac(\delta)$ for every $0 < \delta < \delta_2$, where $\delta_2 \in (0,\delta_1)$ is sufficiently small. Let $u \in \mathrm{PSH}(X,\theta)$ be such that*

$$\sup_U (u_{c(\delta),\delta} - u) \leq B_1(\delta), \tag{2.6}$$

for every $0 < \delta < \delta_2$, where U is a relatively compact subset of Ω. Then we have

$$\sup_U (\rho_{\kappa(\delta)}u - u) \leq 2(B_1 + C_U B_0 + K\delta), \tag{2.7}$$

where

$$C_U = \sup_U (V_\theta - \Psi_0) \text{ and } \kappa(\delta) = \delta \exp\left(\frac{-4A(C_U B_0 + B_1 + 2K\delta)}{\varepsilon_0 B_0}\right).$$

Proof. By the assumption, we have

$$\begin{aligned} B_1 \geq u_{c,\delta}(z) - u(z) &= B_0(\Psi_0(z) - u(z)) \\ &\quad + (1 - B_0)\left(\rho_{t_0}u(z) + K(t_0^2 - \delta^2) + K(t_0 - \delta) - c\log\left(\frac{t_0}{\delta}\right) - u(z)\right), \end{aligned}$$

for every $z \in U$, where $t_0 = t_0(z) \in (0,\delta)$ is a minimum point of $\rho_t u(z) + Kt^2 + Kt - c\log\left(\frac{t}{\delta}\right)$. Since $V_\theta \geq u$ and $\rho_{t_0}u + Kt_0^2 \geq u$ (see Lemma 2.1), it follows that

$$B_0(\Psi_0(z) - V_\theta(z)) - c(1 - B_0)\log\left(\frac{t_0}{\delta}\right) \leq B_1 + K\delta^2 + K\delta \leq B_1 + 2K\delta$$

for every $z \in U$. Hence, for every $z \in U$,

$$\begin{aligned} t_0(z) &\geq \delta \exp\left(\frac{B_0(\Psi_0(z) - V_\theta(z)) - B_1 - 2K\delta}{c(1 - B_0)}\right) \\ &\geq \delta \exp\left(\frac{-4C_U B_0 - 4B_1 - 8K\delta}{A^{-1}\varepsilon_0 B_0}\right) = \kappa(\delta), \end{aligned}$$

using the facts that $B_0\varepsilon_0 = Ac + 2K\delta \le 2Ac$ and $1 - B_0 \ge \frac{1}{2}$. Since $t \mapsto \rho_t u + Kt^2$ is increasing, we get

$$\begin{aligned}\rho_{\kappa(\delta)}u(z) - u(z) &\le \rho_{t_0}u(z) + Kt_0^2 - u(z)\\ &\le \left(\rho_{t_0}u(z) + K(t_0^2-\delta^2) + K(t_0-\delta) - c\log\left(\frac{t_0}{\delta}\right)\right) - u(z) + 2K\delta\\ &\le \Phi_{c,\delta}(z) - u(z) + 2K\delta\\ &= \frac{1}{1-B_0}(u_{c,\delta}(z) - u(z)) + \frac{B_0}{1-B_0}(u(z) - \Psi_0(z)) + 2K\delta\\ &\le \frac{B_1}{1-B_0} + \frac{B_0}{1-B_0}(V_\theta(z) - \Psi_0(z)) + 2K\delta,\end{aligned}$$

for every $z \in U$, where the last inequality, holds due to (2.6). Therefore,

$$\rho_{\kappa(\delta)}u(z) - u(z) \le \frac{B_1}{1-B_0} + \frac{C_U B_0}{1-B_0} + 2K\delta \le 2(B_1 + B_0 C_U) + 2K\delta,$$

for every $z \in U$. The proof is thus complete. □

2.3. **Non-pluripolar product.** Let $\theta^1, \dots, \theta^n$ be closed smooth real (1,1) form representing big cohomology classes, and $\varphi_j \in \mathrm{PSH}(X, \theta^j)$. Following the construction of Bedford–Taylor [BT87], it has been shown in [BEGZ10] that for each $k \in \mathbb{N}$,

$$\mathbf{1}_{\cap_j\{\varphi_j > V_{\theta^j} - k\}}\theta^1_{\max(\varphi_1, V_{\theta^1}-k)} \wedge \cdots \wedge \theta^n_{\max(\varphi_n, V_{\theta^n}-k)}$$

is well-defined as a Borel positive measure with finite total mass. The sequence of these measures is non-decreasing in k and it converges weakly to the so-called *Monge–Ampère product*, denoted by

$$\theta^1_{\varphi_1} \wedge \cdots \wedge \theta^n_{\varphi_n},$$

which does not charge pluripolar sets by definition. In particular, $\theta^1 = \cdots = \theta^n = \theta$ and $\varphi_1 = \cdots = \varphi_n$ we obtain the non-pluripolar Monge–Ampère measure of φ, denoted by $(\theta + dd^c\varphi)^n$ or simply by θ^n_φ.

The *volume* of a big cohomology class $\{\theta\}$ is given by the total mass of the non-pluripolar Monge–Ampère measure of V_θ, i.e.,

$$\mathrm{Vol}(\theta) := \int_X \theta^n_{V_\theta}.$$

We say that $\varphi \in \mathrm{PSH}(X, \theta)$ has *full Monge–Ampère mass* if $\int_X \theta^n_\varphi = \mathrm{Vol}(\theta)$. We let

$$\mathcal{E}(X, \theta) := \left\{\varphi \in \mathrm{PSH}(X, \theta) : \int_X \theta^n_\varphi = \mathrm{Vol}(\theta)\right\}$$

denote the set of θ-psh functions with full Monge–Ampère mass. Note that θ-psh functions with minimal singularities have full Monge–Ampère mass (see [BEGZ10, Theorem 1.16] for more details), but the converse is not true.

Given a measurable function $f : X \to \mathbb{R}$, we define the *θ-psh envelope* of h by

$$P_\theta(f) := (\sup\{u \in \mathrm{PSH}(X, \theta) : u \le f \text{ on } X\})^*,$$

where the star means we take the upper semi-continuous regularization.

Given a θ-psh function ϕ, Ross and Witt-Nyström [RW14] introduced the "rooftop envelope" as follows

$$P_\theta[\phi](f) = \left(\lim_{C \to +\infty} P_\theta(\min(\phi + C, f))\right)^*.$$

When $f = 0$ we simply write $P_\theta[\phi]$.

Definition 2.5. A function $\phi \in \mathrm{PSH}(X,\theta)$ is called a *model potential* if $\int_X \theta_\phi^n > 0$ and $\phi = P_\theta[\phi]$.

We recall a notion of Monge–Ampère capacity, which has been introduced in [DL17, DL15].

Definition 2.6. Given $\phi \in \mathrm{PSH}(X,\theta)$ and $E \subset X$ a Borel subset we define

$$\mathrm{Cap}_\phi(E) := \sup\left\{\int_E \theta_u^n : u \in \mathrm{PSH}(X,\theta), \phi - 1 \le u \le \phi\right\}.$$

We recover the notion of Monge-Ampère capacity used in [BEGZ10] when $\phi = V_\theta$. In this case, we simply denote by Cap_θ.

Definition 2.7. A family of positive measures $\{\mu_i\}_{i\in I}$ on X is said to be uniformly absolutely continuous with respect to ϕ-capacity if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that for each Borel subset $E \subset X$ satisfying $\mathrm{Cap}_\phi(E) < \delta$ the inequality holds $\mu_i(E) < \varepsilon$ for all i. We denote this by $\mu_i << \mathrm{Cap}_\phi$ uniformly for i. In particular, all such measures must vanish on pluripolar sets.

2.4. **Hölder continuous measures.**

Definition 2.8. A positive measure μ on X is said to be $\mathrm{PSH}(X,\theta)$-Hölder continuous (or, for short, Hölder continuous) if there exist $B > 0$ and $0 < \beta \le 1$ such that

$$\int_X (u-v)d\mu \le B\left(\int_X |u-v|\omega_X^n\right)^\beta,$$

for every $u, v \in \mathcal{C} := \{w \in \mathrm{PSH}(X,\theta) : \int_X w\omega_X^n = 0\}$. In this case, we say that B is the Hölder constant and β is the Hölder exponent of μ with respect to dist_{L^1} on $\mathcal{C}$. We let $\mathcal{M}(B,\beta)$ denote the set of such measures.

Lemma 2.9 ([DN14, Lemma 3.3])**.** *Assume μ is a Hölder continuous measure with Hölder constant B and Hölder exponent $0 < \beta \le 1$ with respect to dist_{L^1} on $\mathcal{C}$. Then there exists $C > 0$ depending only on B such that*

$$\|u-v\|_{L^1(\mu)} \le C \max\left\{\|u-v\|_{L^1(X)}, \|u-v\|^\beta_{L^1(X)}\right\}$$

for all $u, v \in \mathrm{PSH}(X,\theta)$. In particular, if a family $\mathcal{F}$ of θ-psh functions is a relatively compact subset of $L^1(X)$ then there exists $C_\mathcal{F} > 0$ such that

$$\|u-v\|_{L^1(\mu)} \le C_\mathcal{F}\|u-v\|^\beta_{L^1(X)},$$

for all $u, v \in \mathcal{F}$.

Proposition 2.10. *Let μ be a Hölder continuous measure on X with Höler constant B and Hölder exponent $0 < \beta \le 1$ with respect to dist_{L^1} on $\mathcal{C}$. Assume that u, v are θ-psh functions satisfying*

$$\int_X e^{m(u-v)}\omega_X^n < C_0,$$

for positive constants m, C_0. Then, for every $0 < \gamma < \beta$, there exists $C > 0$ depending only on C_0, m, B, β and γ such that

$$\int_X e^{\gamma m(u-v)}d\mu < C.$$

Proof. This is more slightly generalized than the one in [DN14, Proposition 4.4]. Denote $w = v - u$. We have

$$\mathrm{Vol}(\{w < -M\}) \le \int_X e^{-m(w+M)}dV \le C_0 e^{-mM}, \tag{2.8}$$

for every $M > 0$. Denote

$$w_M = \max\{w, -M\} = \max\{v, u - M\} - u.$$

We have $w_M - w = \max\{v, u - M\} - v$ is the difference of two θ-psh functions. By Lemma 2.9, there exists $C_1 > 0$ depending only on B such that

$$\int_X (w_M - w) d\mu \leq C_1 \max\left\{\|w_M - w\|_{L^1(X)}, \|w_M - w\|^{\beta}_{L^1(X)}\right\}. \tag{2.9}$$

Moreover, we also have

$$\mu(\{w < -M - 1\}) \leq \int_X (w_M - w) d\mu, \tag{2.10}$$

and by (2.8),

$$\begin{aligned} \int_X (w_M - w) dV &= \int_0^\infty \mathrm{Vol}(\{w < -M - t\}) dt \\ &\leq C_0 \int_0^\infty e^{-m(M+t)} dt = \frac{C_0 e^{-mM}}{m}. \end{aligned} \tag{2.11}$$

Combining (2.9), (2.10) and (2.11), we get

$$\mu(\{w < -M - 1\}) \leq C_1 \max\left\{\frac{C_0 e^{-mM}}{m}, \left(\frac{C_0 e^{-mM}}{m}\right)^{\beta}\right\} \leq C_2 \cdot e^{-\beta m M},$$

where $C_2 = C_1 \max\{\frac{C_0}{m}, (\frac{C_0}{m})^\beta\}$. Then, for every $0 < \gamma < \beta$, we obtain

$$\begin{aligned} \int_X e^{-\gamma m w} d\mu - \mu(X) &= \gamma m \int_0^\infty \mu(w < -t) e^{\gamma m t} dt \\ &\leq \gamma m C_2 e^{\beta m} \int_0^\infty e^{(-\beta+\gamma) m t} dt \\ &= \frac{C_2 \gamma e^{\beta m}}{\beta - \gamma}. \end{aligned}$$

□

Lemma 2.11. *Let $u \in \mathcal{E}(X, \theta)$. Then*

$$\int_X e^{m(V_\theta - u)} d\mu < +\infty,$$

for every $m > 0$.

Proof. It follows from [Dan22, Proposition 2.10] that for any $b > 0$, $P_{\omega_X}(b(u - V_\theta)) \in \mathcal{E}(X, \omega_X)$ where $P_{\omega_X}(f)$ denotes the largest ω_X-psh function lying below f. We remark here that $u - V_\theta$ is well-defined outside a pluripolar set. In particular, for $0 < \gamma < \beta$, we have

$$\int_X e^{m\gamma^{-1}(V_\theta - u)} \omega_X^n \leq \int_X e^{-m\gamma^{-1} P_{\omega_X}(u - V_\theta)} \omega_X^n < +\infty$$

as follows from Skoda's integrability theorem; cf. [GZ17, Theorem 2.50]. We, therefore, apply Proposition 2.10 to conclude. □

Proposition 2.12. *Let $\phi \in \mathrm{PSH}(X, \theta)$ be a model potential with $\int_X \theta_\phi^n > \varrho > 0$. Let $\mu \in \mathcal{M}(B, \beta)$. Then, there exist constants $0 < \gamma < 1$ and $C > 0$ depending only on X, ω_X, θ, B and β such that*

$$\mu(E) \leq C \exp\left(-\gamma \left(\frac{\varrho}{\mathrm{Cap}_\phi(E)}\right)^{1/n}\right), \tag{2.12}$$

for all Borel sets $E \subset X$.

Proof. The case when $\theta = \omega_X$ and $\phi = 0$ was shown in [DN14, Propositions 2.4, 4.4]. For the relative case, the proof relies on the arguments in [DDL18, Proposition 4.30], so we sketch it here. According to [DDL18, Section 4A2], for any Borel set E, we define the global ϕ-extremal function of (E, θ, ϕ) by

$$V_{E,\phi} = \sup\{u \in \mathrm{PSH}(X, \theta, \phi) : u \leq \phi \text{ on } E\}.$$

Set $M_\phi(E) = \sup_X V^*_{E,\phi}$. Thanks to [DDL18, Lemma 4.9], we have

$$\exp(-M_\phi(E)) \leq \exp\left(-\left(\frac{\varrho}{\mathrm{Cap}_\phi(E)}\right)^{1/n}\right).$$

Noticing that we can assume $M_\phi(E) \geq 1$. As follows from [DN14, Proposition 4.4], μ is weakly moderate, i.e. there are constants $\alpha = \alpha(\beta) > 0$ and $C = C(B, \beta) > 0$ such that $\int_X \exp(-\alpha\varphi) d\mu \leq C$ for every $\varphi \in \mathcal{C}$. We apply this to $V^*_{E,\phi} - \int_X V^*_{E,\phi}\omega_X^n$ to obtain

$$\int_X \exp(-\alpha V^*_{E,\phi}) d\mu \leq C \cdot \exp\Big(-\alpha \int_X V^*_{E,\phi}\omega_X^n\Big).$$

Since $V^*_{E,\phi} \leq 0$ on E a.e. and $\int_X V^*_{E,\phi}{\omega_X}^n \geq M_\phi(E) - C_{\omega_X}$ (see, e.g., [GZ17, Proposition 8.5]) we have

$$\mu(E) \leq C \cdot \exp\left(-\gamma\left(\frac{\varrho}{\mathrm{Cap}_\phi(E)}\right)^{1/n}\right)$$

for $\gamma > 0$. □

Proposition 2.13. *Fix $a_0 > 0$, $C_0 > 0$ and $h : (0, \infty) \to (0, \infty)$ is an increasing concave function with $h(\infty) = \infty$. Let $\mathcal{N} = \mathcal{N}(B, \beta, a_0, C_0, h)$ be the set of probability measures ν on X such that $\nu = e^{-\psi}\mu$ with $\mu \in \mathcal{M}(B, \beta)$, $\psi \in \mathrm{PSH}(X, a_0\omega_X)$ and*

$$\int_X h(-\psi)e^{-\psi} \mathrm{d}\mu \leq C_0. \tag{2.13}$$

Then there exists a continuous function $g\colon [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that for all Borel sets E,

$$\nu(E) \leq g(\mathrm{Cap}_{\omega_X}(E)) \quad \textit{for all } \nu \in \mathcal{N}.$$

In particular, the family of measures $(\nu)_{\nu\in\mathcal{N}}$ is uniformly absolutely continuous with respect to capacity.

Proof. For any Borel subset $E \subset X$, we have for $k > 0$

$$\begin{aligned}\int_E e^{-\psi} d\mu &= \int_{E\cap\{\psi\geq -k\}} e^{-\psi} d\mu + \int_{E\cap\{\psi<-k\}} e^{-\psi} d\mu \\ &\leq e^k \mu(E) + \frac{1}{h(k)} \int_{E\cap\{\psi<-k\}} h(-\psi)e^{-\psi} d\mu \\ &\leq C(e^k\, \mathrm{Cap}_{\omega_X}(E) + h(k)^{-1}).\end{aligned}$$

where the last inequality follows from Proposition 2.12. Taking $k := \log \frac{1}{\sqrt{\mathrm{Cap}_{\omega_X}(E)}} > 0$ we get that $e^{-\psi}\mu(E) \leq g(\mathrm{Cap}_{\omega_X}(E))$ where

$$g(t) := C(t^{1/2} + h(\log(t^{-1/2}))^{-1}).$$

Otherwise, since $\frac{1}{\sqrt{\mathrm{Cap}_{\omega_X}(E)}} \leq 1$, the choice of g above ensures that $\nu(E) \leq C \leq g(\mathrm{Cap}_{\omega_X}(E))$. □

Remark 2.14. For each non-pluripolar measure $e^{-\psi}\mu$ with $\int_X e^{-\psi} d\mu < \infty$, there always exists a concave increasing function $h\colon \mathbb{R}^+ \to \mathbb{R}^+$ and $C > 0$ such that $\int_X h(-\psi)e^{-\psi} d\mu \leq C$ as follows from [BEGZ10, Lemma 3.3].

We end this section with some examples of measures as introduced above.

Example 2.15. Dinh and Nguyên [DN14] provided a characterization of Hölder continuous measures. Precisely, a measure μ is Hölder continuous (in the sense of Definition 2.8) if and only if it is a Monge–Ampère measure of Hölder continuous potentials. In particular, we have some explicit examples as follows.

- $\mu = f\nu$ where ν is a Hölder continuous measure and $f \in L^p(\nu)$ for some $p > 1$; cf. [Koł08, DDG+14].
- If there are constants $C > 0$ and $\alpha > 0$ such that $\mu(B(z, r)) \leq Cr^{2n-2+\alpha}$ for $B(z, r)$ denoting the ball centered at z with radius $r > 0$, then μ is Hölder continuous; cf. [Hie10, DDG+14].
- If μ is a positive Radon measure compactly supported on a generic immersed $\mathcal{C}^3$ submanifold K of X of real codimension $d > 0$, then μ is Hölder continuous; cf. [Vu18].

Example 2.16. We provide an example of measure with *radial singularities*, inspiring by [DGL21, Section 1.3] and [GGZ25, Section 5]. Let $X = \mathbb{CP}^n$ be a complex projective manifold of dimension n, equipped with the Fubini–Study metric $\omega_X = \omega_{\mathrm{FS}}$. We assume that φ is a ω_{FS}-psh function on X which has a radial singularity at p, i.e. it is invariant under the group $U(n, \mathbb{C})$ near p. We choose a local chart biholomorphic to the unit ball B of $\mathbb{C}^n$, with p corresponding to the origin. Locally, the function φ can be written as $\varphi = u - \frac{1}{2}\log[1 + \|z\|^2]$ for some psh function u.

We therefore consider a psh function $u := \chi \circ L$ defined near the origin in $\mathbb{C}^n$, where $L(z) := \log \|z\|$ and $\chi\colon \mathbb{R}^- \to \mathbb{R}^-$ is a convex increasing function. A computation shows that

$$(dd^c u)^n = \frac{c_n(\chi' \circ L)^{n-1}\chi'' \circ L}{\|z\|^{2n}}\omega_X^n =: e^{-\psi_\chi}\omega_X^n.$$

To simplify the next computation, we assume that $\chi(t)$ does not go to zero too fast at infinity $-\infty$, that is, $\chi'(t), \chi''(t) \geq e^{Ct}$ near $t = -\infty$. It follows that $\psi_\chi \sim \log \|z\|$. Hence, the function h satisfies $\int_X h(-\psi_\chi)e^{-\psi_\chi}\omega_X^n < \infty$ if and only if

$$\int_{-\infty}^0 h(-t)\chi'(t)^{n-1}\chi''(t)dt < \infty.$$

In what follows, we give several families of examples and check whether the condition mentioned above is satisfied.

(1) Consider $\chi_a(t) = -(-t)^{-a}$ for some $\alpha > 0$. We observe that

$$\int_{-\infty}^0 h(-t)\chi'(t)^{n-1}\chi''(t)dt \lesssim \int_{-\infty}^0 h(-t) \cdot (-t)^{-na-n-1}dt < \infty$$

for $h(s) = s^p$ with $p < n(a + 1)$.

(2) Consider $\chi_a(t) = (-t)^a$ for $a \in (0, 1)$. We have

$$\int_{-\infty}^0 h(-t)\chi'(t)^{n-1}\chi''(t)dt \lesssim \int_{-\infty}^0 h(-t) \cdot (-t)^{-n(1-a)-1}dt < \infty$$

for $h(s) = s^\varepsilon$ with $0 < \varepsilon << 1$ and $a < \frac{n-\varepsilon}{n}$.

(3) Consider $\chi_a(t) = -(\log(-t))^a$, where $a > 0$. We have

$$\int_{-\infty}^{0} h(-t)\chi'(t)^{n-1}\chi''(t)dt \lesssim \int_{-\infty}^{0} h(-t)\cdot(-t)^{-n-1}(\log(-t))^{n\alpha-n}dt < \infty$$

for $h(s) = s^\varepsilon$ with $0 < \varepsilon << 1$.

(4) $\chi(t) = -\log(\log(-t))$. We have

$$\int_{-\infty}^{0} h(-t)\chi'(t)^{n-1}\chi''(t)dt \lesssim \int_{-\infty}^{0} h(-t)\cdot(-t)^{-n-1}(\log(-t))^{-n}dt =$$

for $h(s) = s^\varepsilon$ with $0 < \varepsilon << 1$.

3. Proof of main results

We prove some lemmas which will be used to prove the main theorem.

Lemma 3.1. *Assume u, v are negative θ-psh functions such that*

- $v \le P_\theta[u]$;
- $v = (1-\lambda)v_1 + \lambda v_2$, *where v_1, v_2 are θ-psh functions with v_2 having model singularity type and $\lambda \in [0,1)$.*

Then, for every $s > 0$ and $0 \le t \le \lambda$, we have

$$t^n \operatorname{Cap}_{v_2}\{u < v - s - t\} \le \int_{\{u<v-s\}} \theta_u^n.$$

Proof. Let φ be a θ-psh function such that $v_2 - 1 \le \varphi \le v_2$. For every $s > 0$ and $0 < t < 1$, we have

$$\{u < v - s - \lambda t\} \subset \{u < (1-\lambda)v_1 + (1-t)\lambda v_2 + \lambda t\varphi - s\} \subset \{u < v - s\}.$$

Denoting $\widehat{v} = (1-\lambda)v_1 + (1-t)\lambda v_2 + \lambda t\varphi$, we have

$$\lambda^n t^n \int_{\{u<v-s-\lambda t\}} \theta_\varphi^n \le \int_{\{u<v-s-\lambda t\}} \theta_{\widehat{v}}^n \le \int_{\{u<\tilde{v}-s\}} \theta_{\tilde{v}}^n,$$

for every $s > 0$ and $0 < t < 1$. Since $\widehat{v} \le P_\theta[u]$ it follows from the comparison principle [DDL21, Lemma 2.3] that

$$\int_{\{u<\widehat{v}-s\}} \theta_{\widehat{v}}^n \le \int_{\{u<\widehat{v}-s\}} \theta_u^n \le \int_{\{u<v-s\}} \theta_u^n$$

Since φ was taken arbitrarily, it follows that

$$\lambda^n t^n \operatorname{Cap}_{v_2}\{u < v - s - \lambda t\} \le \int_{\{u<v-s\}} \theta_u^n,$$

for every $s > 0$ and $0 < t < 1$. Substituting $t \mapsto \lambda t$, we obtain the desired inequality. □

The following lemma is due to De Giorgi, which was proved in [EGZ09, pages 614-615].

Lemma 3.2. *Let $g : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-increasing, right-continuous function such that,*

$$tg(t+s) \le C(g(s))^{1+\alpha},$$

for every $s > 0$ and $0 \le t \le \lambda$, where $C > 0$, $\alpha \ge 0$ and $\lambda \in (0,1]$ are given. Assume that there exist $S_0 > 0$ and $0 \le t_0 \le \lambda$ satisfying $g(S_0)^\alpha < \frac{t_0}{2C}$. Then $g(S) = 0$ for all $S \ge S_0 + \frac{t_0}{1-2^{-\alpha}}$.

Lemma 3.3. *Let $u \in \mathcal{E}(X,\theta)$ such that $\theta_u^n = e^{-\psi}d\mu$, where μ is a Hölder continuous measure with Hölder constant B and Hölder exponent $0 < \beta \leq 1$ with respect to dist_{L^1} on $\mathcal{C}$ and ψ is a negative quasi-psh function. Suppose that v is a θ-psh function such that $v \leq V_\theta + a\psi$ for some $a > 0$. Assume that $\int_X \theta_{V_\theta}^n > \varrho > 0$, where ϱ is a constant. Then, there exists $C = C(n, X, \omega_X, \theta, \varrho, B, \beta) > 0$ such that, for every $0 < \lambda < 1/2$,*

$$u \geq \lambda v + (1-\lambda)V_\theta - Ca\lambda\left(1 + \log_+ \int_X e^{\frac{2(V_\theta - u)}{a\lambda}} d\mu\right) - 2.$$

Here we note that $e^{\frac{2(V_\theta-u)}{a\lambda}} \in L^1(\mu)$ by Lemma 2.11. Denote by $\log_+(x) = \max(\log x, 0)$.

Proof. Denote $\widehat{v} := (1-\lambda)V_\theta + \lambda v$. By Lemma 3.1, for every $s > 0$ and $0 \leq t \leq 1-\lambda$, we have

$$t^n \operatorname{Cap}_\theta(u < \widehat{v} - s - t) \leq \int_{\{u < \widehat{v} - s\}} \theta_u^n = \int_{\{u<\widehat{v}-s\}} e^{-\psi} d\mu.$$

Since $v \leq V_\theta + a\psi$, we have $-\psi \leq \frac{V_\theta - \widehat{v}}{a\lambda}$. It follows that

$$\begin{aligned} t^n \operatorname{Cap}_\theta(u < \widehat{v} - s - t) &\leq \int_{\{u<\widehat{v}-s\}} e^{\frac{V_\theta - \widehat{v}}{a\lambda}} d\mu \\ &\leq \int_{\{u<\widehat{v}-s\}} e^{\frac{V_\theta - u - s}{a\lambda}} d\mu. \end{aligned}$$

Then, by Hölder's inequality, we obtain

$$t^n \operatorname{Cap}_\theta(u < \widehat{v} - s - t) \leq \left(\int_X e^{\frac{2(V_\theta - u - s)}{a\lambda}} d\mu\right)^{1/2} \left(\int_{\{u<\widehat{v}-s\}} d\mu\right)^{1/2}. \tag{3.1}$$

Since the measure μ is Hölder continuous, it follows from Proposition 2.12 that there exists $C_1 > 0$ depending on $X, \omega_X, \theta, \varrho, B$ and β such that

$$\mu(K) \leq C_1^{2n} [\operatorname{Cap}_\theta(K)]^4,$$

for every Borel set $K \subset X$, using that $\exp(e^{-1/x}) = O(x^4)$ for all $x > 0$. Then, by (3.1), we have

$$t^n \operatorname{Cap}_\theta(u < \widehat{v} - s - t) \leq C_1^n e^{\frac{-s}{a\lambda}} \left(\int_X e^{\frac{2(V_\theta-u)}{a\lambda}} d\mu\right)^{1/2} [\operatorname{Cap}_\theta(u < \widehat{v} - s)]^2 \tag{3.2}$$

Set $g(s) := [\operatorname{Cap}_\theta(u < \widehat{v} - s)]^{1/n}$ and $C_2 := \left(\int_X e^{\frac{2(V_\theta-u)}{a\lambda}} d\mu\right)^{1/2n}$. By (3.2), we have

$$tg(t+s) \leq C_1 C_2 g(s)^2, \tag{3.3}$$

for every $s > 0$ an $0 \leq t \leq 1 - \lambda$. Moreover, it follows from (3.2) that

$$g(s + 1 - \lambda) \leq \frac{C_3}{1-\lambda} e^{\frac{-s}{a\lambda n}}, \tag{3.4}$$

for every $s > 0$, where $C_3 = C_1 C_2 (\mathrm{vol}(\theta))^{2/n}$. Put $s_0 = a\lambda n \log_+(16 C_1 C_2 C_3)$ so that

$$\frac{C_3}{1-\lambda} e^{\frac{-s_0}{a\lambda n}} < \frac{(1-\lambda)}{4C_1C_2}.$$

Then, by (3.4), we have $g(s_0 + 1 - \lambda) < \frac{(1-\lambda)}{4C_1C_2}$. Applying Lemma 3.2 with $S = s_0 + 1 - \lambda$ and $t_0 = \frac{1-\lambda}{2} \in (0, 1-\lambda)$, we obtain $g(s_0 + 2 - 2\lambda) = 0$. Then

$$u \geq \widehat{v} - s_0 - 2 + 2\lambda \geq \widehat{v} - C_4 a\lambda \left(1 + \log_+ \int_X e^{\frac{2(V_\theta-u)}{a\lambda}} d\mu\right) - 2,$$

where $C_4 > 0$ is a constant depending only on $n, X, \omega_X, \theta, \varrho, B$ and β. □

Lemma 3.4. *Let u, v be negative θ-psh functions such that*

- $v \leq u + M$ *for some* $M \geq 1$;
- $v = (1-\lambda)v_1 + \lambda v_2$, *where* v_1, v_2 *are* θ*-psh functions with* v_2 *has model singularity type and* $\lambda \in (0, 1/2)$;
- *there exist* $C > 0$ *and* $\alpha \geq 1$ *such that, for every* $s > 0$,

$$\int_{\{u<v-s\}} \theta_u^n \leq C[\mathrm{Cap}_{v_2}(u < v - s)]^{1+\alpha}. \tag{3.5}$$

Then, for every $\varepsilon > 0$,

$$\sup_X (v-u) \leq \varepsilon + \frac{2MC^{1/n}}{\lambda}[\mathrm{Cap}_{v_2}(u < v - \varepsilon)]^{\alpha/n}.$$

Proof. We adapt the same arguments as in [EGZ09]. By Lemma 3.1, for every $s > 0$ and $0 < t < \lambda$, we have

$$t^n \,\mathrm{Cap}_{v_2}\{u < v - s - t\} \leq \int_{\{u<v-s\}} \theta_u^n.$$

Therefore, by (3.5), we obtain

$$t^n \,\mathrm{Cap}_{v_2}\{u < v - s - t\} \leq C[\mathrm{Cap}_{v_2}(u < v - s)]^{1+\alpha},$$

for every $s > 0$ and $0 < t < \lambda$. Denote $g(s) = \left[\mathrm{Cap}_{v_2}(u < v - s)\right]^{1/n}$. We have

$$tg(t+s) \leq C^{1/n} g(s)^{1+\alpha},$$

for every $s > 0$ and $0 < t < \lambda$. If there exists $0 \leq t_0 < \lambda$ such that $g(\varepsilon)^\alpha = \frac{t_0}{2C^{1/n}}$ then it follows from Lemma 3.2 that $g\left(\varepsilon + \frac{t_0}{1-2^{-\alpha}}\right) = 0$. Hence

$$\sup_X (v-u) \leq \varepsilon + \frac{t_0}{1-2^{-\alpha}} \leq \varepsilon + \frac{2C^{1/n}g(\varepsilon)^\alpha}{1-2^{-\alpha}},$$

since $1/(1-2^{-\alpha}) \leq 2 \leq M/\lambda$. Otherwise $2C^{1/n}g(\varepsilon)^\alpha \geq \lambda$, we infer that

$$\sup_X (v-u) \leq M \leq \varepsilon + \frac{2MC^{1/n}}{\lambda} g(\varepsilon)^\alpha.$$

This completes the proof. □

Lemma 3.5. *Under the assumption of Lemma 3.4, the following inequality holds:*

$$\sup_X (v-u) \leq \left(2 + \frac{2MC^{1/n}}{\lambda}\right) \|v-u\|_{L^1(\nu)}^{\frac{\alpha}{(n+1)\alpha+n}},$$

where $\nu = \mathbf{1}_{\{u<v\}}\theta_u^n$.

Proof. By Lemma 3.4, for every $\varepsilon > 0$, we have

$$\sup(v-u) \leq 2\varepsilon + \frac{2MC^{1/n}}{\lambda}[\mathrm{Cap}_{v_2}(u < v - 2\varepsilon)]^{\alpha/n}.$$

Then, by Lemma 3.1, we get

$$\begin{aligned}\sup(v-u) &\leq 2\varepsilon + \frac{2MC^{1/n}}{\epsilon^\alpha \lambda}\left(\int_{\{u<v-\epsilon\}} d\nu\right)^{\alpha/n}\\ &\leq 2\varepsilon + \frac{2MC^{1/n}}{\epsilon^{\alpha(n+1)/n}\lambda}\|v-u\|_{L^1(\nu)}^{\alpha/n}.\end{aligned}$$

Choosing $\varepsilon = \|v-u\|_{L^1(\nu)}^{\frac{\alpha}{(n+1)\alpha+n}}$, we obtain the desired inequality. □

Proof of Theorem 1.1. In this proof, we denote by $M_j(a)$, for $j \geq 1$, constants in $[1, \infty)$ only depending on $n, X, \omega_X, \theta, B, \beta$ and an upper bound for $\int_X e^{\frac{2(V_\theta - u)}{a\varepsilon_0}} d\mu$. We assume that $a \mapsto M_j(a)$ is decreasing for every $j \geq 1$, and $\lim_{a\to 0^+} M_j(a) = \infty$.

Assume $0 < \delta < 1$. We define B_0 and $u_{c,\delta}$ as in (2.4) and (2.5):

$$B_0 = B_0(c, \delta) = \frac{Ac + 2K\delta}{\varepsilon_0},$$

and

$$u_{c,\delta} = 2B_0\Psi_0 + (1 - 2B_0)\Phi_{c,\delta},$$

where $c = c(\delta) > 0$ is a small constant that will be chosen later. By Lemma 2.2, we have

$$\theta + dd^c u_{c,\delta} \geq \varepsilon_0 B_0 \omega_X.$$

Let $0 < a \leq a_0$ and denote $v_{a,\delta} = u_{c,\delta} + B_0 a\varepsilon_0\psi$. Then $v_{a,\delta}$ is a θ-psh function satisfying $v_{a,\delta} \leq V_\theta + B_0 a\varepsilon_0\psi$. Using Lemma 3.3 for $\lambda = B_0$ and $v = v_{a,\delta}$, we obtain

$$u \geq B_0 v_{a,\delta} + (1 - B_0)V_\theta - M_2(a) \geq v_{a,\delta} - M_1(aB_0). \tag{3.6}$$

Since $v_{a,\delta} \leq V_\theta + B_0 a\varepsilon_0\psi$, we also have $-\psi \leq \frac{V_\theta - u}{aB_0\varepsilon_0}$ on the set $\{u < v_{a,\delta}\}$ .Combining with Theorem 2.12 yields

$$\begin{aligned}\int_{\{u<v_{a,\delta}-s\}} \theta_u^n = \int_{\{u<v_{a,\delta}-s\}} e^{-\psi} d\mu &\leq \int_{\{u<v_{a,\delta}-s\}} e^{\frac{V_\theta - u}{aB_0\varepsilon_0}} d\mu \\ &\leq \left(\int_X e^{\frac{2(V_\theta - u)}{aB_0\varepsilon_0}} d\mu\right)^{1/2} \left(\int_{\{u<v_{a,\delta}-s\}} d\mu\right)^{1/2} \\ &\leq M_2(aB_0)\left[\mathrm{Cap}_{V_\theta}(\{u < v_{a,\delta} - s\})\right]^2.\end{aligned}$$

On the set $\{u < v_{a,\delta}\}$, we have

$$v_{a/2,\delta} + \frac{B_0 a\varphi_0}{2}\psi = v_{a,\delta} > u,$$

so

$$-\psi \leq \frac{2}{B_0 a\varepsilon_0}(v_{a/2,\delta} - u) \leq \frac{2M_1\left(\frac{aB_0}{2}\right)}{B_0 a\varepsilon_0}$$

using (3.6). Combining this with Lemma 3.5 we infer that

$$\begin{aligned}\sup_X (v_{a,\delta} - u - s) &\leq \frac{M_3(aB_0)}{B_0}\|v_{a,\delta} - u - s\|_{L^1(\nu_s)}^{\frac{1}{2n+1}} \\ &\leq \frac{M_4(aB_0)}{B_0}\|(v_{a,\delta} - u - s)_+\|_{L^1(\mu)}^{\frac{1}{2n+1}},\end{aligned} \tag{3.7}$$

for every $s \geq 0$, where $\nu_s = \mathbf{1}_{\{u<v_{a,\delta}-s\}}\theta_u^n$. Using Lemma 3.3 for $\lambda = \frac{1}{2}$ and $v = \Psi_0 + \varepsilon_0 a\psi$, we also have

$$u(z) \geq \frac{\Psi_0 + \varepsilon_0 a\psi + V_\theta}{2} - M_5(a) \geq \Psi_0 + \frac{\varepsilon_0 a\psi}{2} - M_5(a), \tag{3.8}$$

for every $z \in X$. Hence

$$\begin{aligned}v_{a,\delta} - u &= 2B_0(\Psi_0 + \frac{\varepsilon_0 a\psi}{2} - u) + (1 - B_0)(\Phi_{c,\delta} - u) \\ &\leq 2B_0 M_5(a) + (\Phi_{c,\delta} - u)_+\end{aligned}$$

This combined with (3.7) implies that

$$\begin{aligned}\sup_X(v_{a,\delta}-u) &\le \sup_X(v_{a,\delta}-u-2B_0M_5(a))+2B_0M_5(a)\\ &\le \frac{M_4(aB_0)}{B_0}\|(v_{a,\delta}-u-2B_0M_5(a))_+\|_{L^1(\mu)}^{\frac{1}{2n+1}}+2B_0M_5(a)\\ &\le \frac{M_4(aB_0)}{B_0}\|(\Phi_{c,\delta}-u)_+\|_{L^1(\mu)}^{\frac{1}{2n+1}}+2B_0M_5(a)\\ &\le \frac{M_6(aB_0)}{B_0}\|(\Phi_{c,\delta}-u)_+\|_{L^1(X)}^{\frac{\beta}{2n+1}}+2B_0M_5(a)\\ &\le \frac{M_6(aB_0)}{B_0}\|(\rho_\delta u-u)_+\|_{L^1(X)}^{\frac{\beta}{2n+1}}+2B_0M_5(a)\\ &\le \frac{M_7(aB_0)}{B_0}\delta^{\frac{2\beta}{2n+1}}+2B_0M_5(a),\end{aligned}$$

where the last estimate holds due to Lemma 2.3. Then, for $U \Subset \mathrm{Amp}(\theta)\setminus\{-\psi=-\infty\}$ and $m_U := \sup_U(-\psi)$, we have

$$\sup_U(u_{c,\delta}-u)\le B_0a\varepsilon_0m_U+\frac{M_7(aB_0)}{B_0}\delta^{\frac{2\beta}{2n+1}}+2B_0M_5(a).$$

Denoting by h_1 the function $a\mapsto a\varepsilon_0m_U+2M_5(a)$ and choosing $a=h_1^{-1}\left(\frac{1}{\sqrt{B_0}}\right)$, we have

$$\sup_U(u_{c,\delta}-u)\le\sqrt{B_0}+\frac{M_7\left(h_1^{-1}\left(\frac{1}{\sqrt{B_0}}\right)B_0\right)}{B_0}\delta^{\frac{2\beta}{2n+1}}.$$

Denoting by h_2 the function $t\mapsto\frac{M_7\left(h_1^{-1}\left(\frac{1}{\sqrt{t}}\right)t\right)}{t^{3/2}}$ and choosing c so that $h_2(B_0)=\delta^{\frac{-\beta}{2n+1}}$, we get

$$\sup_U(u_{c,\delta}-u)\le 2\sqrt{B_0(\delta)}.$$

Now, using Lemma 2.4, we obtain

$$\rho_{\kappa(\delta)}u(z)-u(z)\le 4\sqrt{B_0(\delta)}+2C_UB_0(\delta)+2K\delta.$$

where

$$\kappa(\delta)=\delta\exp\left(\frac{-4A(C_UB_0+2\sqrt{B_0(\delta)}+2K\delta)}{\varepsilon_0B_0}\right),\quad C_U=\sup_U(V_\theta-\Psi_0).$$

Replacing δ with $\kappa(\delta)$ we obtain for $\delta\le\kappa(\delta_0)$,

$$\rho_\delta u(z)-u(z)\le 4\sqrt{B_0(\kappa^{-1}(\delta))}+2C_UB_0(\kappa^{-1}(\delta)).$$

The conclusion thus follows. □

We now deal with the special case when $\theta=\omega_X$ is a Kähler form, proving Corollary 1.2. Recall that $\mathcal{N}=\mathcal{N}(B,\beta,a_0,C_0,h)$ is the set of probability measures ν on X such that $\nu=e^{-\psi}\mu$ with $\mu\in\mathcal{M}(B,\beta)$, $\psi\in\mathrm{PSH}(X,a_0\omega_X)$ and

$$\int_X h(-\psi)e^{-\psi}\mathrm{d}\mu\le C_0$$

where $a_0>0$, $C_0>0$ are fixed constants and $h:(0,\infty)\to(0,\infty)$ is an increasing concave function with $h(\infty)=\infty$. We define the set

$$\mathcal{F}=\{u\in\mathcal{E}(X,\omega_X),\ \sup_X u=0\colon(\omega_X+dd^cu)^n\in\mathcal{N}\}.$$

Lemma 3.6. *The closure of $\mathcal{F}$ is contained in $\mathcal{E}(X,\omega_X)$.*

Proof. Let $(u_j)_{j\in\mathbb{N}} \subset \mathcal{F}$ be a sequence of ω_X-psh functions such that $(\omega_X+dd^c u_j)^n \in \mathcal{N}$ and $\sup_X u_j = 0$. By Hartogs' lemma (see, e.g., [GZ17, Proposition 8.4]), we get that $u_j \xrightarrow{L^1} u$ for some $u \in \mathrm{PSH}(X,\omega_X)$. By Proposition 2.13, the family of non-pluripolar measures $(\omega_X+dd^c u_j)^n$ is uniformly absolutely continuous with respect to capacity. Set $v_j = \max(u_j, u)$. We see that v_j converges to u in capacity thanks to Hartogs' lemma and the quasi-continuity of ω_X-psh functions. As follows from [DH12, Propositions 2.10], the sequence of measures $(\omega_X + dd^c v_j)^n$ is uniformly absolutely continuous with respect to capacity. From [DH12, Proposition 2.11], we infer that $u \in \mathcal{E}(X,\omega_X)$. Hence, the closure of $\mathcal{F}$ in the L^1 topology is contained in $\mathcal{E}(X,\omega_X)$. □

Proof of Corollary 1.2. By Theorem 1.1, it suffices to show that there exists a positive constant $C = C(a_0, X, \omega_X, \mu)$ such that

$$\int_X e^{-2u/a_0} d\mu \le C. \tag{3.9}$$

Thanks to Lemma 3.6, the closure of the set $\mathcal{F} = \{u \in \mathcal{E}(X,\omega_X), \sup_X u = 0\colon (\omega_X+dd^c u)^n \in \mathcal{N}\}$ is a compact family of functions in $\mathcal{E}(X,\omega_X)$. Since functions in $\mathcal{E}(X,\omega_X)$ have zero Lelong number at every point on X by [GZ07], the uniform Skoda integrability theorem (see, e.g., [GZ17]) ensures that there exists a constant C depending on a_0 and $\overline{\mathcal{F}}$ such that the inequality (3.9) holds for all $u \in \mathcal{F}$. □

Remark 3.7. Lemma 3.6 still holds in the more general setting where θ is merely semi-positive and big (i.e., $\int_X \theta^n > 0$). Corollary 1.2 also remains valid, but in this case, the modulus of continuity is supported only on $\mathrm{Amp}(\theta) \setminus \{\psi = -\infty\}$.

References

[BBE+19] R. J. Berman, S. Boucksom, P. Eyssidieux, V. Guedj, and A. Zeriahi. Kähler-Einstein metrics and the Kähler-Ricci flow on log Fano varieties. *J. Reine Angew. Math.*, 751:27–89, 2019. 1

[BD12] R. Berman and J.-P. Demailly. Regularity of plurisubharmonic upper envelopes in big cohomology classes. In *Perspectives in analysis, geometry, and topology*, volume 296 of *Progr. Math.*, pages 39–66. Birkhäuser/Springer, New York, 2012. 4

[BEGZ10] S. Boucksom, P. Eyssidieux, V. Guedj, and A. Zeriahi. Monge-Ampère equations in big cohomology classes. *Acta Math.*, 205(2):199–262, 2010. 1, 2, 6, 7, 10

[Bou04] S. Boucksom. Divisorial Zariski decompositions on compact complex manifolds. *Ann. Sci. École Norm. Sup. (4)*, 37(1):45–76, 2004. 3, 5

[BT87] E. Bedford and B. A. Taylor. Fine topology, Šilov boundary, and $(dd^c)^n$. *J. Funct. Anal.*, 72(2):225–251, 1987. 2, 6

[Dan22] Q.-T. Dang. Continuity of Monge-Ampère potentials in big cohomology classes. *Int. Math. Res. Not. IMRN*, (14):11180–11201, 2022. 1, 8

[DDG+14] J.-P. Demailly, S. Dinew, V. Guedj, H. H. Pham, S. Kołodziej, and A. Zeriahi. Hölder continuous solutions to Monge-Ampère equations. *J. Eur. Math. Soc. (JEMS)*, 16(4):619–647, 2014. 1, 4, 10

[DDL18] T. Darvas, E. Di Nezza, and C. H. Lu. Monotonicity of nonpluripolar products and complex Monge-Ampère equations with prescribed singularity. *Anal. PDE*, 11(8):2049–2087, 2018. 9

[DDL21] T. Darvas, E. Di Nezza, and C. H. Lu. Log-concavity of volume and complex Monge-Ampère equations with prescribed singularity. *Math. Ann.*, 379(1-2):95–132, 2021. 11

[Dem82] J.-P. Demailly. Estimations L^2 pour l'opérateur $\bar\partial$ d'un fibré vectoriel holomorphe semi-positif au-dessus d'une variété kählérienne complète. *Ann. Sci. École Norm. Sup. (4)*, 15(3):457–511, 1982. 4

[Dem92] J.-P. Demailly. Regularization of closed positive currents and intersection theory. *J. Algebraic Geom.*, 1(3):361–409, 1992. 3

[Dem94] J.-P. Demailly. Regularization of closed positive currents of type $(1,1)$ by the flow of a Chern connection. In *Contributions to complex analysis and analytic geometry*, Aspects Math., E26, pages 105–126. Friedr. Vieweg, Braunschweig, 1994. 4

[Dem12] J.-P. Demailly. Complex analytic and differential geometry. *available at https://www-fourier.ujf-grenoble.fr/ demailly/manuscripts/agbook.pdf*, 2012. 3

[DGL21] E. DiNezza, V. Guedj, and C. H. Lu. Finite entropy vs finite energy. *Comment. Math. Helv.*, 96(2):389–419, 2021. 10

[DH12] S. Dinew and P. H. Hiep. Convergence in capacity on compact Kähler manifolds. *Ann. Sc. Norm. Super. Pisa, Cl. Sci. (5)*, 11(4):903–919, 2012. 16

[DKN22] T.-C. Dinh, S. Kołodziej, and N. C. Nguyen. The complex Sobolev space and Hölder continuous solutions to Monge-Ampère equations. *Bull. Lond. Math. Soc.*, 54(2):772–790, 2022. 1

[DL15] E. Di Nezza and C. H. Lu. Generalized Monge-Ampère capacities. *Int. Math. Res. Not. IMRN*, (16):7287–7322, 2015. 7

[DL17] E. Di Nezza and C. H. Lu. Complex Monge-Ampère equations on quasi-projective varieties. *J. Reine Angew. Math.*, 727:145–167, 2017. 3, 7

[DN14] T.-C. Dinh and V.-A. Nguyên. Characterization of Monge-Ampère measures with Hölder continuous potentials. *J. Funct. Anal.*, 266(1):67–84, 2014. 1, 7, 9, 10

[EGZ09] P. Eyssidieux, V. Guedj, and A. Zeriahi. Singular Kähler-Einstein metrics. *J. Amer. Math. Soc.*, 22(3):607–639, 2009. 1, 11, 13

[EGZ11] P. Eyssidieux, V. Guedj, and A. Zeriahi. Viscosity solutions to degenerate complex Monge-Ampère equations. *Comm. Pure Appl. Math.*, 64(8):1059–1094, 2011. 1

[FGS20] X. Fu, B. Guo, and J. Song. Geometric estimates for complex Monge-Ampère equations. *J. Reine Angew. Math.*, 765:69–99, 2020. 1

[GGZ25] V. Guedj, H. Guenancia, and A. Zeriahi. Diameter of Kähler currents. *J. Reine Angew. Math.*, 820:115–152, 2025. 1, 10

[GPSS24a] B. Guo, D. H. Phong, J. Song, and J. Sturm. Diameter estimates in Kähler geometry. *Comm. Pure Appl. Math.*, 77(8):3520–3556, 2024. 1

[GPSS24b] B. Guo, D. H. Phong, J. Song, and J. Sturm. Diameter estimates in Kähler geometry II: removing the small degeneracy assumption. *Math. Z.*, 308(3):Paper No. 43, 7, 2024. 1

[GPTW21] B. Guo, D. H Phong, F. Tong, and C. Wang. On the modulus of continuity of solutions to complex Monge-Ampère equations. *arXiv:2112.02354*, 2021. 1

[GS22] B. Guo and J. Song. Local noncollapsing for complex Monge-Ampère equations. *J. Reine Angew. Math.*, 793:225–238, 2022. 1

[GZ07] V. Guedj and A. Zeriahi. The weighted Monge-Ampère energy of quasiplurisubharmonic functions. *J. Funct. Anal.*, 250(2):442–482, 2007. 1, 16

[GZ17] V. Guedj and A. Zeriahi. *Degenerate complex Monge-Ampère equations*, volume 26 of *EMS Tracts in Mathematics*. European Mathematical Society (EMS), Zürich, 2017. 1, 3, 8, 9, 16

[Hie10] Pham Hoang Hiep. Hölder continuity of solutions to the Monge-Ampère equations on compact Kähler manifolds. *Ann. Inst. Fourier (Grenoble)*, 60(5):1857–1869, 2010. 1, 10

[KN19] S. Kołodziej and N. C. Nguyen. Stability and regularity of solutions of the Monge-Ampère equation on Hermitian manifolds. *Adv. Math.*, 346:264–304, 2019. 4

[Koł98] S. Kołodziej. The complex Monge-Ampère equation. *Acta Math.*, 180(1):69–117, 1998. 1

[Koł08] S. Kołodziej. Hölder continuity of solutions to the complex Monge-Ampère equation with the right-hand side in L^p: the case of compact Kähler manifolds. *Math. Ann.*, 342(2):379–386, 2008. 1, 10

[RW14] J. Ross and D. Witt Nyström. Analytic test configurations and geodesic rays. *J. Symplectic Geom.*, 12(1):125–169, 2014. 6

[Vu18] D.-V. Vu. Complex Monge-Ampère equation for measures supported on real submanifolds. *Math. Ann.*, 372(1-2):321–367, 2018. 10

(Quang-Tuan Dang) The Abdus Salam International Centre for Theoretical Physics (ICTP), Str. Costiera, 11, 34151 Trieste, TS, Italy

Email address: qdang@ictp.it

(Hoang-Son Do) Vietnam Academy of Science and Technology, Institute of Mathematics, 18 Hoang Quoc Viet road, Cau Giay, Hanoi, Vietnam

Email address: dhson@math.ac.vn

(Hoang-Hiep Pham) Vietnam Academy of Science and Technology, Institute of Mathematics, 18 Hoang Quoc Viet road, Cau Giay, Hanoi, Vietnam

Email address: phhiep@math.ac.vn