SOME REMARKS ON QUANTUM SPEED LIMIT FOR n-DIMENSIONAL SYSTEMS

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ABSTRACT. Many recent works in quantum information theory focus on the following problem: Given an initial state, a target state, and a driving Hamiltonian H, how quickly can the initial state evolve into the target state? In this work, we demonstrate that it is possible to obtain exact results for the evolution time of any system described by a mixed $n \times n$ state undergoing unitary evolution, given certain conditions on the eigenvalues of the Hamiltonian H. This result extends the work in [14]. Our main finding also provides evidence that the optimal evolution time (the argmin) obtained in [14] cannot be generalized to higher-dimensional systems.

1. INTRODUCTION

Quantum mechanics sets fundamental limits on the processing speed of any device and the communication speed through any channel. When an initial state is transformed into a target state using logic gates, the rate of evolution or time taken becomes a key metric for evaluating the performance of quantum computers. This leads to a natural question: How can the quantum evolution speed be determined? It turns out that the speed of evolution for any quantum system is limited by the principles of quantum mechanics. The quantum speed limit is a fundamental concept in quantum mechanics, and extensive research has been dedicated to this topic.

Mandelstam and Tamm [11], along with Margolus and Levitin [12], discovered elegant and useful bounds for the minimum evolution time between two orthogonal pure states in a unitary process. The quantum speed limit for pure states was later generalized to apply to the evolution between two arbitrary states, not necessarily orthogonal, and was extended to encompass open systems and mixed states. Since then, much work has been done on this subject (e.g., see [13, 3, 14] and the references therein.

In 2018, L. Zhang et al. [14] purse the study in the qubit case:

$$\inf_{t>0} \mathcal{F}_U(\rho, \rho_t)$$

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where $\mathcal{F}_U(\rho,\sigma) = \mathrm{Tr}\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}$ is the fidelity on 2×2 density matrices, and $\rho_t = \mathrm{e}^{-\mathrm{i}tH}\rho\,\mathrm{e}^{\mathrm{i}tH}$. The result are stated as follows:

Proposition 1. [14, Proposition 1] *The following optimization problem*

$$\inf_{t>0} \mathcal{F}_U(\rho\rho_t) = \inf_{t>0} \mathrm{Tr}\left(\sqrt{\sqrt{\rho}\rho_t\sqrt{\rho}}\right) \tag{1}$$

is solved for $t = t_{\min} := \dfrac{\pi}{2\zeta_H}$. *Moreover,* $\rho_{t_{\min}} = \rho - (2\zeta_H)^{-1}[H,[H,\rho]]$ *and* $\mathcal{F}_U(\rho,\rho_{t_{\min}}) = \sqrt{2\det(\rho) + \mathrm{Tr}(\rho^2) - \zeta_H^{-2}\left(\mathrm{Tr}(\rho^2H^2) - \mathrm{Tr}(\rho H\rho H)\right)}$

Along with that, the author also points out that:"Although the problem is exactly in the two dimensional case, the conclusions in Proposition 1 cannot be extended to higher dimensional case" and give the following example to demonstrate it.

Example 1 ([14])**.** Let

$$\rho = \frac{1}{5}\begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

then $\inf\limits_{t>0} \mathcal{F}_U(\rho\rho_t) = \dfrac{\pi}{5}$.
However $\mathcal{F}_U(\rho,\rho_{t_0}) \approx 0.919 > \inf\limits_{t>0} \mathcal{F}(\rho\rho_t)$, with $t_0 = \dfrac{\pi}{2\zeta_H} = \dfrac{\pi}{2\sqrt{3}}$.

In this paper, we try to study the problem mentioned above for Hilbert Schmidt fidelity measures (defined in [10]) and some other well-known ones on the quantum sate spaces of dimension n. Under some technical conditions on the eigenvalues of the hermitian matrix H we obtained the exact minimum value, see Theorem 1 and its corollaries. The result [14, Proposition 1] is a consequence of Theorem 1 for 2-dimensional case and our main finding also provides evidence that the optimal evolution time $t_{\min}$ (the argmin) obtained in [14, Proposition 1] cannot be different in higher-dimensional systems. In particular, if the differences of eigenvalues of H are rational numbers, then the minimal time $t_{\min}$ exists (see Corollary 3 and 4). However, if the differences mentioned above are not all rational, we can give an example to show that the minimal time $t_{\min}$ does not exist (see Example 5). More precisely, applying Kronecker's Theorem, we can show that if the differences of eigenvalues of the Hamiltonian H are independent over the rational numbers, then the infimum value $\inf\limits_{t>0} \mathcal{F}(\rho\rho_t)$ can be obtained while if assumed further that the differences of eigenvalues set contains at least two distinct values then the optimal evolution time (the argmin) $t_{\min}$ does not exist. We also wish to mention that the minimal time $t_{\min}$ obtained in most of the above results holds when $\exp[i(\lambda_j - \lambda_k)t_{\min}] = -1$ for all nonzero eigenvalue differences

$\lambda_j - \lambda_k$ of H. However, Example 4 presents a case where the minimal time $t_{\min}$ exists but $\exp(\lambda_j - \lambda_k)it_{\min} > -1$.

The paper is organized as follows. Subsection 2.1 presents the study of the optimization of the inner product in n-dimensional systems. Applying the results from Subsection 2.1, we present the main results of our work on Hilbert-Schmidt fidelity measures for n-dimensional systems in Subsection 2.2. Using the main result from Subsection 2.2, we discuss examples of 3×3 quantum states in Subsection 2.3, showing that the Argmin obtained in Subsection 2.2 can be incorrect.

2. Evolution limit for $n \times n$ systems

Throughout this section, we assume that ρ, ρ_t are quantum states and $H = H^*$ is an $n \times n$ Hermitian matrix and a unitary U such that $H = U^* D U$, where D is a diagonal matrix of eigenvalues $\lambda_1, \dots, \lambda_n$ of H. Denoted by

$$\begin{aligned} \rho_t &:= \mathrm{e}^{-\mathrm{i}\, tH} \rho\, \mathrm{e}^{\mathrm{i}\, tH}, \\ \tilde{\rho} &:= U\rho U^*, \quad \tilde{\rho}_t := \mathrm{e}^{-\mathrm{i}\, tD}\, \tilde{\rho}\, \mathrm{e}^{\mathrm{i}\, tD}. \end{aligned}$$

In this section, we consider the optimization problem:

$$\inf_{t>0} \mathcal{F}(\rho, \rho_t),$$

where $\mathcal{F}$ are the Hilbert-Schmidt fidelity measures defined in [10]:

$$\mathcal{F}_f(A, B) = \mathrm{Tr}(AB) f(\mathrm{Tr}\, A^2, \mathrm{Tr}\, B^2),$$

and, other well-known ones

$$\begin{aligned} \mathcal{F}_N(A, B) &= \mathrm{Tr}(AB) + \sqrt{1 - \mathrm{Tr}\, A^2}\sqrt{1 - \mathrm{Tr}\, B^2}, \\ \mathcal{F}_C(A, B) &= \frac{n-2}{2(n-1)} + \frac{n}{2(n-1)} \mathcal{F}_N(A, B). \end{aligned}$$

This optimization problem leads to study the one where $\mathcal{F}$ is the Hilbert - Schmidt product.

2.1. Optimization of the inner product on unitary paths of quantum states of order n.

Let's consider the problem:

$$\inf_{t>0} \mathrm{Tr}(\rho\rho_t) = \inf_{t>0} \langle \rho, \rho_t \rangle. \tag{2}$$

Lemma 1. *Suppose that $H = U^* D U$, where D is a diagonal matrix of eigenvalues of H and U is a unitary matrix. Then*

$$\mathrm{e}^{\mathrm{i}\, tH} = U^* \mathrm{e}^{\mathrm{i}\, tD} U. \tag{3}$$

Proof. Substituting $H = U^* D U$ into the Maclaurin expansion of $\mathrm{e}^{\mathrm{i}\,tH}$, we obtain

$$\mathrm{e}^{\mathrm{i}\,tH} = \sum_{n=0}^{\infty} \frac{(\mathrm{i}\,tH)^n}{n!} = \sum_{n=0}^{\infty} \frac{(\mathrm{i}\,tU^*DU)^n}{n!} = \sum_{n=0}^{\infty} \frac{U^*(\mathrm{i}\,tD)^n U}{n!} = U^*\,\mathrm{e}^{\mathrm{i}\,tD}\,U.$$

□

It is clearly that $\mathrm{Tr}(\rho\rho_t) = \mathrm{Tr}(\tilde{\rho}\tilde{\rho}_t)$.

Suppose that $D = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}$. Then $\mathrm{e}^{\mathrm{i}\,tD} = \begin{bmatrix} \mathrm{e}^{\mathrm{i}\,t\lambda_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathrm{e}^{\mathrm{i}\,t\lambda_n} \end{bmatrix}$.

Hence the (j,k) – entry of $\tilde{\rho}_t$ is $\mathrm{e}^{\mathrm{i}\,t(\lambda_j-\lambda_k)}\,\tilde{\rho}_{jk}$. Cleary, (j,j)-entry is $\tilde{\rho}_{jj}$.
Denote by

$$\Im(\tilde{\rho}) := \left\{(j,k) \in \mathbb{N}^2 \mid 1 \le j,k \le n,\ \ \lambda_j \ne \lambda_k \text{ and } \tilde{\rho}_{jk} \ne 0\right\}.$$

By the definition, $\Im(\tilde{\rho})$ does not contain the diagonal (i.e., $(i,i) \notin \Im(\tilde{\rho})$) and is symmetric, i.e,

$$(j,k) \in \Im(\tilde{\rho}) \iff (k,j) \in \Im(\tilde{\rho}).$$

Using Lemma 1 and direct calculation, we get the following lemma.

Lemma 2. *Let's adopt the notations above.*

$$\mathrm{Tr}(\rho\rho_t) = \sum_{(j,k)\notin\Im(\tilde{\rho})} \left|\tilde{\rho}_{jk}\right|^2 + \sum_{(j,k)\in\Im(\tilde{\rho})} \left|\tilde{\rho}_{jk}\right|^2 \cos\left(\left|\lambda_j - \lambda_k\right| t\right). \tag{4}$$

It is clear that if H is a multiple of the identity then $\Im(\tilde{\rho}) = \emptyset$, and by (4), $\mathrm{Tr}(\rho\rho_t)$ is a constant for every t.

Theorem 1. *Let's adopt the notations above. Then, the following statements hold true.*

- **(i):** $\inf_{t>0} \mathrm{Tr}(\rho\rho_t) \ge \sum_{(j,k)\notin\Im(\tilde{\rho})} \left|\tilde{\rho}_{jk}\right|^2 - \sum_{(j,k)\in\Im(\tilde{\rho})} \left|\tilde{\rho}_{jk}\right|^2.$
- **(ii):** *The equality in* **(i)** *holds if and only if there exists a positive number x such that $\left|\lambda_j - \lambda_k\right| x$ are all odd integers for $(j,k) \in \Im(\tilde{\rho})$. In this case,*

$$\inf_{t>0} \mathrm{Tr}(\rho\rho_t) = \mathrm{Tr}(\rho\rho_{t_{\min}}),\ \ \textit{where } t_{\min} = x\pi.$$

- **(iii):** *Denote by $\lambda^{\downarrow}(\rho)$ $\left(\lambda^{\uparrow}(\rho)\right)$ is the vector of eigenvalues of ρ in decreasing (increasing, respectively) order and $\langle\cdot,\cdot\rangle$ the usual inner product in $\mathbb{R}^n$. For any $t > 0$,*

$$\langle\lambda^{\downarrow}(\rho), \lambda^{\uparrow}(\rho)\rangle \le \mathrm{Tr}(\rho\rho_t) \le \langle\lambda^{\downarrow}(\rho), \lambda^{\downarrow}(\rho)\rangle. \tag{5}$$

The equality

$$\langle\lambda^{\downarrow}(\rho), \lambda^{\uparrow}(\rho)\rangle = \inf_{t>0} \mathrm{Tr}(\rho\rho_t)$$

holds if there exist a positive t such that $W^{\downarrow}V^{\uparrow} = e^{itH}$, where

$$W^{\downarrow *}\rho W^{\downarrow} = \lambda^{\downarrow}(\rho) \quad \text{and} \quad V^{\uparrow *}HV^{\uparrow} = \lambda^{\uparrow}(H).$$

Proof. **(i):** By Lemma 2 and $\cos\alpha \geq -1$, we get statement **(i)**.

(ii): The equality **(i)** holds if and only if there is a positive number t_0 such that $\cos\left(\left|\lambda_j - \lambda_k\right| t_0\right) = -1, \ \forall (j,k) \in \Im(\rho)$. Choose $t_{\min}$ to be the smallest such t_0. This implies that $\left|\lambda_j - \lambda_k\right| t_{\min}/\pi$ are all odd positive integers. Equivalently, there is a positive number x such that $\left|\lambda_j - \lambda_k\right| x$ are all odd positive integers.

(iii): Follows from the fact that $\lambda^{\downarrow}(\rho) = \lambda^{\downarrow}(\rho_t)$ and von Neumann's trace inequality (see also [2] or [9, Lemma 2.4]):

$$\langle \lambda^{\downarrow}(\rho), \lambda^{\uparrow}(\sigma)\rangle \leq \langle \rho, V^*\sigma V\rangle \leq \langle \lambda^{\downarrow}(\rho), \lambda^{\downarrow}(\sigma)\rangle, \tag{6}$$

for every unitary V. The left hand side inequality becomes equality at the argmin unitary $V = W^{\downarrow}V^{\uparrow *}$.

□

Comparing (5) and (6), We note that $\mathrm{Tr}(\rho\rho_t)$ can be strictly greater $\langle \lambda^{\downarrow}(\rho), \lambda^{\uparrow}(\rho)\rangle$, as demonstrated by the following example.

Example 2. Let

$$\rho = \begin{bmatrix} 1/3 & 1/6 & 1/6 \\ 1/6 & 1/3 & 1/6 \\ 1/6 & 1/6 & 1/3 \end{bmatrix}, \qquad H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

By Lemma 2 we have $\mathrm{Tr}(\rho\rho_t) = 7/18 + 1/9\cos(t)$. and $\inf_{t>0} \mathrm{Tr}(\rho\rho_t) = \frac{5}{18}$. On the other hand, we have $\lambda^{\downarrow}(\rho) = \big(2/3, 1/6, 1/6\big)$. Therefore,

$$\frac{1}{2} = \langle \lambda^{\downarrow}(\rho), \lambda^{\downarrow}(\rho)\rangle > \inf_{t>0} \mathrm{Tr}(\rho\rho_t) > \frac{1}{4} = \langle \lambda^{\downarrow}(\rho), \lambda^{\uparrow}(\rho)\rangle.$$

Corollary 1. *Suppose that $\left|\lambda_j - \lambda_k\right|$ is either zero or a positive number α, for every $0 \leq j,k \leq n$. Then*

$$\inf_{t>0} \mathrm{Tr}(\rho\rho_t) = \mathrm{Tr}(\rho\rho_{t_{\min}}) = \sum_{(j,k)\notin\Im(\tilde\rho)} \left|\tilde\rho_{jk}\right|^2 - \sum_{(j,k)\in\Im(\tilde\rho)} \left|\tilde\rho_{jk}\right|^2,$$

where $t_{\min} = \dfrac{\pi}{\alpha}$.

Proof. By the hypothesis,

$$\{\left|\lambda_j - \lambda_k\right| \mid 1 \leq j,k \leq n\} = \{0, \alpha\}.$$

Hence,

$$\operatorname{Tr}(\rho\rho_t) = \sum_{(j,k)\notin\Im(\tilde{\rho})} \left|\tilde{\rho}_{jk}\right|^2 + \sum_{(j,k)\in\Im(\tilde{\rho})} \left|\tilde{\rho}_{jk}\right|^2 \cos(\alpha t).$$

Therefore, $\inf_{t>0} \operatorname{Tr}(\rho\rho_t) = \operatorname{Tr}(\rho\rho_{t_{\min}})$, where $t_{\min} = \dfrac{\pi}{\alpha}$. ☐

In particular, if H is a Hermitian 2×2 - matrices and not a multiple of the identity, then $\left|\lambda_j - \lambda_k\right|$ is either zero or a positive number α. By Corrolary 1, we get [14, Proposition 1].

Corollary 2. [14, Proposition 1] *If ρ is a 2×2 – state and H is not a multiple of identity. Then*

$$\inf_{t>0} \operatorname{Tr}(\rho\rho_t) = \operatorname{Tr}(\rho\rho_{t_{\min}}) = \sum_{j=1}^{2} \left|\tilde{\rho}_{jj}\right|^2 - 2\left|\tilde{\rho}_{12}\right|^2,$$

where $t = \dfrac{\pi}{|\lambda_1 - \lambda_2|}$ *and* $\tilde{\rho} = U\rho U^*$.

Proof. We suppose that H has two distinct eigenvalues which are λ_1, λ_2. From Lemma 2, we have

$$\operatorname{Tr}(\rho\rho_t) = \sum_{j=1}^{2} \left|\tilde{\rho}_{jj}\right|^2 + 2\left|\tilde{\rho}_{12}\right|^2 \cos\left(\left|\lambda_1 - \lambda_2\right| t\right).$$

Hence,

$$\inf_{t>0} \operatorname{Tr}(\rho\rho_t) = \operatorname{Tr}(\rho\rho_{t_{\min}}) = \sum_{j=1}^{2} \left|\tilde{\rho}_{jj}\right|^2 - 2\left|\tilde{\rho}_{12}\right|^2,$$

when $\cos\left(\left|\lambda_1 - \lambda_2\right| t\right) = -1$, that is, $t_{\min} = \pi / \left|\lambda_1 - \lambda_2\right|$. ☐

Next, suppose that the differences of eigenvalues $\left|\lambda_j - \lambda_k\right|$ of H are all rational numbers for $(j,k) \in \Im(\rho)$. Then we can write

$$\left|\lambda_j - \lambda_k\right| = \frac{p_{jk}}{q_{jk}},$$

where p_{jk}, q_{jk} are positive integers, for every $(j,k) \in \Im(\rho)$ with the greatest common divisor $\gcd(p_{jk}, q_{jk}) = 1$.

Let q denote the least common multiple of $\{q_{jk} \mid (j,k) \in \Im(\rho)\}$. Then $\left|\lambda_j - \lambda_k\right| q$ are all integers for $(j,k) \in \Im(\rho)$.

Corollary 3. *Let's adopt the notations above. Assume that the differences of eigenvalues $\left|\lambda_j - \lambda_k\right|$ of H are all rational numbers for $(j,k) \in \Im(\rho)$. Then the following statements hold.*

(i). *There exists $t_{min} > 0$ such that*

$$\inf_{t>0} \operatorname{Tr}(\rho\rho_t) = \operatorname{Tr}(\rho\rho_{t_{\min}}).$$

(ii). *Denote by d the greatest common divisor of* $\{\left|\lambda_j-\lambda_k\right|q \mid (j,k)\in\Im(\rho)\}$. *If* $\frac{|\lambda_j-\lambda_k|q}{d}$ *are all odd, for* $(j,k)\in\Im(\tilde{\rho})$, *then*

$$\inf_{t>0}\mathrm{Tr}(\rho\rho_t)=\mathrm{Tr}(\rho\rho_{t_{\min}})=\sum_{(j,k)\notin\Im(\tilde{\rho})}\left|\tilde{\rho}_{jk}\right|^2-\sum_{(j,k)\in\Im(\tilde{\rho})}\left|\tilde{\rho}_{jk}\right|^2,$$

where $t_{\min}=\frac{q}{d}\pi$.

Proof. (i). The period of $\cos(|\lambda_j-\lambda_k|t)$ is $\frac{2\pi}{|\lambda_j-\lambda_k|}=\frac{2q_{jk}\pi}{p_{jk}}$. Hence, by the formula (4), $\mathrm{Tr}(\rho\rho_t)=\mathrm{Tr}(\rho\rho_{t+2q\pi})$, where q is the least common multiple of $\{q_{jk}\mid(j,k)\in\Im(\rho)\}$. Since the interval $[0,2q\pi]$ is a compact set and $\mathrm{Tr}(\rho\rho_t)$ is continuous, there exist a positive $t_{\min}$ in $[0,2q\pi]$ such that

$$\inf_{t>0}\mathrm{Tr}(\rho\rho_t)=\min\{\mathrm{Tr}(\rho\rho_t)\mid 0\le t\le 2q\pi\}=\mathrm{Tr}(\rho\rho_{t_{\min}}).$$

(ii). We see that $\cos\left(\left|\lambda_j-\lambda_k\right|t_{\min}\right)=\cos\left(\frac{|\lambda_j-\lambda_k|q}{d}\pi\right)=-1$ if and only if $\frac{|\lambda_j-\lambda_k|q}{d}$ are all odd, for $(j,k)\in\Im(\tilde{\rho})$. Combine this fact with Theorem 1**(i):**, we get statement (ii). ☐

Remark 1. Assume that $p_1,\cdots,p_m$ are distinct positive integer. Then the system

$$\cos\left(\left|p_i-p_j\right|x\right)=-1,\ 1\le i\ne j\le m \tag{7}$$

has a real solution if and only if $\frac{\left|p_i-p_j\right|}{d}$ is odd for every $1\le i\ne j\le m$, where $d=\gcd(p_1,\cdots,p_m)$.
As a conseguence, if the system (7) has a real solution then $m<3$.

Proof. We only need to prove that if the system (7) has a real solution then $m<3$. Assume, on the contrary, that $m\ge3$. Without loss of generality, suppose that $p_1\ge p_2\ge p_3,\cdots$. Then $\frac{p_1-p_2}{d},\frac{p_2-p_3}{d},\frac{p_1-p_3}{d}$ can not be all odd. Then the system (7) has no solution, we get a contradiction. ☐

To conclude this subsection, by applying Kronecker's Theorem, the independence of the eigenvalue distances of H over the rational numbers can imply the existence of the infimum value $\inf_{t>0}\mathrm{Tr}(\rho\rho_t)$ while the argumin time $t_{\min}$ does not exist provided that there are two distinct eigenvalue distances.

Proposition 2. *Let $\rho,H,\tilde{\rho}$ be given as in Theorem 1. Suppose the set* $\{|\lambda_j-\lambda_k|\mid(j,k)\in\Im(\tilde{\rho})\}$ *is independent over the rational filed* $\mathbb{Q}$. *Then the following statements hold true.*

(1)

$$\inf_{t>0}\mathrm{Tr}(\rho\rho_t)=\sum_{(j,k)\notin\Im(\tilde{\rho})}\left|\tilde{\rho}_{jk}\right|^2-\sum_{(j,k)\in\Im(\tilde{\rho})}\left|\tilde{\rho}_{jk}\right|^2.$$

(2) *If, assumed further that* $\{|\lambda_i - \lambda_j| \mid (i,j) \in \Im(\tilde{\rho})\}$ *has at least two distinct numbers, then for every* $t > 0$, *we have*

$$\inf_{t>0} \operatorname{Tr}(\rho\rho_t) < \operatorname{Tr}(\rho\rho_t).$$

Proof. Using Lemma 1, we get

$$\operatorname{Tr}(\rho\rho_t) = \sum_{(j,k)\notin\Im(\tilde{\rho})} \left|\tilde{\rho}_{jk}\right|^2 + \sum_{(j,k)\in\Im(\tilde{\rho})} \left|\tilde{\rho}_{jk}\right|^2 \exp\big(i(\lambda_j - \lambda_k)t\big).$$

By Kronecker's Theorem ([8, Theorem 9.1]), we obtain (1). To prove (2), we observer that

$$\sum_{(j,k)\in\Im(\tilde{\rho})} \left|\tilde{\rho}_{jk}\right|^2 \exp\big(i(\lambda_j - \lambda_k)t\big) = -\sum_{(j,k)\in\Im(\tilde{\rho})} \left|\tilde{\rho}_{jk}\right|^2$$

if and only if the system

$$\exp i\big((\lambda_j - \lambda_k)t\big) = -1, \quad \forall (j,k) \in \Im(\tilde{\rho}),$$

has a solution. However, without lost of generality, we can assume that $|\lambda_1 - \lambda_2|, |\lambda_2 - \lambda_3|$ are distinct elements in $\Im(\tilde{\rho})$ and so independent over the rational numbers. Then the system $\cos(|\lambda_1 - \lambda_2|t) = 0,\ \&\ \cos(|\lambda_2 - \lambda_3|t) = 0$ has no solution. This ends the proof. □

2.2. **Optimal values of some Hilbert - Schmidt fidelity measures on unitary paths of $n \times n$ states.** Given a real-valued, symmetric and nonnegative function f on $[0,1]^2$, we consider the function $\mathcal{F}_f$ defined on n- dimensional quantum state space as below:

$$\mathcal{F}_f(A,B) = \operatorname{Tr}(AB) f(\operatorname{Tr} A^2, \operatorname{Tr} B^2), \tag{8}$$

for every pair of quantum states of the same size A, B. These fidelity measures are called *Hilbert-Schmidt fidelities*, see [10, 2.6]. This class includes the well-known fidelity measures such as $\mathcal{F}_{\mathrm{II}}, \mathcal{F}_{\mathrm{AM}}, \mathcal{F}_{\mathrm{GM}}$, etc. (see [10]). Precisely,

- If $f(x,y) = 2\frac{\sqrt{xy}}{x+y}$, then

$$\mathcal{F}_f(A,B) = \frac{2\operatorname{Tr}(AB)}{\operatorname{Tr}(A^2) + \operatorname{Tr}(B^2)} =: \mathcal{F}_{\mathrm{AM}}(A,B).$$

- If $f(x,y) = 1$, a constant function then

$$\mathcal{F}_f(A,B) = \frac{\operatorname{Tr}(AB)}{\sqrt{\operatorname{Tr}(A^2)}\sqrt{\operatorname{Tr}(B^2)}} =: \mathcal{F}_{\mathrm{GM}}(A,B).$$

- If $f(x,y) = \frac{\sqrt{xy}}{\max\{x,y\}}$, then

$$\mathcal{F}_f(A,B) = \frac{\operatorname{Tr}(AB)}{\max\{\operatorname{Tr}(A^2), \operatorname{Tr}(B^2)\}} =: \mathcal{F}_{\mathrm{II}}(A,B).$$

It is clear that

$$\mathcal{F}_f(A,B) = \mathcal{F}_{\mathrm{GM}}(A,B) f(\operatorname{Tr}(A^2), \operatorname{Tr}(B^2)).$$

Proposition 3. *Let $\mathcal{F}$ be a fidelity measure defined by (8) and let ρ, ρ_t, H be as specified in Theorem 1. Then the following statements hold true.*

(i) *If there exists a positive number x such that $\left|\lambda_j - \lambda_k\right| x$ are all odd integers for $(j,k) \in \Im(\tilde{\rho})$, then $t_{\min} = x\pi$ and*

$$\inf_{t>0} \mathcal{F}(\rho, \rho_t) = \mathcal{F}(\rho, \rho_{t_{min}}) = f(a,a) \left(\sum_{(j,k)\notin\Im(\tilde{\rho})} \left|\tilde{\rho}_{jk}\right|^2 - \sum_{(j,k)\in\Im(\tilde{\rho})} \left|\tilde{\rho}_{jk}\right|^2 \right),$$

where $a = \operatorname{Tr} \rho^2$.

(ii) *Suppose that $\left|\lambda_j - \lambda_k\right|$ is either zero or a positive number α, for every $0 \le j,k \le n$. Then, there exists $t_{\min} = \dfrac{\pi}{\alpha}$, and the minimal value is the same as that in (i).*

Proof. (i). Let $a = \operatorname{Tr} \rho^2$, $0 \le a \le 1$. Since the trace function is unitarily invariant and $\rho_t = \mathrm{e}^{-\mathrm{i}\,tH} \rho\, \mathrm{e}^{\mathrm{i}\,tH}$, we have $\operatorname{Tr} \rho_t^2 = \operatorname{Tr} \rho^2 = a$. Hence,

$$\inf_{t>0} \mathcal{F}(\rho, \rho_t) = f(a,a) \inf_{t>0} \operatorname{Tr}(\rho\rho_t).$$

Now, by Theorem (1)(ii), we conclude the proof.
(ii). The same argument as in the proof of (i), only replacing 'by Theorem 1(ii)' with 'Corollary 1'. □

In addition, we consider the quantum speed limit for other well-known fidelity measures:

$$\mathcal{F}_N(A,B) = \operatorname{Tr}(AB) + \sqrt{1 - \operatorname{Tr} A^2}\sqrt{1 - \operatorname{Tr} B^2}, \tag{9}$$

$$\mathcal{F}_C(A,B) = \frac{n-2}{2(n-1)} + \frac{n}{2(n-1)} \mathcal{F}_N(A,B). \tag{10}$$

Proposition 4. *Let $\mathcal{F}$ be either $\mathcal{F}_N$ or $\mathcal{F}_C$ (defined by (9, 10)) and let ρ, ρ_t, H be as specified in Theorem 1. Then the following statements hold true.*

(i): *If there exists a positive number x such that $\left|\lambda_j - \lambda_k\right| x$ are all odd integers for $(j,k) \in \Im(\tilde{\rho})$, then there exists $t_{\min} = x\pi$ such that $\inf_{t>0} \mathcal{F}_N(\rho,\rho_t) = \mathcal{F}_N(\rho, \rho_{t_{\min}}) = 1 - \operatorname{Tr}\rho^2 + \sum_{(j,k)\notin\Im(\tilde{\rho})} \left|\tilde{\rho}_{jk}\right|^2 - \sum_{(j,k)\in\Im(\tilde{\rho})} \left|\tilde{\rho}_{jk}\right|^2$, and, $\inf_{t>0} \mathcal{F}_C(\rho,\rho_t) = \mathcal{F}_C(\rho,\rho_{t_{\min}}) = 1 - \dfrac{n}{2(n-1)} \left(\operatorname{Tr}\rho^2 - \sum_{(j,k)\notin\Im(\tilde{\rho})} \left|\tilde{\rho}_{jk}\right|^2 + \sum_{(j,k)\in\Im(\tilde{\rho})} \left|\tilde{\rho}_{jk}\right|^2 \right)$.*

(ii): *Suppose that $\left|\lambda_j - \lambda_k\right|$ is either zero or a positive number α, for every $0 \le j,k \le n$. Then, there exists $t_{\min} = \dfrac{\pi}{\alpha}$, and the minimal value is the same as that in (i).*

Proof. From $\operatorname{Tr}\rho_t^2 = \operatorname{Tr}\rho^2 = a$, (9) and (10), we have

$$\inf_{t>0} \mathcal{F}_N(\rho,\rho_t) = 1 - a + \inf_{t>0} \operatorname{Tr}(\rho\rho_t)$$

$$\inf_{t>0} \mathcal{F}_C(\rho,\rho_t) = \frac{n-2}{2(n-1)} + \frac{n}{2(n-1)} \inf_{t>0} \mathcal{F}_N(\rho,\rho_t).$$

Now, applying Theorem 1(ii) and Corollary 1, we get the statement. □

Using Corollary 3 and the relation between the fidelity measure and the inner product, as discussed in the proofs above, we get the following corollary.

Corollary 4. *Let's adopt the notations specified in Corollary 3 and $\mathcal{F}$ be a fidelity measure defined by (8, 9, 10). Assume that the differences of eigenvalues $\left|\lambda_j - \lambda_k\right|$ of H are all rational numbers for $(j,k) \in \Im(\rho)$. Then the following statements hold.*

(i). *There exists $t_{min} > 0$ such that*

$$\inf_{t>0} \mathcal{F}(\rho,\rho_t) = \mathcal{F}(\rho,\rho_{t_{\min}}).$$

(ii). *Denote by d the greatest common divisor of $\{\left|\lambda_j - \lambda_k\right| q \mid (j,k) \in \Im(\rho)\}$. If $\frac{|\lambda_j - \lambda_k| q}{d}$ are all odd, for $(j,k) \in \Im(\tilde{\rho})$, then*

$$\inf_{t>0} \mathcal{F}(\rho,\rho_t) = \operatorname{Tr}(\rho\rho_{t_{\min}}), \quad \textit{where } t_{\min} = \frac{q}{d}\pi.$$

2.3. **Quantum speed limit for operators on bipartite spaces.** Throughout this section, let ρ^1, ρ^2 be 2×2 quantum states, $\rho = \rho^1 \otimes \rho^2$ be a separable state on $\mathscr{H}_1 \otimes \mathscr{H}_2$. Let $H_1 = H_1^*, H_2 = H_2^*$ be 2×2 Hermitian matrices and

$$H_1 = U_1^* \begin{bmatrix} \lambda_1^1 & 0 \\ 0 & \lambda_2^1 \end{bmatrix} U_1, H_2 = U_2^* \begin{bmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{bmatrix} U_2,$$

U_1, U_2 are 2×2 unitary matrices.
Then $U = U_1 \otimes U_2$ is a unitary matrix on $\mathscr{H}_1 \otimes \mathscr{H}_2$. It follows that

$$\tilde{\rho} = U^*\rho U = U_1^*\rho^1 U_1 \otimes U_2^* \rho^2 U_2 = \tilde{\rho}^1 \otimes \tilde{\rho}^2.$$
$$\text{and } \rho_t = (U_1)_t^* \rho^1 (U_1)_t \otimes (U_2)_t^* \rho^2 (U_2)_t = \mathrm{e}^{-\mathrm{i}tH_1}\,\rho^1\,\mathrm{e}^{\mathrm{i}tH_1} \otimes \mathrm{e}^{-\mathrm{i}tH_2}\,\rho^2\,\mathrm{e}^{\mathrm{i}tH_2} = \rho_t^1 \times \rho_t^2.$$

Lemma 3. *Let's adopt the notation above. For $t > 0$, then*

$$\operatorname{Tr}(\rho\rho_t) \geq \left(\sum_{j=1}^{2} \left|\tilde{\rho}_{jj}^1\right|^2 - 2\left|\tilde{\rho}_{12}^1\right|^2 \right) \left(\sum_{j=1}^{2} \left|\tilde{\rho}_{jj}^2\right|^2 - 2\left|\tilde{\rho}_{12}^2\right|^2 \right).$$

The equality holds if and only if there is a positive number x such that $\left|\lambda_1^1 - \lambda_2^1\right| x$ and $\left|\lambda_1^2 - \lambda_2^2\right| x$ are odd natural numbers.

Proof. We have

$$\operatorname{Tr}(\rho\rho_t) = \operatorname{Tr}(\rho^1\rho_t^1)\operatorname{Tr}(\rho^2\rho_t^2)$$

$$= \left(\sum_{j=1}^{2} \left| \tilde{\rho}_{jj}^{1} \right|^2 + 2\left|\tilde{\rho}_{12}^{1}\right|^2 \cos\left(\left|\lambda_1^1 - \lambda_2^1\right| t\right)\right) \left(\sum_{j=1}^{2} \left| \tilde{\rho}_{jj}^{2} \right|^2 - 2\left|\tilde{\rho}_{12}^{2}\right|^2 \cos\left(\left|\lambda_2^1 - \lambda_2^2\right| t\right)\right)$$

$$\geq \left(\sum_{j=1}^{2} \left| \tilde{\rho}_{jj}^{1} \right|^2 - 2\left|\tilde{\rho}_{12}^{1}\right|^2 \right) \left(\sum_{j=1}^{2} \left| \tilde{\rho}_{jj}^{2} \right|^2 - 2\left|\tilde{\rho}_{12}^{2}\right|^2 \right).$$

The equality holds if and only if

$$\cos\left(\left|\lambda_2^i - \lambda_2^i\right| t\right) = -1,\ i = 1, 2.$$

Then there exists positive x such that $\left|\lambda_2^i - \lambda_2^i\right| x$ are odd natural numbers for $i = 1, 2$ and $t_{\min} = x\pi$. □

Corollary 5. *Let ρ_1, U_1, H_1, H_2 be as in Lemma 3 and $a = \mathrm{Tr}(\rho^2)$.*

(i): *If $\mathcal{F} = \mathcal{F}_f$ is defined by (8), then*

$$\mathcal{F}(\rho, \rho_t) \geq f(a, a) \left(\sum_{j=1}^{2} \left| \tilde{\rho}_{jj}^{1} \right|^2 - 2\left|\tilde{\rho}_{12}^{1}\right|^2 \right) \left(\sum_{j=1}^{2} \left| \tilde{\rho}_{jj}^{2} \right|^2 - 2\left|\tilde{\rho}_{12}^{2}\right|^2 \right).$$

The equality holds if there is a positive number x such that $\left|\lambda_1^1 - \lambda_2^1\right| x$ and $\left|\lambda_1^2 - \lambda_2^2\right| x$ are odd natural numbers.

(ii): $\mathcal{F}_N(\rho, \rho_t) \geq \left(\sum_{j=1}^{2} \left| \tilde{\rho}_{jj}^{1} \right|^2 - 2\left|\tilde{\rho}_{12}^{1}\right|^2 \right) \left(\sum_{j=1}^{2} \left| \tilde{\rho}_{jj}^{2} \right|^2 - 2\left|\tilde{\rho}_{12}^{2}\right|^2 \right) + 1 - a.$

$$\mathcal{F}_C(\rho, \rho_t) \geq 1 + \frac{n}{2(n-1)} \left[\left(\sum_{j=1}^{2} \left| \tilde{\rho}_{jj}^{1} \right|^2 - 2\left|\tilde{\rho}_{12}^{1}\right|^2 \right) \left(\sum_{j=1}^{2} \left| \tilde{\rho}_{jj}^{2} \right|^2 - 2\left|\tilde{\rho}_{12}^{2}\right|^2 \right) - a \right].$$

Proof. **(i):** From $\mathrm{Tr}\,\rho_t^2 = \mathrm{Tr}\,\rho^2 = a$ then

$$\inf_{t>0} \mathcal{F}(\rho, \rho_t) = f(a, a) \inf_{t>0} \mathrm{Tr}(\rho\rho_t)$$

Applying Lemma 3, we get the statement.

(ii): Since $\mathrm{Tr}\,\rho_t^2 = \mathrm{Tr}\,\rho^2 = a$ and Eq. (9),(10), we have

$$\inf_{t>0} \mathcal{F}_N(\rho, \rho_t) = 1 - a + \inf_{t>0} \mathrm{Tr}(\rho\rho_t),$$
$$\inf_{t>0} \mathcal{F}_C(\rho, \rho_t) = \frac{n-2}{2(n-1)} + \frac{n}{2(n-1)} \inf_{t>0} \mathcal{F}_N(\rho, \rho_t).$$

By associate Lemma 3, we obtain the statement.

□

3. Quantum speed limit for 3×3 states

In this section, we consider $H = U^* \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} U$, with $\lambda_1 \le \lambda_2 \le \lambda_3$, $\tilde{\rho} = U\rho U^*$ and

$$\tilde{\rho}_t = \begin{bmatrix} \mathrm{e}^{-\mathrm{i}\, t\lambda_1} & 0 & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\, t\lambda_2} & 0 \\ 0 & 0 & \mathrm{e}^{-\mathrm{i}\, t\lambda_3} \end{bmatrix} \tilde{\rho} \begin{bmatrix} \mathrm{e}^{\mathrm{i}\, t\lambda_1} & 0 & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\, t\lambda_2} & 0 \\ 0 & 0 & \mathrm{e}^{\mathrm{i}\, t\lambda_3} \end{bmatrix}.$$

Let's denote by $\alpha_{ij} = \lambda_i - \lambda_j$. Then

$$\begin{aligned} \mathrm{Tr}(\rho\rho_t) &= \mathrm{Tr}(\tilde{\rho}\tilde{\rho}_t) \\ &= \sum_{j=1}^{3} \left|\tilde{\rho}_{jj}\right|^2 + 2 \sum_{1\le j<k\le 3} \left|\tilde{\rho}_{jk}\right|^2 \cos\left(\left|\lambda_j - \lambda_k\right| t\right) \\ &= \sum_{j=1}^{3} \left|\tilde{\rho}_{jj}\right|^2 + 2|\tilde{\rho}_{12}|^2 \cos(\alpha_{12} t) + 2|\tilde{\rho}_{23}|^2 \cos(\alpha_{23} t) + 2|\tilde{\rho}_{13}|^2 \left(\cos(\alpha_{12}+\alpha_{23})t\right) \end{aligned}$$

It is clear that $\left|\lambda_i - \lambda_j\right| = 0,\ \forall i \neq j$ if and only if H is a multiple of identity. In this case, $\mathrm{Tr}(\rho\rho_t) = \sum\limits_{1\le j,k\le 3} \left|\tilde{\rho}_{jk}\right|^2$. Therefore, throughout this section, we also assume that the Hermitian matrix H is not a multiple of identity.

Proposition 5. *Let's adopt the notations above.*

(i): *If* $\left\{\left|\lambda_i - \lambda_j\right|,\ 1 \le i,j \le 3\right\} = \{0, \alpha\},\ \alpha > 0$. *Then*

$$\inf_{t>0} \mathrm{Tr}(\rho\rho_t) = \mathrm{Tr}(\rho\rho_{t_{\min}}) = \sum_{(j,k)\notin \Im(\tilde{\rho})} \left|\tilde{\rho}_{jk}\right|^2 - \sum_{(j,k)\in \Im(\tilde{\rho})} \left|\tilde{\rho}_{jk}\right|^2,$$

where $t_{\min} = \dfrac{\pi}{\alpha}$.

(ii): *If* $\lambda_2 = \lambda_1 + \alpha, \lambda_3 = \lambda_2 + \alpha$. *Then, there exists* $t_{\min} > 0$ *such that* $\inf\limits_{t>0} \mathrm{Tr}(\rho\rho_t) = \mathrm{Tr}(\rho\rho_{t_{\min}})$, *and,*

$$\begin{aligned} t_{\min} &= \alpha^{-1} \cos^{-1}\left(-\frac{|\tilde{\rho}_{12}|^2 + |\tilde{\rho}_{23}|^2}{4|\tilde{\rho}_{13}|^2}\right) && \text{if } \tilde{\rho}_{13} \neq 0 \ \&\ |\tilde{\rho}_{12}|^2 + |\tilde{\rho}_{23}|^2 < 4|\tilde{\rho}_{13}|^2, \\ t_{\min} &= \frac{\pi}{\alpha} && \text{otherwise.} \end{aligned}$$

Proof. **(i):** Applying Corollary 1 for $n = 3$, we get statement.

(ii): By applying Lemma 2 for $n = 3$, we get

$$\mathrm{Tr}(\rho\rho_t) = \sum_{(j,k)\notin \Im(\tilde{\rho})} \left|\tilde{\rho}_{jk}\right|^2 + \sum_{(j,k)\in \Im(\tilde{\rho})} \left|\tilde{\rho}_{jk}\right|^2 \cos\left(\left|\lambda_j - \lambda_k\right| t\right),$$

where $\Im(\tilde{\rho}) = \left\{(i,j)|1 \le i,j \le 3, \lambda_i \ne \lambda_j, \tilde{\rho}_{ij} \ne 0\right\}$.
By the hypothesis $\lambda_2 = \lambda_1 + \alpha, \lambda_3 = \lambda_2 + \alpha$, we obtain

$$\mathrm{Tr}(\rho\rho_t) = \sum_{j=1}^{3} \left|\tilde{\rho}_{jj}\right|^2 + 2\left(|\tilde{\rho}_{12}|\cos(\alpha t) + |\tilde{\rho}_{13}|\cos(2\alpha t) + |\tilde{\rho}_{23}|\cos(\alpha t)\right).$$

If $\tilde{\rho}_{13} = 0$, then

$$\mathrm{Tr}(\rho\rho_t) = \sum_{j=1}^{3} \left|\tilde{\rho}_{jj}\right|^2 + 2\left(|\tilde{\rho}_{12}| + |\tilde{\rho}_{23}|\right)\cos(\alpha t).$$

Hence, $\mathrm{Tr}(\rho\rho_t)$ achieves its infimum if and only if $\cos(\alpha t) = -1$ and so $t_{\min} = \dfrac{\pi}{\alpha}$.
Therefore $\inf\limits_{t>0} \mathrm{Tr}(\rho\rho_t) = \sum\limits_{j=1}^{3} \left|\tilde{\rho}_{jj}\right|^2 - 2\left(|\tilde{\rho}_{12}| + |\tilde{\rho}_{23}|\right)$.
Now, we consider $\tilde{\rho}_{13} \ne 0$. Let $P_\alpha(t) := |\tilde{\rho}_{12}|\cos(\alpha t) + |\tilde{\rho}_{13}|\cos(2\alpha t) + |\tilde{\rho}_{23}|\cos(\alpha t)$. Then

$$\inf_{t>0} \mathrm{Tr}(\rho\rho_t) = \sum_{j=1}^{3} \left|\tilde{\rho}_{jj}\right|^2 + \inf_{t>0} P_\alpha(t).$$

From $\cos(2\alpha t) = 2\cos^2(\alpha t) - 1$ then we have

$$P_\alpha(t) = 2|\tilde{\rho}_{13}|^2\cos^2(\alpha t) + \left(|\tilde{\rho}_{12}|^2 + |\tilde{\rho}_{23}|^2\right)\cos(\alpha t) - |\tilde{\rho}_{13}|^2.$$

Let $Q_\alpha(x) = 2|\tilde{\rho}_{13}|^2 x^2 + \left(|\tilde{\rho}_{12}|^2 + |\tilde{\rho}_{23}|^2\right) x - |\tilde{\rho}_{13}|^2$. Then

$$\inf_{t>0} P_\alpha(t) = \min_{-1 \le x \le 1} Q_\alpha(x).$$

Since $Q_\alpha(x)$ is parabola with vertex at $x_0 = -\dfrac{|\tilde{\rho}_{12}|^2 + |\tilde{\rho}_{23}|^2}{4|\tilde{\rho}_{13}|^2}$ ($|\tilde{\rho}_{13}| \ne 0$), we obtain the minimal at x_0 if $x_0 \ge -1$ and at $x = -1$ otherwise. Furthermore, the minimum values in these cases are

$$\inf_{t>0} \mathrm{Tr}(\rho\rho_t) = \begin{cases} \sum\limits_{j=1}^{3} \left|\tilde{\rho}_{jj}\right|^2 - 2\left(|\tilde{\rho}_{12}| + |\tilde{\rho}_{23}|\right) & \text{if} \quad \tilde{\rho}_{13} = 0, \\ \sum\limits_{j=1}^{3} \left|\tilde{\rho}_{jj}\right|^2 + 2|\tilde{\rho}_{13}|^2 - \dfrac{\left(|\tilde{\rho}_{12}|^2 + |\tilde{\rho}_{23}|^2\right)^2}{4|\tilde{\rho}_{13}|^2} & \text{if} \quad \tilde{\rho}_{13} \ne 0, \ \dfrac{|\tilde{\rho}_{12}|^2 + |\tilde{\rho}_{23}|^2}{|\tilde{\rho}_{13}|^2} \le 4 \\ \sum\limits_{j=1}^{3} \left|\tilde{\rho}_{jj}\right|^2 + |\tilde{\rho}_{13}|^2 - |\tilde{\rho}_{12}|^2 - |\tilde{\rho}_{23}|^2 & \text{if} \quad \tilde{\rho}_{13} \ne 0, \ \dfrac{|\tilde{\rho}_{12}|^2 + |\tilde{\rho}_{23}|^2}{|\tilde{\rho}_{13}|^2} \ge 4. \end{cases}$$

The proof is complete.

□

According Corollary 3 that if the differences between eigenvalues of H are all rational numbers, then the minimum of $\mathrm{Tr}(\rho\rho_t)$ for $t \in (0, \infty)$ always exists (i.e., there exists the minimal time $t_{\min}$ such that $\inf_{t>0} \mathrm{Tr}(\rho\rho_t) = \mathrm{Tr}(\rho\rho_{t_{\min}})$). The minimal time may not exist if the differences between eigenvalues of H are irrational. The following examples demonstrate this property.

Example 3. Let $H = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{5} \end{bmatrix}$, $\rho = \begin{bmatrix} \frac{1}{4} & \frac{1}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{4} & 0 \\ \frac{1}{8} & 0 & \frac{1}{2} \end{bmatrix}$. From Corrolary 3, we obtain

$$\begin{aligned} \mathrm{Tr}(\rho\rho_t) =& \frac{1}{4^2} + \frac{1}{4^2} + \frac{1}{2^2} + 2\left[\frac{1}{8^2}\cos t + \frac{1}{8^2}\cos\left(\frac{t}{5}\right)\right] \\ =& \frac{3}{8} + \frac{1}{32}\left[\cos t + \cos\left(\frac{t}{5}\right)\right] \end{aligned}$$

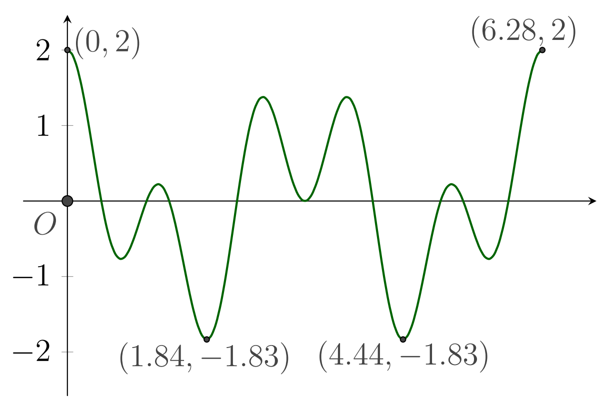
Then, $\inf_{t>0} \mathrm{Tr}(\rho\rho_t) = \min_{t\in[0,10\pi]} \mathrm{Tr}(\rho\rho_t) = \frac{5}{16}$ and $t_{\min} = 5\pi$.

Example 4. Let $H = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{2}{5} \end{bmatrix}$, $\rho = \begin{bmatrix} \frac{1}{4} & \frac{1}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{4} & 0 \\ \frac{1}{8} & 0 & \frac{1}{2} \end{bmatrix}$.

We see that the eigenvalues of H do not satisfy the assumption of Corollary 3. Since Lemma 2, we have

$$\mathrm{Tr}(\rho\rho_t) = \frac{3}{8} + \frac{1}{32}\left[\cos t + \cos\left(\frac{2t}{5}\right)\right]$$

Let $f(x) := \cos 5x + \cos 2x$. Then $f(x)$ is continuous on $\mathbb{R}$. Use Maple version 17 with the command: *plot(f(x), x = 0 .. 2*Pi)*, we obtain the following graph of $f(x)$.



Since f is a periodic function with period 2π, we can deduce from the graph above that

$$\inf_{x>0} f(x) = \min_{x>0} f(x) = \min_{x\in[0,2\pi]} f(x) \approx -1.83$$

Since the system $\cos 5x = -1,\ \cos 2x = -1$ has no solution, so $\min\limits_{x>0} f(x) > -2$, and therefore, $\inf\limits_{t>0} \mathrm{Tr}(\rho\rho_t) > \frac{5}{16}$.

Next, let us consider an example where the difference between eigenvalues of Hermitian H are not all rational. Then, the infimum $\inf\limits_{t>0} \mathrm{Tr}(\rho\rho_t)$ exists while the minimum $\min\limits_{t>0} \mathrm{Tr}(\rho\rho_t)$ does not exist, i.e., there is no $t_{\min}$.

Example 5. Let $H = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \pi & 0 \\ 0 & 0 & \sqrt{7}\pi \end{bmatrix}$, $\rho = \begin{bmatrix} \frac{1}{4} & \frac{1}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{4} & 0 \\ \frac{1}{8} & 0 & \frac{1}{2} \end{bmatrix}$.

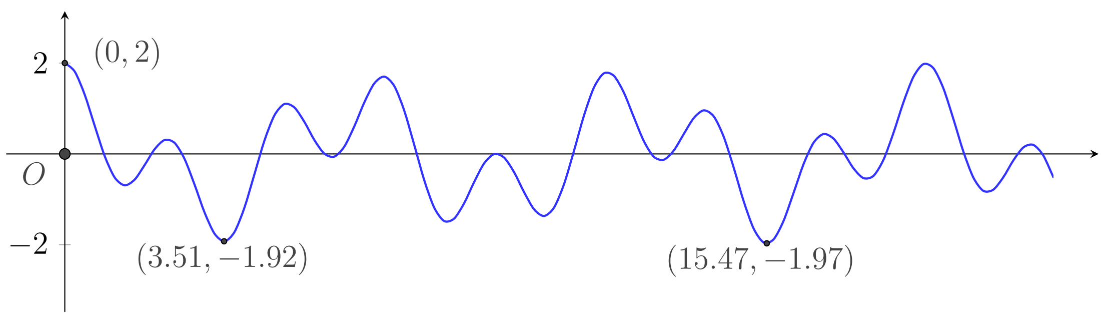
From Lemma 2, we have

$$\mathrm{Tr}(\rho\rho_t) = \frac{3}{8} + \frac{1}{32}\left(\cos \pi t + \cos \sqrt{7}\pi t\right).$$

We can show that

$$\mathrm{Tr}(\rho\rho_t) > \inf_{x>0} \mathrm{Tr}(\rho\rho_t) = \frac{11}{32}.$$

This claim can be justified by consider the function $g(x) = \cos(x) + \cos(\sqrt{7}x)$. Use Maple version 17 with the command: *plot(g(x), x = 0 .. 22)*, we obtain the following graph of $g(t)$.



From the graph above, we can see that as t increases, $g(t)$ gradually approaches -2, but $g(t) > -2$, for every t. In fact, we can show this property as follows.

Proof. Show that $\inf\limits_{x>0} g(x) = -2$.
From the definition of uniform continuity, we have
$\forall \varepsilon > 0, \exists \delta > 0$ such that $\forall t_1, t_2$ satisfied $|t_1 - t_2| < \delta$ then

$$\left|\cos \sqrt{7}t_1 - \cos \sqrt{7}t_2\right| < \varepsilon.$$

Applying Kronecker's theorem[6, Chapter XXIII] for $\delta > 0, \beta = 0, \alpha = \sqrt{7}$ then there exists p, q rational numbers such that

$$\left|\sqrt{7}p - q\right| < \delta.$$

Can assume $0 < p, q$ are odd positive numbers. Then

$$\left|g(p) + 2\right| < \varepsilon.$$

Thus, $\inf_{t>0} g(t) = -2$.
Show that the minimum of g on $(0, \infty)$ does not exist.
Suppose, for the contrary, that the minimum of g on $(0, \infty)$ exists; then $\min_{t>0} g(t) = -2$. This is equivalent to say that the system: $\cos t = -1,\ \cos(\sqrt{7}t) = -1$ has a solution on $(0, \infty)$. Hence, there are positive integer m, n such that $\sqrt{7}+2m\sqrt{7} = 1+2n$; this is a contradiction. Therefore, there does not exist $t_{\min}$ such that $g(t_{\min}) = -2$. We can also apply Proposition 2 to obtain this fact. □

This example can be generalized as Proposition 2.

4. Conclusion

In this work, we present some results on quantum speed limit:

$$\inf_{t>0} \mathcal{F}(\rho, \rho_t),$$

where $\rho_t = \exp(-itH)\rho\exp(itH)$ and where $\mathcal{F}$ is the inner product or a fidelity introduced in [10]. Specifically, we provide a necessary and sufficient condition under which $\inf_{t>0} \mathcal{F}(\rho, \rho_t) = \mathcal{F}(\rho, \rho_{t_{\min}})$, where the infimum is attained at the minimal time $t_{\min}$ such that $\exp i\lambda_j - \lambda_k t_{\min} = -1$ for all distinct eigenvalues Λ_j, λ_k of H, see Theorem 1, Proposition 3 and 4 for details. As a consequence, when the eigenvalue differences of H is either zero or a positive number α then $t_{\min} = \pi/\alpha$, see Corollary 1. In particular, when H is a 2×2 matrix, the eigenvalues of H automatically satisfy this condition and so the main result of [14] become a special case of this corollary.
In the case where all eigenvalue differences of H are all rational numbers, we provide an equivalent condition under which the minimal time $t_{\min}$ exists. In this case, we also compute the exact minimal time; see Corollary 3 and 4.
On the other hand, if the eigenvalue differences of H are independent over the rational numbers, we show that the infimum $\inf_{t>0} \mathcal{F}(\rho, \rho_t)$ exists, although the minimal time $t_{\min}$ may not; see Proposition 2.
As an application of the above result to the low-dimensional case 3×3, we can compute the minimal time $t_{\min}$ when the eigenvalues of H form an arithmetic progression, see Proposition 5, along with some interesting examples and counterexamples that illustrate the results above; see Example 3, 4 and 5.

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