COHOMOLOGY OF THE STRATIFIED FUNDAMENTAL GROUP OF CURVES

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ABSTRACT. Let X be a smooth projective curve over a perfect field k of positive characteristic. This work investigates the relationship between stratified cohomology and group cohomology of the stratified fundamental group of X.

1. INTRODUCTION

Let X be a smooth geometrically connected algebraic variety over an algebraically closed field k of characteristic $p > 0$. A stratified sheaf is a quasi-coherent O_X-module equipped with an action of the sheaf of differential operators $\mathscr{D}(X/k)$. The terminology "stratified" is due to Grothendieck, in connection with the infinitesimal cohomology, see, e.g., [BO78, Section 2] for more details. It is shown that a stratified bundle which is coherent as an O_X-module is automatically locally free, hence will be referred to as a "stratified bundle". As a consequence, the category of stratified bundles is an abelian, rigid tensor category.

An action of $\mathscr{D}(X/k)$ on a sheaf $\mathscr{E}$ determines a descending chain of subsheaves $\mathscr{E}^{(n)}$ which are obtained from one another by the Frobenius pull-back: $\mathscr{E}^{(n)} \cong F^*\mathscr{E}^{(n+1)}$. Such a sequence $(\mathscr{E}^{(n)})$ gives an alternative definition for a stratified bundle and is called a flat bundle, cf. [Gie75, Section 1]. The category of stratified sheaves is denoted by $\mathrm{str}(X/k)$, the full subcategory of stratified bundles is denote by $\mathrm{str}^{\mathrm{coh}}(X/k)$. The cohomology of stratified sheaves as the right derived functors of the hom-set functor in $\mathrm{str}(X/k)$ was studied by Ogus in [Ogu75], see also [Hai13].

Let x be a k-point of X. The fiber functor at x:

$$x^* : \mathrm{str}^{\mathrm{coh}}(X/k) \longrightarrow \mathrm{Vec}_k, \quad (\mathcal{V}, \nabla) \mapsto \mathcal{V}_x,$$

is faithfully exact (since the kernel and cokernel of a map in $\mathrm{str}^{\mathrm{coh}}(X)$ are always locally free). Hence, Tannakian duality (cf. [DM82, Theorem 2.11]) yields an affine group scheme, denoted by $\pi(X, x)$, with the property that x^* gives an equivalence:

$$x^* : \mathrm{str}^{\mathrm{coh}}(X/k) \xrightarrow{\cong} \mathrm{Rep}^{\mathrm{f}}(\pi(X, x)). \tag{1}$$

This equivalence extends to an equivalence between the ind-category $\mathrm{str}^{\mathrm{ind}}(X/k)$ and the category $\mathrm{Rep}(\pi(X, x))$. Here $\mathrm{str}^{\mathrm{ind}}(X/k)$ denotes the category of stratified bundles which can be presented as inductive limits of stratified bundles. We shall refer to them as *ind-stratified bundles.* The group $\pi(X, x)$ can be seen as an analog in positive characteristic of the differential fundamental group. It was implicitly studied in the work [Gie75] of Giesker and more recently

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in the work of dos Santos [dSa07] and subsequently by many authors, see, e.g., [EM10, Kin15, vdP19, Sun19].

In this work, we are interested in comparing the stratified cohomology of a stratified bundle with the group cohomology of the corresponding representation of the stratified fundamental group. The equivalence in (1) implies that the group cohomology of $\pi(X)$ with coefficients in a representation V is the same as the cohomology of the corresponding stratified bundle $\mathcal{V}$ computed in the category $\mathrm{str}^{\mathrm{ind}}(X/k)$ (that is the derived functors of the hom-set functor computed in $\mathrm{str}^{\mathrm{ind}}(X/k)$). Consequently, there is a natural map

$$\delta^i : \mathrm{H}^i(\pi(X), V) \longrightarrow \mathrm{H}^i_{\mathrm{str}}(X, \mathcal{V}), \tag{2}$$

where $\mathcal{V}$ is a stratified bundle and V its fiber at x, since the functors on both sides of this natural map are derived functors of the hom-set functors in the corresponding categories. In particular δ^0 is an isomorphism. Further, as extensions of vector bundles are again vector bundles, δ^1 is also an isomorphism.

However, $\mathrm{str}^{\mathrm{ind}}(X/k)$ is much smaller than the ambient category $\mathrm{str}(X/k)$. Therefore, the map δ^i is not expected to be injective or surjective. Our first main finding is that, for curves, we have the following.

Theorem 1 (Theorem 3.1.1, Proposition 3.2.1and Corollary 3.3.1)**.** *For for smooth projective curve X over k we have:*

(1) *The comparison maps δ^i in (2) are isomorphisms for all $i \geq 0$ and all stratified bundles $\mathcal{V}$. As a consequence, the cohomology of $\pi(X)$ vanishes from degree $i \geq 2$.*
(2) *The restriction map $\rho^i : \mathrm{H}^i(\pi^{\acute{e}t}(X), V) \longrightarrow \mathrm{H}^i(\pi(X), V)$ given by the canonical surjective map $\pi(X) \longrightarrow \pi^{\acute{e}t}(X)$, is bijective for any $i \geq 0$ and any representation V of $\pi^{\acute{e}t}(X)$. As a consequence, the cohomology of $\pi^{\acute{e}t}(X)$ vanishes from degree $i \geq 2$.*
(3) *Let $\pi^{\mathrm{uni}}(X)$ be the Tannakian dual to the full subcategory of nilpotent stratified bundles. Then $\mathrm{H}^i(\pi^{\mathrm{uni}}(X/k), V) \cong \mathrm{H}^i(\pi(X), V)$ As a consequence, the cohomology of $\pi^{\mathrm{uni}}(X)$ vanishes from degree $i \geq 2$.*

The vanishing of group cohomology in degree at least 2 does not agree with that of the differential fundamental group in characteristic 0 (cf. [BTH25]) where the cohomology vanishes only from degree larger than 2. Such a phenomenon was already known in the work of Gieseker, who showed the "rigidity" of regular singular stratified bundles in the formal neighborhood of a point [Gie75, Theorem 3.3].

Being motivated by the non-vanishing of the first de Rham cohomology of smooth projective curves of genus > 0 [BTH25], we ask a similar question for stratified cohomology of curves in positive characteristic. It turns out that, unlike the de Rham cohomology for curves in characteristic 0, in positive characteristic, the first stratified cohomology group may vanish. To see that, we carry out some computations for finite stratified bundles, that is étale trivial vector bundles. Our explicit computation on the curves in Section 4.2 also shows that the non-vanishing of stratified cohomology in degree 1 depends on some invariants of the curves. This point of view suggests many interesting questions, which we hope to address in the future. Our second main finding is the following.

Proposition 2 (Corollary 3.5.4 and Appendix A.6)**.** (1) *Let X be an ordinary curve of genus larger than 1. Let $\mathcal{E}$ be an ordinarily étale trivial vector bundle. Then $\mathrm{h}^1_{\mathrm{str}}(X, \mathcal{E}) \neq 0$.*

(2) *There exists an étale trivial vector bundle on a hyper-elliptic curve for which the first stratified cohomology is equal to zero.*

There is another difference between the two cohomologies. For curves of genus at least 2, the maps δ^i do not automatically extend to isomorphisms at ind-stratified bundles. In fact, the stratified cohomology and the group cohomology differ in their nature. While group cohomology commutes with inductive limits, stratified cohomology is computed in terms of inverse limit functors (see Section 2.2) which usually do not commute with direct limits. We exhibit in Section 4 an explicit example of an ind-stratified bundle, at which the map δ^1 is not bijective. To this end, we use the Cartier map and introduce the concept of order of Cartier nilpotency (cf. Proposition 4.3.3). Our explicit computation for a hyperelliptic curve of genus 2 over $k = \overline{\mathbb{F}}_3$ suggest that the nilpotency might be a finer invariant of curves in positive characteristic. Our finding is:

Proposition 3 (Proposition 4.3.4)**.** *Let X be a hyperelliptic curve of genus 2 and Hasse-Witt rank 1. There exists an ind-stratified bundle $\mathscr{E}$, which is an infinite extension of the trivial stratified bundle, such that the canonical map*

$$\delta^1 : \mathrm{H}^1(\pi(X), E) \longrightarrow \mathrm{H}^1_{\mathrm{str}(X)}(X, \mathscr{E})$$

is not bijective. In fact, the source space is non-zero while the target space is zero.

Let us briefly describe the structure of our work. In Section 2, we recall the definitions of stratified bundles and the stratified cohomology following Giesker [Gie75] and Ogus [Ogu75]. Each stratified bundle $\mathscr{E}$ can be equivalently given as a sequence of vector bundles $\mathscr{E}^{(n)}$ such that $F^*\mathscr{E}^{(n+1)} \cong \mathscr{E}^{(n)}$, F being the absolute Frobenius map. The i-th stratified cohomology is computed from the inverse limit of the pro-system $\mathrm{H}^i(X, \mathscr{E}^{(n)})$, in which the transfer maps are given by the action of the Frobenius (Proposition 2.2.2). Thus, the computation of stratified cohomology is reduced to the study of the Frobenius action on the cohomology groups of Frobenius divided vector bundles. We show the finiteness and vanishing properties of this cohomology, cf. Proposition 2.2.2.

In Section 3, we define a natural map from the group cohomology of the stratified fundamental group to the stratified cohomology. We show that for smooth projective curves the maps are bijective, cf. Theorem 3.1.1. Since stratified cohomology on projective curves vanishes in degree larger than 1, we obtain a similar result for the fundamental group cohomology. This vanishing theorem also allows us to compare the cohomology of the stratified fundamental group with the cohomology of the étale fundamental group and its pro-unipotent quotient in k-linear representations (Proposition 3.2.1 and Corollary 3.3.1). A key ingredient here is the fact, due to dos Santos [dSa07], that any unipotent quotient of the stratified fundamental group is finite. In Subsection 3.2 we compare the stratified cohomology of finite stratified bundles with the étale cohomology of their p-torsion sub-sheaves of Frobenius invariant sections. Then we compare the stratified cohomology of finite stratified bundles with cohomology of the corresponding p-torsion étale sheaves. Our motivation is to study the non-vanishing of the first cohomology group.

In Section 4, we consider the case of ind-stratified bundles. We check the bijectivity of the comparison maps δ^i for elliptic curves. For general curves, the claim turns out to be wrong. Section 4.2 introduces the notion of Cartier nilpotency of Frobenius invariant vector bundles.

We construct, on a hyperelliptic curve of genus $g = 2$, a sequence $\mathscr{E}_n$ of iterated extensions of the trivial bundles, which are Frobenius invariant. Assuming that the ground field has characteristic 3, we show that the order of Cartier nilpotency of $\mathscr{E}_n$ tends to infinity together with n. As a consequence, the first stratified cohomology of $\mathscr{E} = \varprojlim \mathscr{E}_n$ is infinite dimensional and hence the comparison map δ^1 is not bijective.

2. Stratified bundles and stratified cohomology

2.1. **Stratified bundles.** Let k be an algebraically closed field of characteristic $p > 0$ and let X/k be a smooth, geometrically connected scheme.

2.1.1. Let $\mathscr{D}(X/k)$ be the sheaf of differential operators on X/k, cf. [EGA4, IV.16.8]. A stratified sheaf on X/k is a quasi-coherent $\mathscr{O}_X$-module $\mathscr{V}$ equipped with an action of $\mathscr{D}(X/k)$, that is, a k-linear map of sheaves of algebras

$$\nabla : \mathscr{D}(X/k) \longrightarrow \mathscr{E}nd_k(\mathscr{V}).$$

∇ is also called a stratification on $\mathscr{V}$. The category of stratified sheaves over X/S is denoted by $\mathrm{str}(X/S)$. We denote by $\mathrm{str}^{\mathrm{coh}}(X/k)$ the full subcategory of $\mathscr{O}_X$-coherent stratified sheaves. According to a theorem of Grothendieck, an $\mathscr{O}_X$-coherent module equipped with a stratification is always locally free [BO78, Prop. 2.16] or [dSa07, Lemma 6]. We shall therefore address objects of $\mathrm{str}^{\mathrm{coh}}(X/k)$ as stratified bundles.

Thus $\mathrm{str}^{\mathrm{coh}}(X/k)$ is a rigid tensor category with the unit object being the trivial connections $(\mathscr{O}_X, d)$ where d denotes the natural inclusion of $\mathscr{D}(X/k)$ into $\mathscr{E}nd_k(\mathscr{O}_X)$.

2.1.2. Let $(\mathscr{E}, \nabla)$ be a stratified sheaf on X/k. We define $\mathscr{E}^{(i)}$ the subsheaf of sections of $\mathscr{E}$ annihilated by differential operators of order less than p^i and vanishing on constant functions. In other words, let $\mathscr{D}^{<p^i}$ be the subsheaf of differential operators of order less than p^i. Then

$$\mathscr{E}^{(i)} = \mathscr{H}om_{\mathscr{D}^{<p^i}}(\mathscr{O}_X, \mathscr{E}),$$

in particular, $\mathscr{E}^{(0)} = \mathscr{E}$. We define the sheaf of horizontal sections of $(\mathscr{E}, \nabla)$ to be

$$\mathscr{E}^{\nabla} := \bigcap_i E^{(i)} = \varprojlim_i E^{(i)}.$$

We have

$$\mathrm{Hom}_{\mathrm{str}}((\mathscr{O}_X, d), (\mathscr{E}, \nabla)) = \mathrm{H}^0(X, E^{\nabla}) = \bigcap_i \mathrm{H}^0(X, \mathscr{E}^{(i)}) = \varprojlim_i \mathrm{H}^0(X, \mathscr{E}^{(i)}),$$

here $\mathrm{Hom}_{\mathrm{str}}$ is the abbreviation of $\mathrm{Hom}_{\mathrm{Str}(X/k)}$. This space is always finite dimensional, cf. [Hai13, Lemma 1.5] or 3.1 below.

2.1.3. According to a result of Cartier, the Frobenius pull-back of $\mathscr{E}^{(1)}$ is isomorphic to $\mathscr{E}$, that is there is an isomorphism

$$\sigma_0 : F^*\mathscr{E}^{(1)} \cong \mathscr{E},$$

where $F : X \longrightarrow X$ is the absolute Frobenius map, which on the structure sheaf is given by $F : \mathscr{O}_X \longrightarrow \mathscr{O}_X$, $s \mapsto s^p$. There is a p-linear map from $\mathscr{E}^{(1)}$ to $F^*\mathscr{E}^{(1)}$. Compose it with σ_0 we obtain a p-linear map $\psi_1 : \mathscr{E}^{(1)} \longrightarrow \mathscr{E}$.

More generally, the Frobenius pull-back of $\mathscr{E}^{(i+1)}$ is isomorphic to $\mathscr{E}^{(i)}$. As a consequence, there is an alternative definition of stratified sheaves as infinite sequences of sheaves $(\mathscr{E}^{(i)})_{i\geq 0}$ together with isomorphisms $\sigma_i : F^*\mathscr{E}^{(i+1)} \cong \mathscr{E}^{(i)}$, for all $i \geq 0$, cf. [Gie75]. We shall denote by ψ_n the resulting p^n-linear map $\mathscr{E}^{(n)} \longrightarrow \mathscr{E}$. Following dos Santos, we shall call a sequence of bundles and isomorphisms $(\mathscr{E}^{(n)}, \sigma_n)$ an F-divided bundle. We notice that, on a projective smooth variety, the stratified bundle associated to an F-divided bundle is (up to an isomorphism) independent of the choice of the isomorphism σ_i [Gie75, Lemma 1.7].

2.1.4. We notice that $\mathrm{str}^{\mathrm{ind}}(X/k)$ is strictly smaller than $\mathrm{str}(X/k)$. For instance, consider $\mathscr{D}(X/k)$ as a stratified sheaf on itself, then it cannot be presented as the union of its coherent stratified subsheaf, since any proper subsheaf cannot contain the unit section.

2.2. **Stratified cohomology.** When it is clear which stratification is referred to, we shall omit the symbol for it in the notation of a stratified sheaf, thus we shall write simply $\mathscr{E}$ instead of $(\mathscr{E}, \nabla)$. The 0-th stratified cohomology sheaf of a stratified sheaf $\mathscr{E}$ is defined to be

$$\mathrm{H}^0_{\mathrm{str}}(X/k, \mathscr{E}) := \mathrm{Hom}_{\mathrm{str}}(\mathscr{O}, \mathscr{E}) = \mathrm{H}^0(X, E^{\nabla}),$$

where $\mathscr{O}$ is equipped with the trivial stratification 2.1.1. This is a left exact functor from $\mathrm{str}(X/k)$ to the category of k-vector spaces. We notice that $\mathrm{str}(X/k)$ has enough injectives. Hence, we can define the derived functors $\mathrm{H}^i_{\mathrm{str}}(X/k, -)$, which are the higher stratified cohomology functors. In particular, we have

$$\mathrm{H}^i_{\mathrm{str}}(X/k, -) \cong \mathrm{Ext}^i_{\mathrm{str}}(\mathscr{O}_X, -).$$

2.2.1. *Computation.* Following Ogus (cf. [Ogu75]), we provide some computations of the stratified cohomology in terms of the inverse limits. As in 2.1.2, the functor $\mathrm{H}^0_{\mathrm{str}}(X/k, -)$ can be seen as the composed functor

$$\mathrm{H}^0_{\mathrm{str}}(X/k, \mathscr{E}) = \varprojlim_n \mathrm{H}^0(X, \mathscr{E}^{(n)}).$$

Therefore, there is a spectral sequence abutting it. As a consequence, we have exact sequences (cf. [Ogu75, Theorem 2.4]) for any $i \geq 1$:

$$0 \to \varprojlim_n{}^1 \mathrm{H}^{i-1}(X, \mathscr{E}^{(n)}) \to \mathrm{H}^i_{\mathrm{str}}(X/k, (\mathscr{E}, \nabla)) \to \varprojlim_n \mathrm{H}^i(X, \mathscr{E}^{(n)}) \to 0.$$

2.2.2. Recall that the limit on inverse systems of vector spaces (or modules) is a left exact functor. Its derived functors, except possibly for the first one, vanish. Further, for an inverse system $(X_i, t_i : X_{i+1} \longrightarrow X_i)_{i\geq 0}$ of vector spaces (or modules over a ring), we have an exact sequence

$$0 \longrightarrow \varprojlim X_i \longrightarrow \prod_i X_i \xrightarrow{d} \prod_i X_i \longrightarrow \varprojlim{}^1 X_i \longrightarrow 0,$$

where the map d sends a tuple (x_i) to the tuple $(x_i - t_i(x_{i+1}))$. As a consequence, $\varprojlim{}^1(X_i) = 0$ if and only if the map d is surjective. And we have the following sufficient condition for that.

Proposition 2.2.1. *If X_i is complete w.r.t the filtration $F^k(X_i) = \mathrm{Im}(X_{i+k} \to A_i)$ for all i, then d is surjective.*

Proof. Given $(y_0, y_1, ...) \in \prod_i X_i$, we need to find $(x_0, x_1, ...) \in \prod_i X_i$ such that $y_i = x_{i+1} - x_i$ for all i. We only have 1 choice:

$$x_i = -(y_i + y_{i+1} + ...$$

By hypothesis, X_i is complete, so $x_i \in X_i$ for all i. □

In particular, if the spaces X_i are finite dimensional then the first derived limit vanishes. Moreover, if the dimensions of the spaces X_i are uniformly bounded (i.e., $\dim(X_i)$ is bounded above by a constant c for all i), then $\varprojlim X_i$ is finitely dimensional.

Proposition 2.2.2. *Let X be a smooth connected projective variety over k, and $\mathscr{E} = \{\mathscr{E}^{(n)}\}$ be a stratified bundle. Then the stratified cohomology group can be computed as the inverse limit of the Zariski cohomology of $\mathscr{E}^{(n)}$:*

$$\mathrm{H}^i_{\mathrm{str}}(X/k, \mathscr{E}) = \varprojlim_n \mathrm{H}^i(X, \mathscr{E}^{(n)}).$$

Thus $\mathrm{H}^i_{\mathrm{str}}(X/k, \mathscr{E})$ are finite dimensional and vanish for $i > \dim X$.

Proof. We show that the cohomology groups $\mathrm{H}^i_{\mathrm{str}}(X/k, \mathscr{E})$ are finite dimensional for all i and $\mathscr{E}$. For this, we first recall some properties of the Hilbert polynomial of $\mathscr{E}$. Let $\mathscr{O}_X(1)$ be an ample line bundle. For a bundle B of rank $r > 0$, set $B(m) = B \otimes_{\mathscr{O}_X} \mathscr{O}_X(m)$, and define the Hilbert polynomial of B relative to $\mathscr{O}_X(1)$ to be

$$p_B(m) = \frac{1}{r}\chi(X, B(m)) \in \mathbb{Q}[m] \quad \text{for } m \gg 0,$$

Additionally, we define the slope of B to be $\mu(B) = \deg(B)/r$. B is said to be semistable (or μ-semistable) if for all subsheaves $U \subset B$ we have $\mu(U) \leq \mu(B)$.

Consider the stratified bundle $\mathscr{E} = (\mathscr{E}^{(n)}, \sigma_n)_{n\in\mathbb{N}}, \sigma_n : F^*\mathscr{E}^{(n+1)} \cong \mathscr{E}^{(n)}$, of rank $r > 0$. Since

$$\mu((F^n)^*\mathscr{G}) = p^n \mu(\mathscr{G}),$$

we see that $\mathscr{E}^{(n)}$ is μ-semistable when n is large enough. Moreover, by [EM10, Corollary 2.2], the Hilbert polynomials of $\mathscr{E}^{(n)}$ are the same and equal to the one of $\mathscr{O}_X$. Now Langer's boundedness theorem [La04, Theorem 4.2] tells us that the family $\{\mathscr{E}^{(n)}\}$ is bounded, i.e., there exists a scheme S of finite type over k and an S-flat family $\mathscr{F}$ of coherent sheaves on the fibers of $X \times S \to S$, which contains $\{\mathscr{E}^{(n)}\}$. From the above argument, we have that the family of bundles $\{\mathscr{E}^{(n)}\}_{n\in\mathbb{N}}$ comes from a S-flat coherent sheaf, say $\mathscr{F}$, over $X \times S$, where S is a scheme of finite type over k. Hence, to show the uniformly boundedness of $\mathrm{H}^i(X, \mathscr{E}^{(n)})$, it is enough to consider the case S is affine. In that case, set $S = \mathrm{Spec}(A)$, for some finitely generated k-algebra A. Since X is proper, the projection map $X \times S \to S$ is proper too. Hence, by (the Grothendieck complex), there is a finite complex $K^\bullet$ of finitely generated A-modules such that

$$\mathrm{H}^i(K^\bullet \otimes_A B) = \mathrm{H}^i((X \times \mathrm{Spec}(B), \mathscr{F} \otimes_A B),$$

for any A-algebra B. So we deduce our first claim by taking B to be $k(s)$ for any point $s \in S$. □

Remark 2.2.3. i) We note that the fundamental difference between stratified cohomology in positive characteristic and its characteristic zero counterpart, i.e., the de Rham cohomology. While the de Rham cohomology vanishes only in degrees larger than two times the dimension of the variety, stratified cohomology vanishes already in degrees larger than the dimension of the variety.

ii) In a recent preprint [Xia25], the author also proves the finiteness of the stratified cohomology on a smooth and proper scheme by using a similar method. Since he does not assume the projectivity of X, the proof there is more technical.

3. Comparison with fundamental group cohomology

3.1. **The stratified fundamental group and cohomology-comparison theorem.** From the assumption that X is geometrically connected, we have

$$\mathrm{End}_{\mathrm{str}}((\mathscr{O}_X, d)) = \mathrm{H}^0_{\mathrm{str}}(X, (\mathscr{O}_X, d)) = \bigcap_n (\mathscr{O}_X)^{p^n} = k.$$

On the other hand, as the underlying sheaves of objects of $\mathrm{str}^{\mathrm{coh}}(X/k)$ are locally free, the fiber functor at any point is exact, hence is faithful. Consequently, $\mathrm{str}^{\mathrm{coh}}(X/k)$ is a Tannakian category with respect to the fiber functor at any point $x \in X(k)$. In particular, for any stratified bundle, the space $\mathrm{H}^0_{\mathrm{str}}(X, (E, \nabla))$ is finite dimensional. Tannakian duality yields an affine group scheme $\pi(X)$ (we shall not mention the base point x in what follows) with the property: the functor x^* yields an equivalence:

$$x^* : \mathrm{str}^{\mathrm{coh}}(X/k) \xrightarrow{\cong} \mathrm{Rep}^{\mathrm{f}}(\pi(X)), \tag{3}$$

which extends to an equivalence

$$x^* : \mathrm{str}^{\mathrm{ind}}(X/k) \xrightarrow{\cong} \mathrm{Rep}(\pi(X)). \tag{4}$$

According to the Tannakian duality, the functor x^* yields an equivalence between $\mathrm{str}^{\mathrm{ind}}(X/k)$ and the representation category of an affine group scheme which is called the stratified fundamental group of X at the base point x and is denoted by $\pi(X, x)$:

$$x^* : \mathrm{str}^{\mathrm{ind}}(X/k) \cong \mathrm{Rep}(\pi(X, x)).$$

We will simplify the notation to $\pi(X)$. The category $\mathrm{Rep}(\pi(X))$ has enough injectives. Hence we can define the cohomology of $\pi(X)$ with coefficients in its representation V to be the right derived functors of the invariant-subspace functor

$$\mathrm{H}^0(\pi(X), V) := \mathrm{Hom}_{\pi(X)}(k, V) = V^{\pi(X)},$$

where k stands for the trivial representation. The i-th cohomology group is denoted, as usual, by $\mathrm{H}^i(\pi(X), V)$. We also have

$$\mathrm{H}^i(\pi(X), V) \cong \mathrm{Ext}^i_{\pi(X)}(k, V).$$

These isomorphisms yield natural maps

$$\delta^i : \mathrm{H}^i(\pi(X), E) \longrightarrow \mathrm{H}^i_{\mathrm{str}}(X/k, \mathscr{E}),$$

where $E := x^*\mathscr{E} = \mathscr{E}|_x$. As x^* induces an equivalence between $\mathrm{str}^{\mathrm{ind}}(X/k) \cong \mathrm{Rep}(\pi(X))$ and $\mathrm{str}^{\mathrm{ind}}(X/k)$ is stable under taking extension, we immediately have that δ^0 and δ^1 are isomorphisms. Moreover, for degree 2 cohomology, we also have the isomorphism in the case that X is a curve.

Theorem 3.1.1. *Let X be a smooth projective curve over an algebraically closed field k of positive characteristic, $x \in X(k)$. Then for any stratified bundle $(\mathscr{E}, \nabla)$ and $E := \mathscr{E}|_x$, all the maps*

$$\delta^i : \mathrm{H}^i(\pi(X), E) \longrightarrow \mathrm{H}^i_{\mathrm{str}}(X/k, \mathscr{E})$$

are bijective, moreover, δ^i for $i \geq 2$ are all zero-maps, that is

$$\mathrm{H}^i(\pi(X), E) = \mathrm{H}^i_{\mathrm{str}}(X/k, \mathscr{E}) = 0$$

for all $i \geq 2$.

Proof. We have δ^0 and δ^1 are isomorphisms. According to Proposition 2.2.2 we have

$$\mathrm{H}^i_{\mathrm{str}}(X/k, \mathscr{E}) = 0$$

for all $i \geq 2$. Thus, it remains to show that $\mathrm{H}^i(\pi(X), E) = 0$, for any $i \geq 2$.

Since δ^1 is an isomorphism and since $\mathrm{H}^2_{\mathrm{str}}(X/k, -)$ vanishes on any stratified vector bundle, any epimorphism $\mathscr{F} \twoheadrightarrow \mathscr{F}$ in $\mathrm{str}^{\mathrm{ind}}(X/k)$ yields a surjective map $\mathrm{H}^1_{\mathrm{str}}(X/k, \mathscr{F}) \twoheadrightarrow \mathrm{H}^1_{\mathrm{str}}(X/k, \mathscr{F}')$ and hence, taking the fibers at x, we obtain a surjective map

$$\mathrm{H}^1(\pi(X), F) \twoheadrightarrow \mathrm{H}^1(\pi(X), F').$$

Let now $(\mathscr{J}, \nabla)$ be the injective envelope of $(\mathscr{E}, \nabla)$ in $\mathrm{str}^{\mathrm{ind}}(X/k)$ and $J := \mathscr{J}|_x$. Then we have a surjective map

$$0 = \mathrm{H}^1(\pi(X), J) \twoheadrightarrow \mathrm{H}^1(\pi(X), J/E)$$

forcing $\mathrm{H}^1(\pi(X), J/E) = 0$. Hence, the exact sequence

$$0 = \mathrm{H}^1(\pi(X), J/E) \longrightarrow \mathrm{H}^2(\pi(X), E) \longrightarrow \mathrm{H}^2(\pi(X), J) = 0,$$

implies that

$$\mathrm{H}^2(\pi(X), E) = 0.$$

The same discussion holds for higher cohomology groups showing that they all vanish at E. □

3.2. **Comparison with cohomology of the étale fundamental group.** We call a vector bundle on X *étale finite* if it can be trivialized by an étale finite torsor (i.e. torsor under an étale finite group scheme) on X. The category $\mathscr{C}^{\text{ét}}(X)$ of étale finite vector bundles equipped with the fiber functor x^* is Tannakian (the fixed $k-$ point x will be omitted) and its dual is precisely the (geometric) étale fundamental group scheme $\pi^{\text{ét}}(X)$ (considered as a pro-finite group scheme). The reader is referred to [No82], [MS02] and [EHS08] for more details. Further, there exists a natural functor

$$\mathscr{C}^{\text{ét}}(X) \longrightarrow \mathrm{str}(X/k)$$

in the sense that each étale finite vector bundle is equipped in a canonical way a stratification making it a stratified bundle with finite monodromy (cf. [dSa07]). This functor is fully faithful, exact and closed under taking subquotients (which is equivalent to the surjectivity of the map $\pi(X) \longrightarrow \pi^{\text{ét}}(X)$). For convenience, we shall identify $\mathscr{C}^{\text{ét}}(X)$ with the full subcategory of stratified bundle with finite monodromy $\mathrm{str}^{\mathrm{fin}}(X/k)$.

Let $\mathrm{str}^{\mathrm{uni}}(X)$ be the full subcategory of $\mathrm{str}^{\mathrm{coh}}(X)$ of objects which are iterated extension of the trivial object $(\mathscr{O}, d)$. The corresponding Tannakian group is a pro-unipotent group, denoted $\pi^{\mathrm{uni}}(X/k)$. It is a quotient group scheme of $\pi(X)$, and, according to [dSa07], it is pro-finite, hence is a quotient of $\pi^{\text{ét}}(X/k)$.

Summarizing, we have surjective (i.e. faithfully flat) maps

$$\pi(X) \longrightarrow \pi^{\text{ét}}(X/k) \longrightarrow \pi^{\mathrm{uni}}(X/k).$$

Proposition 3.2.1. *Let X/k be a smooth projective curve. For any finite representation V of $\pi^{\text{ét}}(X)$ the natural maps induced by restriction*

$$\text{res}^i : \mathrm{H}^i(\pi^{\text{ét}}(X), V) \longrightarrow \mathrm{H}^i(\pi(X), V)$$

are isomorphisms.

Proof. The claim for $i = 0$ follows from the surjectivity of $\pi(X) \longrightarrow \pi^{\text{ét}}(X/k)$. The surjectivity also ensures the injectivity of res^1.

The claim for $i = 1$ amounts to the thickness of $\mathscr{C}^{\text{ét}}(X)$ in $\text{str}(X/k)$, that is, for any extension

$$\mathscr{O} \longrightarrow \mathscr{W} \longrightarrow \mathscr{V}$$

in $\text{str}(X/k)$ with $\mathscr{V}$ in $\text{str}^{\text{fin}}(X/k)$ we also have $\mathscr{W}$ in $\text{str}^{\text{fin}}(X/k)$. Indeed, let $Y \xrightarrow{p} X$ be the G-torsor constructed from $\mathscr{V}$. By assumption, G is an étale finite group scheme. Then $p^*\mathscr{V}$ is a trivial vector bundle on Y equipped with the trivial connection. According to dos Santos' result mentioned above ($\pi^{\text{uni}}(X/k)$ is a quotient of $\pi^{\text{ét}}(X/k)$), $p^*\mathscr{W}$ is also a finite connection on Y. But this implies the push forward $p_*p^*\mathscr{W}$ is also a finite connection that forces $\mathscr{W}$ to be a finite connection as we have an inclusion of connections

$$\mathscr{W} \hookrightarrow p_*p^*\mathscr{W}.$$

Thus res^1 is an isomorphism.

For $i \geq 2$, it is to show that $\mathrm{H}^i(\pi(X), V) = 0$. We use the same trick as in the proof of 3.1.1. First notice that the isomorphism res^1 extends to any representations of $\pi^{\text{ét}}(X, x)$. Let J be the injective envelope of V in $\text{Rep}(\pi^{\text{ét}}(X, x)) \cong \text{Ind} - \mathscr{C}^{\text{ét}}(X)$. Then we have

$$\mathrm{H}^2(\pi^{\text{ét}}(X,x), V) \cong \mathrm{H}^1(\pi^{\text{ét}}(X,x), J/V) \overset{(\text{res}^1)}{\cong} \mathrm{H}^1(\pi(X), J/V) \cong \mathrm{H}^2(\pi(X), V) = 0.$$

The proof is complete. □

Corollary 3.2.2. *The cohomology groups $\mathrm{H}^i(\pi^{\text{ét}}(X), V)$ vanish for $i \geq 2$.*

3.3. **Cohomology of the pro-unipotent fundamental group.** By definition, the cohomology of $\pi^{\text{uni}}(X/k)$ with coefficients in a representation V is the same as the cohomology of $(\mathscr{V}, \nabla)$ in $\text{str}^{\text{uni}}(X/k)$ with respect to its ind-category, where $(\mathscr{V}, \nabla)$ is the nilpotent stratified bundle corresponding to the representation V:

$$\mathrm{H}^i(\pi^{\text{uni}}(X/k), V) \cong \mathrm{H}^i_{\text{str}^{\text{uni}}}(X/k, (\mathscr{V}, \nabla)).$$

Using Tannakian duality and the fact that extensions of nilpotent stratified bundles in $\text{str}(X/k)$ is again nilpotent, we have, for $i = 0, 1$,

$$\mathrm{H}^i_{\text{str}^{\text{uni}}}(X/k, (\mathscr{V}, \nabla)) \cong \mathrm{H}^i_{\text{str}}(X/k, (\mathscr{V}, \nabla))$$

and

$$\mathrm{H}^i(\pi^{\text{uni}}(X/k), V) \cong \mathrm{H}^i(\pi(X), V)$$

for any nilpotent stratified bundle $(\mathscr{V}, \nabla)$ and its fiber V. Recall that the latter isomorphism extends to ind-representations, while the former one may not extend to ind-stratified bundles.

Corollary 3.3.1. *The group cohomology $\mathrm{H}^i(\pi^{\text{uni}}(X/k), V)$, $i \geq 2$, vanishes for all representation V.*

Proof. Let J be an injective envelope of V in $\mathrm{Rep}(\pi^{\mathrm{uni}}(X/k)$. Then we have a natural isomorphism (cf. proof of Proposition 3.2.1):

$$\mathrm{H}^i(\pi^{\mathrm{uni}}(X/k),V) \cong \mathrm{H}^{i-1}(\pi^{\mathrm{uni}}(X/k),J/V).$$

Employing the isomorphism

$$\mathrm{H}^i(\pi^{\mathrm{uni}}(X/k),V) \cong \mathrm{H}^i(\pi(X),V)$$

which holds for any representation of $\pi^{\mathrm{uni}}(X/k)$, we obtain an isomorphism

$$\mathrm{H}^i(\pi^{\mathrm{uni}}(X/k),V) \cong \mathrm{H}^i(\pi(X),V).$$

Whence we obtain the vanishing claim, by means of Theorem 3.1.1. □

3.4. **Application: a computation for cohomology of curves of genus 1.** Let X have genus 1. Then, according to dos Santos [dSa07], $\pi(X)$ is commutative and decomposes into a direct product of a unipotent part and a diagonal part. Since the group cohomology of the diagonal part vanishes (being linearly reductive),

$$\mathrm{H}^i(\pi(X),V)) = \mathrm{H}^i(\pi^{\mathrm{uni}}(X),V),$$

for $i \geq 1$ and any unipotent representation V. In categorical terms, we have:

- If X is super-singular, the absolute Frobenius is the zero map. Hence $\pi(X)$ is unipotent, that is, all stratified bundles are nilpotent.
- If X is ordinary, that is, the absolute Frobenius acts by a bijective map, then according to [dSa07], any object of $\mathrm{str}^{\mathrm{coh}}(X/k)$ decomposes into a direct sum of objects of the form

$$\mathscr{L} \otimes \mathscr{N}$$

  where $\mathscr{L}$ is a rank one stratified bundle and $\mathscr{N}$ is a nilpotent object, i.e., all of its irreducible subquotients are trivial objects. Such an object is also called to be $\mathscr{L}$-isotypical.

Now, dos Santos tells us that (loc.cit. Theorem 21)

$$\pi^{\mathrm{uni}}(X) = \mathbb{Z}_p^r,$$

where r is the p-rank of X (i.e., the rank of the Frobenius map acting on $\mathrm{H}^1(X,\mathscr{O}_X)$, which can be either 1 or 0). Thus we have a complete understanding of the stratified cohomology of stratified bundles on an elliptic curve.

Proposition 3.4.1. *Let X be a smooth projective curve of genus 1.*

(1) *If X has p-rank 0 then $\mathrm{h}^1_{\mathrm{str}}(X,\mathscr{E}) = 0$ for any stratified bundle $\mathscr{E}$.*
(2) *If X has p-rank 1, then $\mathrm{h}^1_{\mathrm{str}}(X,\mathscr{O}_X) = 1$ and hence this function is non-zero at any unipotent stratified bundle. On the other hand, for any non-trivial stratified line bundle $\mathscr{L}$, we have $\mathrm{h}^1_{\mathrm{str}}(X,\mathscr{V}) = 0$ if $\mathscr{V}$ is an $\mathscr{L}$-isotypical stratified bundle.*

Proof. If X has p-rank 0, that is, X is super-singular, the action of the Frobenius of X is zero, hence $\pi(X)$ is diagonal. So its cohomology vanishes from degree 1.

If X has p-rank 1, that is X is ordinary, the absolute Frobenius acts by a bijective map on the 1-dimensional space $\mathrm{H}^1(X,\mathscr{O}_X)$. Hence $\mathrm{H}^1_{\mathrm{str}}(X,\mathscr{O}_X) \cong k$. If $\mathscr{V}$ is a nilpotent object, then there is a surjective map $\mathscr{V} \longrightarrow \mathscr{O}_X$ as objects in $\mathrm{str}(X/k)$. As the $\mathrm{H}^2_{\mathrm{str}}$-group vanishes, we have a surjective map $\mathrm{H}^1_{\mathrm{str}}(X,\mathscr{V}) \longrightarrow \mathrm{H}^1_{\mathrm{str}}(X,\mathscr{O}_X)$, whence $\mathrm{H}^1_{\mathrm{str}}(X,\mathscr{V}) \neq 0$.

On the other hand, as explained above, any object of $\mathrm{str}^{\mathrm{coh}}(X/k)$ decomposes into a direct sum of objects of the form $\mathscr{L}\otimes\mathscr{N}$ where $\mathscr{L}$ is a rank one stratified bundle and $\mathscr{N}$ is a nilpotent object, i.e., all of its irreducible subquotients are trivial. Hence, if $\mathscr{L}$ is a non-trivial rank 1 object $\mathrm{H}^1_{\mathrm{str}}(X,\mathscr{L}) \cong \mathrm{Ext}^1_{\mathrm{str}(X/k)}(\mathscr{O}_X,\mathscr{L}) = 0$. Using the long exact sequence of cohomology groups, we obtain $\mathrm{Ext}^1_{\mathrm{str}(X/k)}(\mathscr{N},\mathscr{L}) = 0$ for any nilpotent stratified bundle $\mathscr{N}$, whence the last claim of Proposition. □

Corollary 3.4.2. *Consider $\mathbb{Z}_p$ as a pro-finite group scheme over k. Then $\mathrm{h}^i(\mathbb{Z}_p,-)$ vanishes for any $i \geq 2$ and*

$$\mathrm{h}^i(\mathbb{Z}_p,k) = 1,$$

for $i = 0,1$.

3.5. **The non-vanishing of the first cohomology group.** Our aim in this section is to discuss the vanishing property of the first cohomology group. This question is motivated by the non-vanishing property for the first de Rham cohomology on curves over a field of characteristic zero [BTH25]. We shall restrict ourselves to *finite stratified bundles,* that is étale trivial vector bundles, cf. 3.2.

A vector bundle is said to be F-periodic if it is isomorphic to the pull-back of itself by some powers of the Frobenius map. According to Lange-Stuhler [LS77], F-periodic vector bundles are étale trivial. The converse implication holds if the base field is finite or if the vector bundle is stable [BD07]. Now we restrict further our problem to F-periodic vector bundles.

Let $\mathscr{E}$ be an F-periodic vector bundle, we shall use the same symbol for the associated stratified bundle. Fix an isomorphism $\tau : F^{*n}\mathscr{E} \longrightarrow \mathscr{E}$. The composition of τ's and the canonical inclusion $\mathscr{E} \longrightarrow F^{*n}\mathscr{E}$ yields a p^n-linear endomorphism

$$f : \mathscr{E} \longrightarrow F^{*n}\mathscr{E} \xrightarrow{\tau} \mathscr{E}.$$

Denote by $\mathscr{H}_{\mathscr{E}}$ the subsheaf of $\mathscr{E}$ fixed by f in the étale topology on X. The lemma below goes back to [Kat69, Propositions 1.1 and 1.2].

Lemma 3.5.1. *Assume that the p-rank of X is equal to r. Then $\mathscr{H}_{\mathscr{E}}$ is a locally constant sheaf with fiber a vector space over $\mathbb{F}_{p^n}$ with dimension equal to the rank of $\mathscr{E}$, and there is an exact sequence*

$$0 \to \mathscr{H}_{\mathscr{E}} \longrightarrow \mathscr{E} \xrightarrow{f-1} \mathscr{E} \longrightarrow 0$$

of sheaves on $X_{ét}$.

Remark. $\mathscr{H}_{\mathscr{E}}$ can be seen as a finite additive group scheme on X.

Proposition 3.5.2. *Given an F-periodic stratified bundle $\mathscr{E}$ of period n. Let $\mathscr{H}_{\mathscr{E}}$ be the associated finite étale group scheme defined as above. Then we have a canonical isomorphism:*

$$\mathrm{H}^i_{ét}(X,\mathscr{H}_{\mathscr{E}}) \otimes_{\mathbb{F}_q} k \xrightarrow{\cong} \mathrm{H}^i_{\mathrm{str}}(X,\mathscr{E}),$$

for all i, where $q = p^n$.

Proof. From the short exact sequence of sheaves on $X_{\text{ét}}$:

$$0 \to \mathscr{H}_{\mathscr{E}} \to \mathscr{E} \xrightarrow{F^{*n}-1} \mathscr{E} \to 0,$$

we have the following long exact sequence of *étale* cohomologies:

$$\cdots \to \mathrm{H}^i_{\text{ét}}(X, \mathscr{H}_{\mathscr{E}}) \to \mathrm{H}^i_{\text{ét}}(X, \mathscr{E}) \xrightarrow{F^{*n}-1} \mathrm{H}^i_{\text{ét}}(X, \mathscr{E}) \to \cdots \tag{5}$$

Since E is a coherent sheaf on X, we have $\mathrm{H}^i_{\text{ét}}(X, \mathscr{E}) \cong \mathrm{H}^i(X, \mathscr{E})$. Since the induced morphism on cohomology $F^n : \mathrm{H}^i_{\text{ét}}(X, \mathscr{E}) \to \mathrm{H}^i_{\text{ét}}(X, \mathscr{E})$ is q-linear, the argument before this theorem implies that $F^{*n} - 1 : \mathrm{H}^i_{\text{ét}}(X, \mathscr{E}) \to \mathrm{H}^i_{\text{ét}}(X, \mathscr{E})$ is surjective (cf. [Kat69, Proposition 1.2]). Hence, for each i, we have the following short exact sequence:

$$0 \to \mathrm{H}^i_{\text{ét}}(X, \mathscr{H}_{\mathscr{E}}) \to \mathrm{H}^i_{\text{ét}}(X, \mathscr{E}) \xrightarrow{F^{*n}-1} \mathrm{H}^i_{\text{ét}}(X, \mathscr{E}) \to 0.$$

This finishes the proof since we have $\mathrm{H}^i_{\text{str}}(X, \mathscr{E}) = \mathrm{H}^i(X, \mathscr{E})_{\text{ss}} \cong \mathrm{H}^i(X, \mathscr{E})^{F^{*n}-1} \otimes_{\mathbb{F}_q} k$ which is isomorphic to $\mathrm{H}^i_{\text{ét}}(X, \mathscr{H}_{\mathscr{E}}) \otimes_{\mathbb{F}_q} k$ by the above short exact sequence. □

Assume that X is a curve of genus larger than one. The method of [BTH25] relies on the Euler characteristic formula which is not always available in positive characteristic. More precisely, let $\mathscr{F}$ be a locally constant sheaf on X and $f : Y \longrightarrow X$ be an étale covering map that trivializes $\mathscr{F}$, that is, $f^*\mathscr{F}$ is a constant sheaf on Y. Then it holds [Pin00, Theorem 0.2]:

$$\chi(X, \mathscr{F}) \geq (1 - g_X) \cdot \operatorname{rank} \mathscr{F},$$

where $\operatorname{rank} \mathscr{F}$ denote the $\mathbb{F}_p$-dimension of the fiber of $\mathscr{F}$ at a point of X. The equality takes place when Y is ordinary. We introduce the following notation.

Definition 3.5.3. A vector bundle on a smooth projective curve X is said to be ordinarily étale trivial if there exists a Galois étale covering map $f : Y \longrightarrow X$ that trivializes $\mathscr{F}$, where Y is an ordinary curve.

Corollary 3.5.4. *Let X be an ordinary curve of genus larger than* 1. *Let $\mathscr{E}$ be an ordinarily étale trivial vector bundle. Then* $\mathrm{h}^1_{\text{str}}(X, \mathscr{E}) \neq 0$.

Proof. Étale finite vector bundles are semi-stable of degree 0, hence possess composition series such that the successive quotients are stable of degree zero. Using the long exact sequence of cohomology and induction on the length of the above composition series, the problem is essentially reduced to considering only étale finite *stable* vector bundles. According to [BD07], étale trivial stable vector bundles are F-periodic. Thus, we can assume that $\mathscr{E}$ is F-periodic with F-period, say, n.

By Theorem 3.5.2, it is enough to show that $\mathrm{h}^1_{\text{ét}}(X, \mathscr{H}_{\mathscr{E}}) \neq 0$. By [Pin00, Theorem 0.2], we have that:

$$\chi(H_{\mathscr{E}}) = (1 - \mathrm{h}^1_{\text{str}}(X, \mathscr{O}_X)) \cdot \operatorname{rank} \mathscr{H}_{\mathscr{E}} \neq 0.$$

This proves our claim. □

We shall give in the Appendix A.6 an example of an F-periodic vector bundle whose first stratified cohomology vanishes.

4. Comparison for cohomologies of ind-stratified bundles

Unlike group scheme cohomology and de Rham cohomology in characteristic 0, stratified cohomology *does not commute* with direct limits. Therefore, the comparison morphisms in Theorem 3.1.1 on ind-stratified bundles may not be bijective. More explicitly, the group scheme cohomology commutes with direct limits (cf. [Jan87, Lemma 4.17]):

$$\mathrm{H}^i(\pi(X), \varinjlim_i V_i) \cong \varinjlim_i \mathrm{H}^i(\pi(X), V_i).$$

But the similar property for stratified cohomology is not obvious, as its definition involves inverse limits. Thus, we see that the two cohomology theories are different in nature.

4.1. **The case of elliptic curves.** We first show that for elliptic curves, the comparison map is nevertheless bijective. Some other cases are treated in the next section. The absolute Frobenius of X acts by a p-linear endomorphism of the one-dimensional space $\mathrm{H}^1(X, \mathscr{O})$. According to the work of dos Santos, [dSa07], the structure of the category $\mathrm{str}^{\mathrm{coh}}(X/k)$ depends on the rank of this map.

Proposition 4.1.1. *Let X/k be an elliptic curve. Then the maps δ^i defined as in 3.1.1 for ind-stratified bundles are isomorphisms.*

Proof. If X is supersingular, that is, if the action of the Frobenius of X is zero, then $\pi(X)$ is diagonal. Consequently, an ind-stratified bundle on X is a direct sum of rank one bundles. Therefore, the stratified cohomology $\mathrm{H}^i(X/k, -)$ vanishes on any ind-stratified bundles, for all $i \geq 1$.

If X is ordinary, that is, the absolute Frobenius acts by a bijective map, then according to [dSa07], any object of $\mathrm{str}^{\mathrm{coh}}(X/k)$ decomposes into a direct sum of objects of the form

$$\mathscr{L} \otimes \mathscr{N}$$

where $\mathscr{L}$ is a rank one stratified bundle and $\mathscr{N}$ is a nilpotent object, i.e., all of its irreducible subquotients are trivial objects. Such an object is also called a $\mathscr{L}$-isotypical object. Passing to direct limits, we conclude that any ind-stratified bundle decomposes into the direct sum of $\mathscr{L}$-isotypical ind-objects for some rank one stratified bundles $\mathscr{L}$. Thus, it leads to computing the stratified cohomology of $\mathscr{L}$-isotypical ind-objects.

First, let $\mathscr{J}$ be a nilpotent ind-stratified bundle (i.e., an $(\mathscr{O}, d)$-isotypical object). We shall show that all transfer maps in the inverse system

$$\mathrm{H}^i(X, \mathscr{J}^{(n)})$$

for any ind-stratified bundle $\mathscr{J}$ are bijective. Indeed, by assumption, the maps

$$f_i^n : \mathrm{H}^i(X, \mathscr{O}^{(n+1)}) \longrightarrow \mathrm{H}^i(X, \mathscr{O}^{(n)}), \quad i = 0, 1,$$

are bijective. Hence, using the long exact sequence of cohomology, we conclude that for any nilpotent object $\mathscr{E}$ the maps

$$f_i^n : \mathrm{H}^i(X, \mathscr{E}^{(n+1)}) \longrightarrow \mathrm{H}^1(X, \mathscr{E}^{(n)})$$

for $i = 0, 1$ are also bijective. Consequently, we have, for $i = 0, 1$,

$$\varprojlim \mathrm{H}^i(X, \mathscr{E}^{(n)}) \cong \mathrm{H}^i(X, \mathscr{E}^{(0)}),$$

and

$$\varprojlim{}^1 \mathrm{H}^i(X, \mathscr{E}^{(n)}) = 0.$$

Therefore, by Ogus' theorem (cf. 2.2.1) we conclude that

$$\mathrm{H}^1_{\mathrm{str}}(X/k, \mathscr{J}) \cong \mathrm{H}^1(X, \mathscr{J}) = \varinjlim_{\alpha} \mathrm{H}^1(X, \mathscr{J}_\alpha) = \varinjlim_{\alpha} \mathrm{H}^1_{\mathrm{str}}(X/k, \mathscr{J}_k),$$

where $(\mathscr{J}_\alpha)$ is the ind-system of stratified subbundles of $\mathscr{J}$, and

$$\mathrm{H}^2_{\mathrm{str}}(X/k, \mathscr{J}) = 0.$$

Consequently the maps δ^i, $i = 0, 1, 2$, are isomorphisms.

For $\mathscr{J}$ and $\mathscr{L}$-isotypical object, since

$$\mathrm{H}^0(X, \mathscr{L}) = \mathrm{H}^1(X, \mathscr{L}) = 0,$$

all the limits mentioned above vanish. Hence, on these objects, all the higher stratified cohomology groups vanish. □

4.2. **The order of Cartier nilpotency of a Frobenius invariant vector bundle.** Next we show that the comparison map δ^i for curves of genus at least 2 may be non-bijective. For a counterexample, we construct an infinitely iterated extension of the trivial vector bundle which is Frobenius invariant and investigate its stratified cohomology. For this, we introduced the concept of the order of Cartier nilpotency of a Frobenius invariant vector bundle, which seems interesting on its own.

Our approach is based on the Cartier map. Let Ω_X denote the sheaf of differential forms on X. Let $\mathscr{B}_X$ denote the image of $F_{X*}\mathscr{O}_X$ in $F_{X*}\Omega_X$ under the differential $F_{X*}d$. This is called the sheaf of locally exact forms. Then we have exact sequences

$$0 \longrightarrow \mathscr{O}_X \longrightarrow F_{X*}\mathscr{O}_X \longrightarrow \mathscr{B}_X \longrightarrow 0$$

and

$$0 \longrightarrow \mathscr{B}_X \longrightarrow F_{X*}\Omega_X \xrightarrow{C} \Omega_X \longrightarrow 0,$$

where C denotes the Cartier map [Car57]. Roughly speaking, the differential forms that are not locally exact have the form $u^{p-1}du$ where u is not a p-power (in other words, K is a separable extension of $k(u)$). Such a form is mapped to du by the Cartier map C, other forms are mapped to zero by C. Serre's duality yields a perfect pairing given by the residue map (cf. [Ser58]):

$$H^1(X, \mathscr{O}) \times \Omega(X) \longrightarrow k, \quad (e, \mu) \longmapsto \langle s, \mu \rangle = \mathrm{Res}(e_{ij}\mu) := \sum_{P \in |X|} \mathrm{res}_P(f_i \mu).$$

Then C is conjugate to F_X^* with respect to this pairing in the sense that $\langle F_X^* e, \mu \rangle = \langle e, C\mu \rangle^p$. In fact, we have, for any closed point P, $\mathrm{res}_P(\mu) = \mathrm{res}_P(C\mu)^p$.

Consider the restriction of C on the space $\Omega(X)$ of global 1-forms. C is p^{-1}-linear. Hence $\Omega(X)$ decomposes into the direct sum of a "C-bijective" subspace and a "C-nilpotent" subspace. This decomposition is compatible with the decomposition of $\mathrm{H}^1(X, \mathscr{O})$ given above.

Let $\mathscr{E}$ be a vector bundle. Tensoring the second sequence with $\mathscr{E}$ and taking the cohomology we get

$$\mathrm{H}^0(X, \mathscr{E} \otimes F_{X*}\Omega_X) \longrightarrow \mathrm{H}^0(X, \mathscr{E} \otimes \Omega_X).$$

By the projection formula
$$\mathscr{E}\otimes F_{X*}\Omega_X \cong F_{X*}(F^*\mathscr{E}\otimes\Omega_X)$$
and, as F_X is affine,
$$\mathrm{H}^0(X, F_{X*}\mathscr{H}) \cong \mathrm{H}^0(X,\mathscr{H})$$
for any quasi-coherent sheaf on X ([Har77, Ex. 8.1]), hence we obtain a map
$$C_{\mathscr{E}}:\mathrm{H}^0(X,F^*\mathscr{E}\otimes\Omega_X)\longrightarrow \mathrm{H}^0(X,\mathscr{E}\otimes\Omega_X),$$
called *the Cartier map* for $\mathscr{E}$. This map is conjugate to the Frobenius map on $\mathrm{H}^1(X,\mathscr{E}^\vee)\longrightarrow \mathrm{H}^1(X,\mathscr{F}^*E^\vee)$ by Serre's duality.

Let now $\mathscr{E}$ be Frobenius invariant. Fix an isomorphism $\sigma:\mathscr{E}\longrightarrow F^*\mathscr{E}$. We obtain a p^{-1}-linear map

$$C_\sigma: H^0(\Omega\otimes\mathscr{E})\xrightarrow{\sigma} H^0(\Omega\otimes F^*\mathscr{E})\xrightarrow{C} H^0(\Omega\otimes\mathscr{E}). \tag{6}$$

Definition 4.2.1. The order of Cartier nilpotency of $\mathscr{E}$ is the smallest integer n such that $C_\sigma^n=0$ on the nilpotent part $\mathrm{H}^0(\Omega\otimes\mathscr{E})$.

The following result will provide a more comfortable way to compute the order of Cartier nilpotency.

Proposition 4.2.2. *Let $C^{(n)}$ denote the composition of n Cartier morphisms:*
$$C_i := C_{F^{*i}\mathscr{E}}: H^0(\Omega\otimes F^{*i}\mathscr{E})\to H^0(\Omega\otimes F^{*(i-1)}\mathscr{E}),$$
$$C^{n,0}:=C_1\circ C_2\circ\ldots\circ C_n: H^0(\Omega\otimes F^{*n}\mathscr{E})\to H^0(\Omega\otimes\mathscr{E}).$$
The order of Cartier nilpotency of $\mathscr{E}$ is equal to the smallest integer n satisfying that $C^{n,0}$ vanishes on the nilpotent part of $H^0(\Omega\otimes F^{*n}\mathscr{E}_2)$.

Proof. The construction of the Cartier map $C_{\mathscr{E}}$ is functorial. In particular, the functoriality with respect to the map $F^{*(i-1)}\sigma: F^{*(i-1)}\mathscr{E}\longrightarrow F^{*i}\mathscr{E}$ amounts to the following commutative diagram:

$$\begin{array}{ccc} H^0(\Omega\otimes F^{*i}\mathscr{E}) & \xrightarrow{C} & H^0(\Omega\otimes F^{*(i-1)}\mathscr{E}) \\ {\scriptstyle F^{*i}\sigma}\downarrow & & \downarrow{\scriptstyle F^{*(i-1)}\sigma} \\ H^0(\Omega\otimes F^{*(i+1)}\mathscr{E}) & \xrightarrow{C} & H^0(\Omega\otimes F^{*i}\mathscr{E}), \end{array} \tag{7}$$

for any i. Hence, one can check easily by induction that
$$(C\circ\sigma)^n = C_1\circ C_2\circ\ldots\circ C_n\circ F^{*n}\sigma\circ\ldots\circ\sigma.$$
The claim follows. □

4.3. **The construction of the example.** Our aim in this section is to exhibit an ind-stratified sheaf at which the comparison map δ^1 is not bijective. To do this, we shall construct a sequence of Frobenius invariant vector bundles $\mathscr{E}_n$, $n=1,2,\ldots$, with $\mathscr{E}_1=\mathscr{O}$ and $\mathscr{E}_{n+1}$ is an extension of $\mathscr{O}$ by $\mathscr{E}_n$, such that $\mathrm{h}^1(X,\mathscr{E}_n)$ tends to infinity. Being Frobenius invariant, each $\mathscr{E}_n$ is naturally equipped with a stratification. The desired ind-stratified sheaf is the inductive limit of $\mathscr{E}_n$.

The extensions of $\mathscr{O}$ by $\mathscr{E}$:
$$0\longrightarrow\mathscr{E}\xrightarrow{p}\mathscr{F}\xrightarrow{q}\mathscr{O}\longrightarrow 0$$

is classified by the first cohomology group $H^1(X,\mathscr{E})$ as follows. First we recall that two extensions $\mathscr{F},\mathscr{F}'$ are same if there is an isomorphism $\mathscr{F}\longrightarrow\mathscr{F}'$ which becomes the identity when restricted to $\mathscr{E}$ and to $\mathscr{O}$.

An extension of $\mathscr{O}$ by itself is said to be Frobenius invariant if it is isomorphic to its pullback by the absolute Frobenius. It corresponds to a Frobenius invariant element in $\mathrm{H}^1(X,\mathscr{O}_X)$. Since the action of F_X on $\mathrm{H}^1(X,\mathscr{O}_X)$ is p-linear, this space decomposes as a direct sum of F_X-stable parts:

$$\mathrm{H}^1(X,\mathscr{O}_X)=\mathrm{H}^1(X,\mathscr{O}_X)_{\mathrm{ss}}\oplus\mathrm{H}^1(X,\mathscr{O}_X)_{\mathrm{nil}}.$$

On the *semi-simple* part $\mathrm{H}^1(X,\mathscr{O}_X)_{\mathrm{ss}}$ F_X acts bijectively and the *nilpotent* part $\mathrm{H}^1(X,\mathscr{O}_X)_{\mathrm{nil}}$ F_X acts nilpotently. The dimension of the semi-simple subspace $\mathrm{H}^1(X,\mathscr{O}_X)_{\mathrm{ss}}$ is called the *Hasse-Witt rank* of the curve X. If this space is non-zero, it has a basis consisting of Frobenius invariant elements. See, e.g., [Mu70, §14] or [SGA7, Exp. XXII].

Assume now that X is a hyperelliptic curve of genus 2 and Hasse-Witt rank 1. Then, there exists a cohomology class e_2 invariant under the Frobenius map: $F^*e_2=e_2$, which is unique up to an element in $\mathbb{F}_p^\times$. It determines an extension $\mathscr{E}_2$ of $\mathscr{O}$ by itself with the property $F^*\mathscr{E}_2\cong\mathscr{E}_2$. The long exact sequence associated with the extension $\mathscr{O}\longrightarrow\mathscr{E}_2\longrightarrow\mathscr{O}$, which is F_X-equivariant. This sequence then splits into the direct sum of an exact sequence of semi-simple subspaces and an exact sequence of nilpotent subspaces:

$$0\longrightarrow k\longrightarrow\mathrm{H}^1(X,\mathscr{O})_{\mathrm{ss}}\longrightarrow\mathrm{H}^1(X,\mathscr{E}_2)_{\mathrm{ss}}\longrightarrow\mathrm{H}^1(X,\mathscr{O})_{\mathrm{ss}}\longrightarrow 0.$$

We conclude that the semi-simple part of $\mathrm{H}^1(X,\mathscr{E}_2)$ is also one-dimensional and contains a Frobenius invariant element e_3 that projects to e_2 through the rightmost map. This element e_3 yields an extension $\mathscr{E}_3$ of $\mathscr{O}$ by $\mathscr{E}_2$ which is Frobenius invariant:

$$\mathscr{E}_2\longrightarrow\mathscr{E}_3\longrightarrow\mathscr{O}.$$

Iterating this, we obtain an increasing chain of Frobenius invariant vector bundles

$$\mathscr{O}=\mathscr{E}_1\subset\mathscr{E}_2\subset\ldots\subset\mathscr{E}_n\subset\ldots,$$

the successive quotients are isomorphic to $\mathscr{O}$. Further the spaces $\mathrm{H}^1(X,\mathscr{E}_n)_{\mathrm{ss}}$ are all one-dimensional and the maps

$$\mathrm{H}^1(X,\mathscr{E}_n)_{\mathrm{ss}}\longrightarrow\mathrm{H}^1(X,\mathscr{E}_{n+1})_{\mathrm{ss}}$$

are all zero maps.

We first need the following properties of the sequence $\{\mathscr{E}_n\}_{n\in\mathbb{N}^{\geq 1}}$:

Proposition 4.3.1. *For any positive integers m and n, we have a natural exact sequence of vector bundles over X:*

$$0\longrightarrow\mathscr{E}_n\longrightarrow\mathscr{E}_{m+n}\longrightarrow\mathscr{E}_m\longrightarrow 0. \tag{8}$$

Proof. We will prove by induction on m. If $m=1$, this follows from the definition of $\mathscr{E}_n$. Assume that we have the sequence 8 for $m=t$:

$$0\longrightarrow\mathscr{E}_n\longrightarrow\mathscr{E}_{t+n}\longrightarrow\mathscr{E}_t\longrightarrow 0, \tag{9}$$

we will show the existence of a sequence for $m=t+1$. From 9, we have a long exact sequence of cohomology groups:

$$0\longrightarrow H^0(X,\mathscr{E}_n)_{\mathrm{ss}}=k\longrightarrow H^1(X,\mathscr{E}_{t+n})_{\mathrm{ss}}=k\longrightarrow H^1(X,\mathscr{E}_t)_{\mathrm{ss}}=k\longrightarrow 0.$$

The isomorphism $H^1(X,\mathscr{E}_{t+n})_{\rm ss} = k \longrightarrow H^1(X,\mathscr{E}_t)_{\rm ss} = k$ means that we have the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{E}_n & \xrightarrow{i_n} & \mathscr{E}_{t+n} & \xrightarrow{\mathrm{pr}_1} & \mathscr{E}_t & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle i_{t+n}} & & \downarrow{\scriptstyle i_t} & & \\
0 & \longrightarrow & \mathscr{E}' & \xrightarrow{i'} & \mathscr{E}_{t+1+n} & \xrightarrow{\mathrm{pr}_2} & \mathscr{E}_{t+1} & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle \mathrm{pr}_4} & & \downarrow{\scriptstyle \mathrm{pr}_3} & & \\
 & & & & \mathscr{O} & \xrightarrow{=} & \mathscr{O} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

where the top exact sequence comes from the inductive hypothesis, $\mathscr{E}'$ is the kernel of the map $\mathscr{E}_{t+1+n} \to \mathscr{E}_{t+1}$, and $\phi : \mathscr{E}_n \to \mathscr{E}'$ is the induced map. We will finish the proof by showing that ϕ is an isomorphism.

ϕ is injective. Let a be any element in the kernel of ϕ. Then $i_n(a) \in \mathrm{Ker}(i_{t+n}) = 0$. Since i_n is injective, we have that $a = 0$.

ϕ is surjective. For any $a \in \mathscr{E}'$, then $\mathrm{pr}_3(\mathrm{pr}_2(i'(a))) = 0 = \mathrm{pr}_4(i'(a))$. This implies that $i'(a) \in \mathrm{Im}(i_{t+n})$, so let say $i'(a) = i_{t+n}(b)$ for some $b \in \mathscr{E}_{t+n}$. Since $\mathrm{pr}_2(i_{t+n}(b)) = 0$, and i_t is injective, we have that $\mathrm{pr}_1(b) = 0$. So $b \in \mathrm{Im}(i_n)$. This finishes our proof. □

Lemma 4.3.2. i) *For any $m, n \in \mathbb{N}^{>0}$, we have the following exact sequence of nilpotent forms:*

$$0 \to H^0(\Omega \otimes \mathscr{E}_n)_{\rm nil} \xrightarrow{i_{n,m+n}} H^0(\Omega \otimes \mathscr{E}_{n+m})_{\rm nil} \xrightarrow{\mathrm{pr}_{n+m,m}} H^0(\Omega \otimes \mathscr{E}_m)_{\rm nil} \to 0. \tag{10}$$

In particular, there exists an element in $H^0(\Omega \otimes \mathscr{E}_n)_{\rm nil}$ that (under the projection $\mathrm{pr}_{n,1}$) maps to $\omega \in H^0(\Omega)$.

ii) *For any $s \in \mathrm{pr}_{n,1}^{-1}(\omega)$, we have that $C(s) = a.i_{n-2,n}(s')$ for some $s' \in \mathrm{pr}_{n-2,1}^{-1}(\omega) \subset H^0(\Omega \otimes \mathscr{E}_{n-2})$ and $a \in k^*$ (here we assume that $n > 2$).*

Proof. The statement i) is an easy corollary of the exact sequence given in Lemma 4.3.1.
To show ii), let consider the following commutative diagrams:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^0(\Omega \otimes \mathscr{E}_{n-3}) & \xrightarrow{i_{n-3,n}} & H^0(\Omega \otimes \mathscr{E}_n) & \xrightarrow{\mathrm{pr}_{n,3}} & H^0(\Omega \otimes \mathscr{E}_3) & \longrightarrow & 0\ , \\
 & & \downarrow{\scriptstyle C} & & \downarrow{\scriptstyle C} & & \downarrow{\scriptstyle C} & & \\
0 & \longrightarrow & H^0(\Omega \otimes \mathscr{E}_{n-3}) & \xrightarrow{i_{n-3,n}} & H^0(\Omega \otimes \mathscr{E}_n) & \xrightarrow{\mathrm{pr}_{n,3}} & H^0(\Omega \otimes \mathscr{E}_3) & \longrightarrow & 0
\end{array}$$

and

$$\begin{array}{ccc}
H^0(\Omega \otimes \mathscr{E}_n) & \xrightarrow{\mathrm{pr}_{n,3}} & H^0(\Omega \otimes \mathscr{E}_3)\ . \\
{\scriptstyle \mathrm{pr}_{n,1}} \searrow & & \swarrow {\scriptstyle \mathrm{pr}_{3,1}} \\
 & H^0(\Omega) &
\end{array}$$

So for any $s \in pr_{n,1}^{-1}(\omega)$, set $s_3 = \mathrm{pr}_{n,3}(s)$ then we have

$$\mathrm{pr}_{n,3}(C(s)) = C(s_3)$$
$$\mathrm{pr}_{31}(s_3) = \omega.$$

The computations for $H^0(\Omega \otimes \mathscr{E}_3)$ (see the paragraph before Example 4.4.7) tell us that $\mathrm{pr}_{3,2}(C(s_3)) = 0$, and $C(s_3) = a.i_{1,3}(\omega)$ for some $a \in k^*$. We then imply from the commutative diagram

$$\begin{array}{ccc} H^0(\Omega \otimes \mathscr{E}_n) & \xrightarrow{\mathrm{pr}_{n,3}} & H^0(\Omega \otimes \mathscr{E}_3) \\ & {\scriptstyle \mathrm{pr}_{n,2}} \searrow \quad \swarrow {\scriptstyle \mathrm{pr}_{3,2}} & \\ & H^0(\Omega \otimes \mathscr{E}_2) & \end{array}$$

that $\mathrm{pr}_{n,2}(C(s)) = 0$, and $C(s) = a.i_{n-2,n}(s')$ for some $s' \in \mathrm{pr}_{n-2,1}^{-1}(\omega) \subset H^0(\Omega \otimes \mathscr{E}_{n-2})$ and $a \in k^*$. □

As a corollary, we have the following result which implies that the order of Cartier nilpotency of $\mathscr{E}_n$ goes to infinity when n goes to infinity.

Proposition 4.3.3. *Assume that the order of Cartier nilpotency of $\mathscr{E}_3$ is at least 1. Then for any integer $n \geq 1$, the order of Cartier nilpotency of $\mathscr{E}_n$ is at least $\lfloor \frac{n+1}{2} \rfloor$. Further, this is the case for a hyperelliptic curve of genus 2 and Hasse-Witt rank 1 and the ground field k has characteristic 3.*

Proof. We prove by induction on n. The case $n = 1$ is the assumption. Assume that the statement holds for $n = t$, we will show that the statement also holds for $n = t + 1$. We will argue the case of $\mathscr{E}_{2t+2}$ and the case of $\mathscr{E}_{2t+3}$ is similar.

By Lemma 4.3.2 i) we have an exact sequence:

$$0 \to H^0(\Omega \otimes \mathscr{E}_{2t+1})_{\mathrm{nil}} \to H^0(\Omega \otimes \mathscr{E}_{2t+2})_{\mathrm{nil}} \xrightarrow{\mathrm{pr}_{2t+2,1}} H^0(\Omega)_{\mathrm{nil}} \to 0.$$

This implies that the order of Cartier nilpotency of $\mathscr{E}_{2t+2}$ is equal to the order of Cartier nilpotency of some element in $H^0(\Omega \otimes \mathscr{E}_{2t+2})_{\mathrm{nil}}$ whose image under the projection $\mathrm{pr}_{2t+2,1}$ is ω. Now the inductive hypothesis and Lemma 4.3.2 will finish the proof.

The last claim is verified in 4.4.6. □

Proposition 4.3.4. *Let X be a hyperelliptic curve of genus 2 and Hasse-Witt rank 1. Then for the Frobenius invariant vector bundles $\mathscr{E}_n$ with inductive limit $\mathscr{E}$ constructed above. Then the comparison map*

$$\delta^1_{\mathscr{E}} : \mathrm{H}^1_{\mathrm{str}}(X, \mathscr{E}) = \varinjlim_n \mathrm{H}^1_{\mathrm{str}}(X, \mathscr{E}_n) \longrightarrow \mathrm{H}^1_{\mathrm{str}}(X, \mathscr{E})$$

is not bijective. In fact, the target space is infinite dimensional while the source space is zero.

Proof. The spaces $\mathrm{H}^1(X, \mathscr{E}_n)_{\mathrm{ss}}$ are all one-dimensional and the maps $\mathrm{H}^1(X, \mathscr{E}_n)_{\mathrm{ss}} \longrightarrow \mathrm{H}^1(X, \mathscr{E}_{n+1})_{\mathrm{ss}}$ are all zero maps. Consequently we have $\mathrm{H}^1_{\mathrm{str}}(X, \mathscr{E}) = \varinjlim_n \mathrm{H}^1(X, \mathscr{E}_n) = 0$.

On the other hand, according to the exact sequence in 2.2.2, there is a surjective map

$$\mathrm{H}^1_{\mathrm{str}}(X, \mathscr{E}) \twoheadrightarrow \varprojlim_n \mathrm{H}^1_{\mathrm{str}}(X, \mathscr{E}^{(n)}).$$

By construction, the ind-vector bundles $\mathscr{E}^{(n)}, n \geq 0$ are isomorphic. Therefore, the vector spaces in the inverse system on the right-hand side are canonically isomorphic, and the transfer maps are given by the Frobenius. Using Proposition 4.3.3 and Lemma 4.4.6, the transfer maps are locally nilpotent, but the order of nilpotency is not bounded. As a consequence, the limit of that system is infinite dimensional. □

4.4. **Explicit computation for hyperelliptic curves of genus 2 over a field of characteristic 3.** Let X be a hyperelliptic curve of genus 2 with two marked Weierstrass points, and $\mathrm{char}(k) = 3$. By fixing two points, such curve can be defined by the following affine covering: $X = U_0 \cup U_1$, where

$$U_0 = X \setminus \{\infty\} : y^2 = a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x;$$
$$U_1 = X \setminus \{0\} : w^2 = a_1v^5 + a_2v^4 + a_3v^3 + a_4v^2 + a_5v,$$

and on the intersection $U_{01} := U_0 \cap U_1$, we have the following relations: $y = wv^{-3}$ and $x = v^{-1}$. We first provide the necessary and sufficient conditions for X to have Hasse-Witt rank 1.

Lemma 4.4.1. *Let X be the curve obtained by patching the two affine curves*

$$U_0 = X \setminus \{\infty\} : y^2 = a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x;$$
$$U_1 = X \setminus \{0\} : w^2 = a_1v^5 + a_2v^4 + a_3v^3 + a_4v^2 + a_5v,$$

subject to the relations: $y = wv^{-3}$ and $x = v^{-1}$. Then X is smooth (hence has genus 2 and is hyper-elliptic) and has Hasse-Witt rank one if and only if

$$\begin{cases} a_4a_2 = a_5a_1 \\ a_1a_5(a_2^4 + a_1^3a_5) \neq 0 \\ a_3(a_2a_5^2 + a_4^3 - a_3a_4a_5) \neq 0. \end{cases}$$

The proof will be given in the Appendix A.1. □

Lemma 4.4.2 (The vector bundle $\mathscr{E}_2$)**.** *The 1-cocycle $e_{01} \in \mathscr{O}(U_{01})$*

$$e_{01} = (a + bx)x^{-2}y = a.\frac{w}{v} + b.\frac{y}{x}, \tag{11}$$

is invariant under the Frobenius map if $a, b \in k$ satisfy:

$$\begin{bmatrix} a_4 & a_1 \\ a_5 & a_2 \end{bmatrix} \cdot \begin{bmatrix} a^3 \\ b^3 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}. \tag{12}$$

The proof will be given in the Appendix A.2. □

Lemma 4.4.3. *The Cartier map $C : \mathrm{H}^0(\Omega \otimes F^*\mathscr{E}_2) \to \mathrm{H}^0(\Omega \otimes \mathscr{E}_2)$ has nilpotency* 1.

The proof will be given in the Appendix A.3. □

Example 4.4.4. *If $a_i = -1$ for all $1 \leq i \leq 5$, then we can choose*

$$\omega = \frac{(x-1)dx}{y}; \quad \eta = \frac{(x+1)dx}{y}$$
$$e_{01} = (1+x)\frac{y}{x^2} = \frac{w}{v} + \frac{w}{v^2}.$$

Lemma 4.4.5 (The vector bundle $\mathscr{E}_3$)**.** *The bundle $\mathscr{E}_3$ is given by a $GL_3(\mathscr{O}(U_{01}))$-cocycle of the form:*

$$A_3 = \begin{bmatrix} 1 & e_{01} & f_{01} \\ 0 & 1 & e_{01} \\ 0 & 0 & 1 \end{bmatrix}.$$

with $f_{01} = -e_{01}^2$.

The proof will be given in the Appendix A.4. □

Now, each element in $H^0(\Omega \otimes \mathscr{E}_3)$: is a pair of vectors of local sections subject to the relation

$$\begin{bmatrix} {}^3s_0 \\ {}^2s_0 \\ {}^1s_0 \end{bmatrix} = \begin{bmatrix} 1 & e_{01} & 0 \\ 0 & 1 & e_{01} \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} {}^3s_1 \\ {}^2s_1 \\ {}^1s_1 \end{bmatrix}$$

for $i = 0,1,2$, where ${}^js_i \in \Omega(U_i)$, for any j, that is

$$\begin{cases} {}^3s_0 - {}^3s_1 = e_{01}.{}^2s_1 \\ {}^2s_0 - {}^2s_1 = e_{01}.{}^1s_1 \\ {}^1s_0 - {}^1s_1 = 0. \end{cases}$$

Since ${}^1s_0 - {}^1s_1 = 0$, these local sections (of the sheaf Ω_X) glue together to obtain a global section s in $\Omega(X)$.

Our finding here is the following

Lemma 4.4.6. *The Cartier map $C : H^0(\Omega \otimes F^*\mathscr{E}_3) \longrightarrow H^0(\Omega \otimes \mathscr{E}_3)$ has nilpotency at least* 2.

The proof will be given in the Appendix A.5. □

From the above calculations and Lemma 4.4.3, we see that the image under the Cartier map of any nilpotent element of $H^0(\Omega \otimes F^*\mathscr{E}_3)$ of the form

$$\begin{bmatrix} {}^3s_i \\ {}^2s_i \\ \omega \end{bmatrix} \quad \text{is} \quad \lambda \cdot \begin{bmatrix} \omega \\ 0 \\ 0 \end{bmatrix}, \text{ where } \lambda \in k^\times.$$

Example 4.4.7. (i) *If $a_i = a_j$ for all $i \neq j$ then $C({}^3s_1) \in \Omega(X)$ is always non-zero.*

(ii) *If $a_5 = a_2 = -1$ and $a_4 = a_3 = a_1 = 1$, then $y^2 = -x^5 + x^4 + x^3 - x^2 + x$ defines a hyperelliptic curve, and by taking $\omega_1 = 0$, we have $C({}^3s_1) = 0$.*

APPENDIX A.

A.1. **Proof of Lemma 4.4.1.** X is smooth iff the polynomials $f(x)$ of degree 5 has no double roots. This is the same as saying that $a_1a_5 \neq 0$ and the discriminant of the quartic polynomial $a_5x^4 + a_4x^3 + a_3x^2 + a_2x + a_1$,

$$\Delta := a_5^3a_1^3 + a_5^2a_3^2a_1^2 + a_5a_4a_3^2a_2a_1 + a_5a_3^4a_1 - a_5a_3^3a_2^2 - a_4^3a_2^3 - a_4^2a_3^3a_1 + a_4^2a_3^2a_2^2 \neq 0.$$

The space of global differential 1-forms $\Omega(X)$ is spanned by the holomorphic differential

$$\omega_1 = y^{-1}dx \quad \text{and} \quad \omega_2 = y^{-1}xdx.$$

The Cartier operator in this basis is given by:

$$C = \begin{bmatrix} a_2^{1/3} & a_1^{1/3} \\ a_5^{1/3} & a_4^{1/3} \end{bmatrix}.$$

For X to have Hasse-Witt rank one, it is necessary that $\det(C) = 0$, that is

$$a_1 a_5 = a_2 a_4.$$

Under this equality, the condition $\Delta \neq 0$ can be expressed as

$$a_3(a_2 a_5^2 + a_4^3 - a_3 a_4 a_5) \neq 0.$$

Further, there is a basis $\{\omega, \eta\}$ of the k-vector space $\Omega(X)$ satisfying that $C(\omega) = 0$, and $C(\eta) = \eta$. We can choose

$$\omega = a_4\omega_1 - a_5\omega_2.$$

If we present η in the form $\eta = \lambda_1^3\omega_1 + \lambda_2^3\omega_2$ then

$$C(\eta) = C(\lambda_1^3\omega_1 + \lambda_2^3\omega_2) = \lambda_1 C(\omega_1) + \lambda_2 C(\omega_2).$$

So λ_1, λ_2 need to satisfy:

$$\begin{bmatrix} a_2^{1/3} & a_1^{1/3} \\ a_5^{1/3} & a_4^{1/3} \end{bmatrix} \cdot \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} \lambda_1^3 \\ \lambda_2^3 \end{bmatrix}$$

Elementary computation shows that the existence of $\eta \neq 0$ is equivalent to

$$a_2^4 + a_1^3 a_5 \neq 0.$$

A.2. **Proof of Lemma 4.4.2.** One needs to check $e_{01}^3 - e_{01} = 0$ in $H^1(X, \mathcal{O}_X)$. Indeed, we have

$$\begin{aligned}
e_{01}^3 &= a^3\frac{w^3}{v^3} + b^3\frac{y^3}{x^3} \\
&= a^3\frac{w(a_1v^5 + a_2v^4 + a_3v^3 + a_4v^2 + a_5v^1)}{v^3} + b^3\frac{y(a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x^1)}{x^3} \\
&= a^3w(a_1v^2 + a_2v + a_3) + b^3y(a_5x^2 + a_4x + a_3) + \left(a^3a_4\frac{w}{v} + a^3a_5\frac{w}{v^2} + b^3a_2\frac{y}{x} + b^3a_1\frac{y}{x^2}\right) \\
&= f_1 + f_0 + \left(a^3a_4\frac{w}{v} + a^3a_5\frac{y}{x} + b^3a_2\frac{y}{x} + b^3a_1\frac{w}{v}\right) \\
&= f_1 + f_0 + \left(\frac{w}{v}(a^3a_4 + b^3a_1) + \frac{y}{x}(a^3a_5 + b^3a_2)\right) \\
&= f_1 + f_0 + \left(\frac{w}{v}.a + \frac{y}{x}.b\right) \quad \text{by (12)} \\
&= f_1 + f_0 + e_{01}
\end{aligned}$$

where $f_0 = b^3y(a_5x^2 + a_4x + a_3) \in \mathcal{O}(U_0)$, and $f_1 = a^3w(a_1v^2 + a_2v + a_3) \in \mathcal{O}(U_1)$.

A.3. **Proof of Lemma 4.4.3.** We now compute the image of $\begin{bmatrix} {}^2 s_i \\ \omega \end{bmatrix} \in H^0(\Omega \otimes F^* \mathscr{E}_2)$ under C, where recall that $\omega = a_4\omega_1 - a_5\omega_2$. According to (11), $e_{01}^3 = \frac{y^3}{x^6}(a^3 + b^3x^3)$. Hence

$$\begin{aligned} e_{01}^3.\omega &= \frac{y^3}{x^6}(a^3+b^3x^3).\omega = \frac{y^2}{x^6}(a^3+b^3x^3).(a_4 - a_5x)dx \\ &= \frac{(b^3x^3+a^3)(a_4-a_5x)(a_5x^4+a_4x^3+a_3x^2+a_2x+a_1)}{x^5}dx \\ &= \big(-b^3a_5^2x^3 + b^3(a_4^2 - a_5a_3)x + b^3(a_4a_3 - a_5a_2) - a^3a_5^2\big)dx \\ &\quad - \big(b^3a_4a_1 + a^3(a_4^2 - a_5a_3) + a^3(a_4a_3 - a_5a_2)v + a^3a_4a_1v^3\big)dv. \end{aligned}$$

Consequently

$$\begin{aligned} {}^2s_0 - {}^2s_1 &= e_{01}^3.\omega \\ &= \big(-b^3a_5^2x^3 + b^3(a_4^2 - a_5a_3)x + b^3(a_4a_3 - a_5a_2) - a^3a_5^2\big)dx \\ &\quad - \big(b^3a_4a_1 + a^3(a_4^2 - a_5a_3) + a^3(a_4a_3 - a_5a_2)v + a^3a_4a_1v^3\big)dv. \end{aligned}$$

So, up to an element of $\Omega(X)$, we have that

$$ {}^2s_0 = \big(-b^3a_5^2x^3 + b^3(a_4^2 - a_5a_3)x + b^3(a_4a_3 - a_5a_2) - a^3a_5^2\big)dx \tag{13}$$

$$ {}^2s_1 = \big(b^3a_4a_1 + a^3(a_4^2 - a_5a_3) + a^3(a_4a_3 - a_5a_2)v + a^3a_4a_1v^3\big)dv. \tag{14}$$

With the above choices, we see that $C({}^2s_0) = C({}^2s_1) = 0$.

A.4. **Proof of Lemma 4.4.5.** We have

$$e_{01}^3 - e_{01} = f_0 + f_1$$

with $f_0 = b^3y(a_5x^2 + a_4x + a_3) \in \mathscr{O}(U_0)$ and $f_1 = a^3w(a_1v^2 + a_2v + a_3) \in \mathscr{O}(U_1)$. Hence

$$\begin{aligned} \begin{bmatrix} 1 & f_0 & -f_0^2 \\ 0 & 1 & f_0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & e_{01} & -e_{01}^2 \\ 0 & 1 & e_{01} \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & f_1 & -f_1^2 \\ 0 & 1 & f_1 \\ 0 & 0 & 1 \end{bmatrix} &= \begin{bmatrix} 1 & e_{01}^3 & -(f_1+f_0)^2 - 2e_{01}(f_0+f_1) - e_{01}^2 \\ 0 & 1 & e_{01}^3 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & e_{01}^3 & -e_{01}^6 \\ 0 & 1 & e_{01}^3 \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

Now

$$e_{01}^2 = \frac{(a+bx)^2y^2}{x^4} = \frac{(a+bx)^2(a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x)}{x^4} = g_0 - g_1,$$

for suitable $g_0 \in \mathscr{O}(U_0), g_1 \in \mathscr{O}(U_1)$. Hence $e_{01}^6 = g_0^3 - g_1^3 = 0$ as an element in $H^1(\mathscr{O}_X)$. So the cocycle A_3 is equivalent to

$$\begin{bmatrix} 1 & e_{01} & 0 \\ 0 & 1 & e_{01} \\ 0 & 0 & 1 \end{bmatrix}. \tag{15}$$

A.5. **Proof of Lemma 4.4.6.** We need to construct a section in $H^0(\Omega\otimes F^*\mathscr{E}_3)$ which is not annihilated by C but is annihilated by a power of C. Thus we choose $s=\omega$ and ${}^2s_i$ as in (13), (14). Then

$$\begin{aligned}
e_{01}^3\,{}^2s_1 &= \frac{w^3}{v^6}(a^3v^3+b^3)\big(b^3a_4a_1+a^3(a_4^2-a_5a_3)+a^3(a_4a_3-a_5a_2)v+a^3a_4a_1v^3\big)dv\\
&= \frac{w^4}{wv^6}(a^3v^3+b^3)\big(b^3a_4a_1+a^3(a_4^2-a_5a_3)+a^3(a_4a_3-a_5a_2)v+a^3a_4a_1v^3\big)dv\\
&= \frac{(a_1v^4+a_2v^3+a_3v^2+a_4v+a_5)^2}{wv^4}(a^3v^3+b^3)\big(c_0+c_1v+c_3v^3\big)dv,
\end{aligned}$$

here we set $c_0=b^3a_4a_1+a^3(a_4^2-a_5a_3)$; $c_1=a^3(a_4a_3-a_5a_2)$; and $c_3=a^3a_4a_1\neq 0$.

$$\begin{aligned}
&= \frac{(a_1v^4+a_2v^3+a_3v^2+a_4v+a_5)^2}{wv^4}(a^3c_3v^6+a^3c_1v^4+(a^3c_0+b^3c_3)v^3+b^3c_1v+b^3c_0)dv\\
&= \frac{(a_1v^4+a_2v^3+a_3v^2+a_4v+a_5)^2}{wv^4}(a^3c_3v^6+a^3c_1v^4)dv\\
&\quad+\frac{(a_1v^4+a_2v^3+a_3v^2+a_4v+a_5)^2}{wv}(a^3c_0+b^3c_3)dv\\
&\quad+\frac{(a_1v^4+a_2v^3+a_3v^2+a_4v+a_5)^2}{wv^4}(b^3c_1v+b^3c_0)dv\\
&= s_v+\frac{a_5^2(a^3c_0+b^3c_3)}{wv}dv+\frac{(b^3c_1v+b^3c_0)(a_4^2v^2+a_5^2-a_2a_5v^3-a_3a_4v^3-a_3a_5v^2-a_4a_5v)}{w.v^4}dv\\
&= s_v+\frac{a_5^2(a^3c_0+b^3c_3)}{wv}dv+\frac{b^3(c_1v+c_0)(-(a_2a_5+a_3a_4)v^3+(a_4^2-a_3a_5)v^2-a_4a_5v+a_5^2)}{w.v^4}dv\\
&= s_v'+\frac{a_5^2(a^3c_0+b^3c_3)+b^3c_1(a_4^2-a_3a_5)-b^3c_0(a_2a_5+a_3a_4)}{wv}dv+\frac{b^3(-c_1a_4a_5+c_0(a_4^2-a_3a_5))}{wv^2}dv\\
&\quad+\frac{b^3(c_1a_5^2-c_0a_4a_5)}{wv^3}dv+\frac{b^3c_0a_5^2}{wv^4}dv\\
&= s_v'-\frac{a_5^2(a^3c_0+b^3c_3)+b^3c_1(a_4^2-a_3a_5)-b^3c_0(a_2a_5+a_3a_4)}{y}x^2dx-\frac{b^3(-c_1a_4a_5+c_0(a_4^2-a_3a_5))}{y}x^3dx\\
&\quad-\frac{b^3(c_1a_5^2-c_0a_4a_5)}{y}x^4dx-\frac{b^3c_0a_5^2}{y}x^5dx,
\end{aligned}$$

where s_v and s_v' are in $\Omega(U_1)$. Thus,

$$\begin{aligned}
{}^3s_1 = \Big(&\frac{a_5^2(a^3c_0+b^3c_3)+b^3c_1(a_4^2-a_3a_5)-b^3c_0(a_2a_5+a_3a_4)}{y}x^2+\frac{b^3(-c_1a_4a_5+c_0(a_4^2-a_3a_5))}{y}x^3+\\
&+\frac{b^3(c_1a_5^2-c_0a_4a_5)}{y}x^4+\frac{b^3c_0a_5^2}{y}x^5\Big)dx+\gamma,
\end{aligned}$$

for some $\gamma \in \Omega(X)$. So

$$\begin{aligned}
C(^3s_1) &= \frac{1}{y}C\Big(y^2\big((a_5^2(a^3c_0+b^3c_3)+b^3c_1(a_4^2-a_3a_5)-b^3c_0(a_2a_5+a_3a_4))x^2+b^3(-c_1a_4a_5+\\
&\quad c_0(a_4^2-a_3a_5))x^3+b^3(c_1a_5^2-c_0a_4a_5)x^4+b^3c_0a_5^2x^5\big)dx\Big)+C(\gamma)\\
&= \frac{1}{y}C\Big(\big(a_3x^5\big(a_5^2(a^3c_0+b^3c_3)+b^3c_1(a_4^2-a_3a_5)-b^3c_0(a_2a_5+a_3a_4)\big)+\\
&\quad +(a_5x^8+a_2x^5)b^3(-c_1a_4a_5+c_0(a_4^2-a_3a_5))+(a_4x^8+a_1x^5)b^3(c_1a_5^2-c_0a_4a_5)+\\
&\quad +a_3x^8b^3c_0a_5^2\big)dx\Big)+C(\gamma)\\
&= \frac{1}{y}C\Big(\big(a_3x^5\big(a_5^2(a^3c_0+b^3c_3)+b^3c_1(a_4^2-a_3a_5)-b^3c_0(a_2a_5+a_3a_4)\big)+\\
&\quad +a_2x^5b^3(-c_1a_4a_5+c_0(a_4^2-a_3a_5))+a_1x^5b^3(c_1a_5^2-c_0a_4a_5)\big)dx\Big)+C(\gamma)
\end{aligned}$$

Substitute $c_0 = b^3a_4a_1 + a^3(a_4^2 - a_5a_3)$; $c_1 = a^3(a_4a_3 - a_5a_2)$; $c_3 = a^3a_4a_1 \neq 0$, and using the fact that $\frac{b}{a_5} = \frac{a}{a_4}$, by replacing $a = t.a_4$, and $b = t.a_5$ into the expression above, we obtain that there is a non-zero constant $\lambda \in k^*$ such that

$$C(^3s_1) = \lambda \cdot \frac{x}{y}C\big((-a_4^3a_5^2a_2 + a_4^6 - a_4^4a_5a_3 - a_5^3a_2a_3a_4 + a_5^4a_2^2)x^2dx\big) + C(\gamma) \tag{16}$$

$$= \lambda \cdot \big((-a_4^3a_5^2a_2 + a_4^6 - a_4^4a_5a_3 - a_5^3a_2a_3a_4 + a_5^4a_2^2)\big)^{1/3}\omega_1 + C(\gamma). \tag{17}$$

We have

$$-a_4^3a_5^2a_2 + a_4^6 - a_4^4a_5a_3 - a_5^3a_2a_3a_4 + a_5^4a_2^2 = (a_5^2a_2 + a_4^3)(a_5^2a_2 + a_4^3 - a_3a_4a_5) \neq 0,$$

by Lemma 4.4.1. Indeed, since $a_1a_5 = a_2a_4$, we have $a_2^3(a_5^2a_2 + a_4^3) = a_5^2(a_2^4 + a_1^3a_5)$. To summary, $C(^3s_1) = c.\omega_1 + d.C(\eta)$, for some $c \in k^*$ and $d \in k$, here recall that $\{\omega, \eta\}$ is a basis of the k-vector space $\Omega(X)$ satisfying that $C(\omega) = 0$, and $C(\eta) = \eta$. And we can take $\omega = a_4\omega_1 - a_5\omega_2$, and present $\eta = \lambda_1^3\omega_1 + \lambda_2^3\omega_2$, for some non-zero constants λ_1, λ_2. Thus $C(^3s_1) = C(^3s_0)$ is always non-vanishing.

A.6. **A non-example for the vanishing of the first stratified cohomology group.** An example of a hyperelliptic curve in characteristic 3 which is an F-periodic line bundle whose first stratified cohomology group vanishes.

Let X be the curve given in 4.4. Let $\mathscr{L} := \mathscr{O}(\infty - O)$ be the line bundle corresponding to the Weierstrass divisor $\infty - O$, where $O = [0:0:1]$ and $\infty = [0:1:0]$ are two points of X. Since $2\cdot\infty - 2\cdot O$ is the divisor of x^{-1}, $\mathscr{L}^{\otimes 2}$ is trivial. Hence $\mathscr{L}^3 \cong \mathscr{L}$, that is $\mathscr{L}$ is an F-periodic line bundle. Thus, $\mathscr{L}$ determines a stratified line bundle which will also be denoted by $\mathscr{L}$.

We will investigate the vanishing $H^1_{\mathrm{str}}(X,\mathscr{L})$ by studying the Frobenius action on $H^1(X,\mathscr{L})$, or equivalently the Cartier map on $H^0(X,\Omega\otimes\mathscr{L}^\vee) = H^0(X,\Omega(O-\infty))$.

Since $g(X) = 2$, by Riemann-Roch theorem $h^0(X,\Omega(O-\infty)) = 1$, so there is a unique (up to scalars) differential form which has a pole at O of order at most 1, vanishes at ∞ and is regular elsewhere. Using the notations of Section 4.4, the form is:

$$\omega_1 = \frac{dx}{y},$$

and the Cartier map on $H^0(X,\Omega(O-\infty))$ is the following composition (cf. (6)):

$$C_\sigma : H^0(X,\Omega(O-\infty)) \xrightarrow{\cdot x^{-1}} H^0(X,\Omega(3\cdot(O-\infty))) \xrightarrow{C} H^0(X,\Omega(O-\infty)). \tag{18}$$

The image of ω_1 under this map is

$$\begin{aligned} C_\sigma(\omega_1) = C\left(\frac{y^2x^2dx}{x^3y^3}\right) &= \frac{1}{xy}\cdot C((a_5x^5+a_4x^4+a_3x^3+a_2x^2+a_1x)x^2dx) \\ &= \frac{1}{y}\cdot a_3^{1/3}dx = a_3^{1/3}\cdot\omega_1. \end{aligned}$$

We conclude that the stratified cohomology of $\mathscr{L}$ vanishes if and only if the coefficient a_3 is 0. Notice that, according to the proof of 4.4.1, if $a_3 = 0$ then X has to be ordinary. Indeed, if $a_3 = 0$, $\Delta = a_5^3a_1^3 - a_4^3a_2^3$. Thus, if X is smooth, it is ordinary.

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