An open set of unbounded cocycles with simple Lyapunov spectrum and no exponential separation

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Abstract

We give an example of an open set of unbounded cocycles satisfying the integrability condition of the multiplicative ergodic theorem such that all the cocycles in this open set have simple Lyapunov spectrum but have no exponential separation. Thus, unlike the bounded case, the exponential separation property is nongeneric in the space of unbounded cocycles.

1 Introduction

A discrete-time linear deterministic dynamical system is defined by a single matrix A, and its Lyapunov spectrum is nothing but the set of the logarithms of the moduli of the eigenvalues of A. It is well-known that the eigenvalues depend continuously on A, hence the matrices with simple Lyapunov spectrum form an open (but not dense because the multiplicity 2 of pairs of conjugate complex eigenvalues is stable under small perturbation) set in the space of all nonsingular matrices. Millionshchikov [13] studied linear systems of non-autonomous differential equations and proved that the systems with so-called *integral separateness* form an open and dense set in the space of all linear systems equipped with uniform topology. A system with integral separateness has simple Lyapunov spectrum, hence the set of linear systems of differential equations with simple Lyapunov spectrum contains an open and dense set. We note that besides having simple Lyapunov spectrum systems with integral separateness exhibit an exponentially dichotomous behavior of trajectories which leads to a stronger property that all systems in a neighborhood of it have a similar property. This uniform property is crucial in investigation of structural stability of dynamical systems.

The object of our interest in this paper are products of random matrices (linear cocycles). Thanks to the multiplicative ergodic theorem of Oseledets [17], the Lyapunov spectrum of a cocycle is well defined (under some integrability conditions) and it is a generalization of the Lyapunov spectrum in the deterministic case and the Oseledets subspaces are generalization of the eigenspaces.

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The study of the Lyapunov spectrum of linear cocycles is one of central tasks of the theory of random dynamical systems (see Arnold [2]). In various situations it is of great theoretical and practical importance to know when the Lyapunov spectrum is simple and the Oseledets splitting is exponentially separated. The result of Millionshchikov mentioned above support the conjecture that cocycles with integral separateness, hence with simple spectrum, form a "big" set. Recently, Arbieto and Bochi [1], Bochi [6], Bochi and Viana [7, 8], Bonatti and Viana [9] and Nguyen Dinh Cong [16] have derived some new results on genericity of hyperbolicity of several classes of dynamical systems including smooth dynamical systems and linear cocycles. Let us mention here a result of Nguyen Dinh Cong [16] stating that the set of cocycles with integral separateness is open and dense in the space of all bounded $Gl(d, \mathbb{R})$-cocycles equipped with uniform topology. As a consequence, a generic bounded linear cocycle has simple Lyapunov spectrum and exponentially separated Oseledets splitting. In this paper we show that this result cannot be extended to the case of unbounded cocycles. Namely, we construct an open set of cocycles with simple Lyapunov spectrum but having no exponentially separated splitting.

This paper is organized as follows. Following a brief introduction in Section 1, we present the notions of exponential dichotomy and exponential separation of linear cocycles as well as their properties in Section 2, where we also discuss the difference between the bounded and unbounded cases. Section 3 is the main section of the paper where we state and prove the main results of the paper: there exists an open set of unbounded linear cocycles with simple Lyapunov spectrum but having no exponentially separated splitting. Thus, unlike the case of bounded cocycles, the exponential separation is nongeneric in the space of unbounded cocycles.

2 The space of linear cocycles and exponential separation properties

2.1 The space of linear cocycles

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a non-atomic Lebesgue probability space, and θ an ergodic automorphism of $(\Omega, \mathcal{F}, \mathbb{P})$ preserving the probability measure $\mathbb{P}$. Throughout this paper we will fix and consider this dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$. A measurable mapping A from the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to the topological space $Gl(d, \mathbb{R})$ (for short: $Gl(d)$) of linear nonsingular operators of $\mathbb{R}^d$ equipped with its Borel σ-algebra is called a *random linear map*. A generates a *linear cocycle* over the dynamical system θ via

$$\Phi_A(n, \omega) := \begin{cases} A(\theta^{n-1}\omega) \circ \cdots \circ A(\omega), & n > 0, \\ \mathrm{id}, & n = 0, \\ A^{-1}(\theta^{n}\omega) \circ \cdots \circ A^{-1}(\theta^{-1}\omega), & n < 0. \end{cases}$$

Conversely, if we are given a linear cocycle over θ, then its time-one map is a linear random map. Therefore, we usually speak of a linear cocycle A, meaning the cocycle Φ_A generated by A. The above construction applies to any topological group G in place of $Gl(d)$ (in particular, G can be a Lie subgroup of $Gl(d)$, for instance $Sl(d)$). For simplicity of notation we denote by $\| \cdot \|$ both the standard Euclidean norm of $\mathbb{R}^d$ and the operator norm of linear operators of $\mathbb{R}^d$. We shall look at linear cocycles as linear operators of $\mathbb{R}^d$ and identify them with their matrix representations in the standard Euclidean basis of $\mathbb{R}^d$. Thus we denote by $Gl(d)$ also the group of nonsingular d-dimensional matrices.

The multiplicative ergodic theorem of Oseledets [17] (see also Arnold [2]) states that if $A(\cdot)$ satisfies the integrability conditions

$$\log^+ \|A(\cdot)^{\pm 1}\| \in \mathbb{L}^1(\mathbb{P}), \tag{1}$$

then the cocycle Φ_A has Lyapunov exponents $\lambda_1 < \cdots < \lambda_p$ with multiplicities $d_1, \ldots, d_p$, which are independent of ω due to the ergodicity of θ, and the phase space $\mathbb{R}^d$ is decomposed into the direct sum of subspaces $E_i(\omega)$ of dimensions d_i corresponding to the Lyapunov exponents λ_i, $i = 1, \ldots, p$, i.e.

$$\lim_{n\to\pm\infty} n^{-1} \log \|\Phi_A(n,\omega)x\| = \lambda_i \iff x \in E_i(\omega)\backslash\{0\}.$$

The above splitting is called *Oseledets splitting* of Φ_A, and the subspaces $E_i(\omega)$ are called *Oseledets subspaces* of Φ_A, they are measurable and *invariant with respect to* A, i.e., $A(\omega)E_i(\omega) = E_i(\theta\omega)$. The collection $\{(\lambda_i, d_i), i = 1, \ldots, p\}$ is called the *Lyapunov spectrum* of Φ_A. It is said to be *simple* if $p = d$. The cocycle A is called *hyperbolic* if none of its Lyapunov exponents vanishes. We note that the statements of the multiplicative ergodic theorem hold on an invariant set of full $\mathbb{P}$-measure. Since we deal with discrete-time cocycles we can always neglect sets of null measure, and we shall identify the random mappings which coincide $\mathbb{P}$-almost surely, and when needed we shall assume, without loss of generality, that the assertions of the Oseledets' theorem hold on the whole of Ω.

Denote by $\mathcal{G}(d)$ the set of all $Gl(d)$-valued random maps. Let $\mathcal{G}_{IC}(d) \subset \mathcal{G}(d)$ denote the subset of those random maps which satisfy the integrability conditions (1), and $\mathcal{G}_\infty(d) \subset \mathcal{G}(d)$ the subset of those which are essentially bounded together with their inverses. Clearly $\mathcal{G}_\infty(d) \subset \mathcal{G}_{IC}(d)$. We define a L^∞ metric ρ on $\mathcal{G}(d)$ as follows. For $A, B \in \mathcal{G}(d)$ set

$$\delta(A,B) := \operatorname*{ess\,sup}_{\omega\in\Omega} \|A(\omega) - B(\omega)\| + \operatorname*{ess\,sup}_{\omega\in\Omega} \|A^{-1}(\omega) - B^{-1}(\omega)\|$$

and

$$\rho(A,B) := \begin{cases} \delta(A,B)(1+\delta(A,B))^{-1} & \text{if } \delta(A,B) < \infty, \\ 1 & \text{if } \delta(A,B) = \infty. \end{cases}$$

It is known that $(\mathcal{G}(d), \rho)$ is a complete metric space, and $\mathcal{G}_{IC}(d)$ and $\mathcal{G}_\infty(d)$ are its closed subsets, hence are complete metric spaces, too (see Arnold and Cong [3]). Moreover, if $A \in \mathcal{G}_{IC}(d)$ (respectively, $\mathcal{G}_\infty(d)$), $B \in \mathcal{G}(d)$ and $\rho(A,B) < 1$, which is equivalent to $\delta(A,B) < \infty$, then $B \in \mathcal{G}_{IC}(d)$ (respectively, $\mathcal{G}_\infty(d)$). Being complete metric spaces, $\mathcal{G}(d)$, $\mathcal{G}_{IC}(d)$ and $\mathcal{G}_\infty(d)$ are Baire spaces. A subset of them is *residual* if it contains a countable intersection of open and dense sets. A property which holds on a residual set is called *generic*.

The angle between two non-vanishing vectors x, y of $\mathbb{R}^d$ is defined by

$$\measuredangle(x,y) := \arccos \frac{\langle x, y\rangle}{\|x\|\,\|y\|} \in [0,\pi].$$

The (minimal) angle between two subspaces $E_1, E_2 \subset \mathbb{R}^d$ is defined by

$$\measuredangle(E_1,E_2) := \inf\{\measuredangle(x,y) \mid 0 \neq x \in E_1, 0 \neq y \in E_2\} \in [0,\pi/2].$$

2.2 Exponential dichotomy

Definition 2.1. A linear cocycle $A \in \mathcal{G}(d)$ is said to exhibit an *exponential dichotomy* if there exist positive numbers $K > 0, \alpha > 0$ and a family of projections P_ω of $\mathbb{R}^d$ depending measurably on $\omega \in \Omega$ such that

$$(i) \quad \|\Phi_A(n,\omega)P_\omega\Phi_A^{-1}(m,\omega)\| \leqslant K\exp(-\alpha(n-m)) \text{ for all } \quad n \geqslant m, \omega \in \Omega,$$

$$(ii) \quad \|\Phi_A(n,\omega)(\mathrm{id}-P_\omega)\Phi_A^{-1}(m,\omega)\| \leqslant K\exp(\alpha(n-m)) \text{ for all } \quad n \leqslant m, \omega \in \Omega.$$

Remark 2.2. (i) If $A \in \mathcal{G}(d)$ has an exponential dichotomy with constants $K > 0, \alpha > 0$ and a family of projections P_ω of $\mathbb{R}^d$, then the angle between the subspaces $P_\omega\mathbb{R}^d$ and $(\mathrm{id} - P_\omega)\mathbb{R}^d$ is separated from zero by a positive constant independent of $\omega \in \Omega$.
(ii) The random subspaces $E_1(\omega) := P_\omega\mathbb{R}^d$ and $E_2(\omega) := (\mathrm{id} - P_\omega)\mathbb{R}^d$ are invariant with respect to A, i.e., $\Phi_A(n,\omega)E_i(\omega) = E_i(\theta^n\omega)$ for all $n \in \mathbb{Z}$, $\omega \in \Omega$ and $i = 1, 2$.
(iii) Exponential dichotomy is also called uniform hyperbolicity.

The properties of exponential dichotomy is robust under small perturbation.

Theorem 2.3. *If $A \in \mathcal{G}(d)$ exhibits an exponential dichotomy then there exists $\varepsilon > 0$ such that any cocycle $B \in \mathcal{G}(d)$ satisfying $\operatorname{ess\,sup}_{\omega\in\Omega} \|A(\omega) - B(\omega)\| < \varepsilon$ also exhibits an exponential dichotomy. Moreover, for small ε the constants and family of projection of the exponential dichotomy of B are close to those of A.*

Proof. This is a two-sided random version of the roughness theorem on exponential dichotomy of ordinary differential equations proved by Coppel [11, Proposition 1, Chapter 4] (see also Sacker and Sell [21]). It is basically an ω-wise application of Palmer's theorem [19, Proposition 2.10] on exponential dichotomy of difference equations (see also Nguyen Dinh Cong [14]). Note that the main tool used for the deterministic case is the contraction mapping principle which can be easily adapted to the random case because the fixed point given by the contraction principle is unique and clearly measurable if all the inputs are measurable. □

Corollary 2.4. *If $A \in \mathcal{G}(d)$ exhibits an exponential dichotomy then there exists $\varepsilon > 0$ such that any cocycle $B \in \mathcal{G}(d)$ satisfying $\rho(A, B) < \varepsilon$ also exhibits an exponential dichotomy.*

2.3 Exponentially separated splitting of bounded cocycles

Definition 2.5. Let $A \in \mathcal{G}_\infty(d)$ and

$$\mathbb{R}^d = E_1(\omega) \oplus \cdots \oplus E_k(\omega), \quad k \geqslant 2, \tag{2}$$

be an invariant splitting of A, i.e. for almost all $\omega \in \Omega$ and any $i = 1, \ldots, k$ we have $A(\omega)E_i(\omega) = E_i(\theta\omega)$. The splitting (2) is called *exponentially separated* if there exist positive numbers $\alpha, K > 0$ such that for any $n \in \mathbb{N}$, $\omega \in \Omega$, and any $i = 1, \ldots, k-1$ the inequality

$$\frac{\|\Phi_A(n,\omega)x\|}{\|x\|} \leqslant Ke^{-n\alpha} \cdot \frac{\|\Phi_A(n,\omega)y\|}{\|y\|}$$

holds for all $0 \neq x \in E_1(\omega) \oplus \cdots \oplus E_i(\omega)$ and $0 \neq y \in E_{i+1}(\omega) \oplus \cdots \oplus E_k(\omega)$.

The notion of *exponential separation* given in Definition 2.5 (for discrete-time bounded cocycles) is equivalent to the notion of *domination* introduced by Viana and his co-workers [7, 8, 9] for classical dynamical systems on compact manifolds and cocycles over them. It is also a random version of the notion of exponential separation of ordinary differential equations which originated from the works of Bylov, Coppel, Sacker and Sell, Palmer and others (see [10, 11, 21, 18, 20]). For linear cocycles there is also a notion of *integral separateness* introduced by Nguyen Dinh Cong [16] which is a random version of the notion of integral separateness of linear systems of differential equations and is equivalent to the notion of exponential separation (see Bylov et al. [10] for the case of ordinary differential equations). Although these terms are equivalent in a sense, for linear cocycles we prefer the term *exponential separation* which has longer history and emphasizes the "separation" of invariant subspaces.

Note that, like exponential dichotomy, in case of bounded cocycle, exponential separation implies boundedness from below of the angle between invariant subspaces of the splitting (see also Bochi and Viana [7]).

Proposition 2.6. *If $A \in \mathcal{G}_\infty(d)$ has an exponentially separated splitting*

$$\mathbb{R}^d = E_1(\omega) \oplus \cdots \oplus E_k(\omega),$$

then for any $i = 1, \ldots, k-1$ the angle between $E_i(\omega)$ and $E_1(\omega)\oplus\cdots\oplus E_{i-1}(\omega)\oplus E_{i+1}(\omega) \oplus \cdots \oplus E_k(\omega)$ is separated from zero by a positive constant independent of $\omega \in \Omega$.

Proof. Put $F_i(\omega) := E_1(\omega) \oplus \cdots \oplus E_i(\omega)$ and $G_{i+1}(\omega) := E_{i+1}(\omega) \oplus \cdots \oplus E_k(\omega)$. Further, $M := \operatorname*{ess\,sup}_{\omega\in\Omega} \|A(\omega)\| + \operatorname*{ess\,sup}_{\omega\in\Omega} \|A^{-1}(\omega)\| < \infty$. Let m be a positive integer such that $2Ke^{-m\alpha} < 1$, where K, α are the constants provided by the definition of exponential separation of A. Let $x \in F_i(\omega)$ and $y \in G_{i+1}(\omega)$ be arbitrary unit vectors. We have

$$2\|\Phi_A(m,\omega)x\| \leqslant \|\Phi_A(m,\omega)y\| \leqslant \|\Phi_A(m,\omega)x\| + \|\Phi_A(m,\omega)(y-x)\|,$$

hence $\|\Phi_A(m,\omega)x\| \leqslant \|\Phi_A(m,\omega)(y-x)\| \leqslant \|\Phi_A(m,\omega)\|\|y-x\|$. Consequently,

$$\|y-x\| \geqslant \|\Phi_A(m,\omega)\|^{-1}\|\Phi_A(m,\omega)^{-1}\|^{-1} \geqslant M^{-2m},$$

which implies $\measuredangle(x,y) \geqslant M^{-2m}$ and the angle between $F_i(\omega)$ and $G_{i+1}(\omega)$ is separated from zero by $M^{-2m} > 0$.

Now, let $z \in E_i(\omega)$ and $h \in F_{i-1}(\omega) \oplus G_{i+1}(\omega)$ be arbitrary unit vectors. Then $h = a_1u + a_2v$ for some $u \in F_{i-1}(\omega)$, $v \in G_{i+1}(\omega)$, $a_1, a_2 \in \mathbb{R}$, $\|u\| = \|v\| = 1$. Clearly, $\max\{|a_1|, |a_2|\} \geqslant 1/2$. Suppose $|a_1| \geqslant 1/2$, setting $r := \frac{a_2}{a_1}v - \frac{1}{a_1}z$ we have

$$\|h-z\| = \|a_1u + a_2v - z\| = |a_1|\|u + \frac{a_2}{a_1}v - \frac{1}{a_1}z\| = |a_1|\|u+r\|.$$

Since $\measuredangle(F_{i-1}(\omega), G_i(\omega)) \geqslant M^{-2m}$, $r \in G_i(\omega)$, $u \in F_{i-1}(\omega)$, $\|u\| = 1$, we have $\|u+r\| \geqslant \sin M^{-2m} \geqslant M^{-2m}/2$, hence $\|h-z\| \geqslant M^{-2m}/4$. Similarly for the case $|a_2| \geqslant 1/2$. Thus, $\measuredangle(E_i(\omega), F_{i-1}(\omega) \oplus G_{i+1}(\omega)) \geqslant M^{-2m}/4$. □

The property of having an exponentially separated splitting is also robust.

Proposition 2.7. *If $A \in \mathcal{G}_\infty(d)$ has an exponentially separated splitting then there exists $\varepsilon > 0$ such that any cocycle $B \in \mathcal{G}(d)$ satisfying $\operatorname{ess\,sup}_{\omega\in\Omega} \|A(\omega) - B(\omega)\| < \varepsilon$ also has an exponentially separated splitting. Moreover, for small ε the constants and the splitting of the exponential separation B are close to those of A.*

The proof of this proposition is analogous to that of robustness of exponential dichotomy in Theorem 2.3 above (see also Bochi and Viana [7]).

Corollary 2.8. *If $A \in \mathcal{G}_\infty(d)$ has an exponentially separated splitting then there exists $\varepsilon > 0$ such that any cocycle $B \in \mathcal{G}(d)$ satisfying $\rho(A, B) < \varepsilon$ also has an exponentially separated splitting.*

Remark 2.9. Corollary 2.8 is in fact equivalent to Proposition 2.7 because in $\mathcal{G}_\infty(d)$ the distance defined by $\operatorname{ess\,sup}_{\omega\in\Omega} \|A(\omega) - B(\omega)\|$ is equivalent to ρ. To see this, use the formula $B(\omega)^{-1} - A(\omega)^{-1} = B(\omega)^{-1}(A(\omega) - B(\omega))A(\omega)^{-1}$.

2.4 Exponential dichotomy is strictly stronger than exponential separation

Theorem 2.10. *Suppose that $A \in \mathcal{G}(d)$ exhibits an exponential dichotomy with constants $K > 0, \alpha > 0$ and a family of projections P_ω of $\mathbb{R}^d$. Then the invariant splitting*

$$\mathbb{R}^d = E_1(\omega) \oplus E_2(\omega),$$

where $E_1(\omega) := P_\omega \mathbb{R}^d$ and $E_2(\omega) := (\mathrm{id} - P_\omega)\mathbb{R}^d$, is exponentially separated.

Proof. As noted in Remark 2.2, the spaces $E_1(\omega)$ and $E_2(\omega)$ are invariant with respect to A. For any nonvanishing vectors $x \in E_1(\omega)$ and $y \in E_2(\omega)$ and $n \in \mathbb{N}$ we have

$$\begin{aligned} \frac{\|\Phi_A(n,\omega)x\|}{\|x\|} &\leqslant \|\Phi_A(n,\omega)P_\omega\| \leqslant Ke^{-\alpha n}, \\ \frac{\|\Phi_A(n,\omega)y\|}{\|y\|} &\geqslant \frac{1}{\|(I - P_\omega)\Phi_A(n,\omega)^{-1}\|} \geqslant \frac{e^{\alpha n}}{K}. \end{aligned}$$

Therefore the splitting $\mathbb{R}^2 = E_1(\omega) \oplus E_2(\omega)$ is exponentially separated. □

The following proposition (based on a one-dimensional example by Nguyen Dinh Cong [15, Proposition 3.2]) shows us a bounded linear two-dimensional cocycle which has an exponentially separated splitting but does not exhibit exponential dichotomy. First we recall from [15] a technical lemma which will be needed later.

Lemma 2.11. *There exists a measurable set $U \subset \Omega$ which can be represented in the form*

$$U = \bigcup_{k=4}^{\infty} \bigcup_{j=0}^{3k-1} \theta^j U_k, \tag{3}$$

where the sets $\theta^j U_k$, $k = 0, 1, \ldots,$ $j = 0, \ldots, 3k-1$, are pairwise disjoint and are all of positive $\mathbb{P}$-measure.

Proposition 2.12. *There exists $A \in \mathcal{G}_\infty(2)$ and $0 < \varepsilon < 1$ such that any cocycle $B \in \mathcal{G}_\infty(2)$ satisfying $\rho(B, A) < \varepsilon$ is hyperbolic and the Oseledets splitting of B is exponentially separated but B exhibits no exponential dichotomy.*

Proof. Let $U \subset \Omega$ be a set with representation (3) provided by Lemma 2.11. It is easily seen that we may assume $\mathbb{P}(U) < \frac{1}{4}$. We construct a cocycle

$$A(\omega) := \begin{pmatrix} a_1(\omega) & 0 \\ 0 & a_2(\omega) \end{pmatrix} \in \mathcal{G}_\infty(2)$$

by setting

$$\begin{aligned} a_1(\omega) &= \begin{cases} \frac{1}{4} & \text{for } \omega \in U, \\ 1 & \text{for } \omega \in \Omega \setminus U \ , \end{cases} \\ a_2(\omega) &= \begin{cases} \frac{1}{2} & \text{for } \omega \in U, \\ 2 & \text{for } \omega \in \Omega \setminus U \ . \end{cases} \end{aligned}$$

Clearly,

$$\mathbb{R}^2 = \mathbb{R}e_1 \oplus \mathbb{R}e_2 \tag{4}$$

is an invariant and exponentially separated splitting of A. The Lyapunov exponents of A are easily computed and since $\mathbb{P}(U) < 1/4$ we have

$$\begin{aligned} \lambda_1(A) &= \int_\Omega \log |a_1(\omega)| d\mathbb{P}(\omega) < 0, \\ \lambda_2(A) &= \int_\Omega \log |a_2(\omega)| d\mathbb{P}(\omega) > 0, \end{aligned}$$

hence A is hyperbolic. By Proposition 2.7, there is $\varepsilon > 0$ such that any $B \in \mathcal{G}_\infty(2)$ satisfying $\rho(B, A) < \varepsilon$ has an exponentially separated splitting which is close to the splitting (4). By making $\varepsilon > 0$ smaller if necessary we can show that B is also hyperbolic. Note that along the orbit segments on U the norm of $A(\omega)$ equals $1/2$, hence, for $\varepsilon < 1/4$, we have $\|B(\omega)\| < 3/4 < 1$. The set U contains arbitrarily long orbit segments and on these segments there are no expanding direction for B. Hence B has no uniformly expanding direction, and thus it cannot exhibit exponential dichotomy. ☐

Remark 2.13. (i) A higher dimensional example is easily constructed in a similar way. Thus the converse of Theorem 2.10 is false, hence exponential dichotomy is strictly stronger than exponential separation.
(ii) In the case of ordinary differential equation the concepts of exponential dichotomy and integral separateness are equivalent modulo the use of shifted equations (see Palmer [18]).

2.5 Exponential separation of unbounded cocycles

For the general case of unbounded cocycles we have to be more careful with definition of exponentially separated splitting. In Subsection 2.3 we gave definition of exponentially separated splitting for bounded cocycles and presented some important properties concerning robustness and boundedness from zero of the angles between subspaces. These properties will be no longer true if we apply directly without modification Definition 2.5 to the unbounded case as will be shown in Proposition 2.15. Besides, it is not difficult to construct an unbounded cocyce which has an invariant splitting satisfying the properties stated in Definition 2.5 but the angles between subspaces are not bounded from zero. Therefore, it is reasonable to require additionally an angle condition in the definition of exponential separation for unbounded cocycles. Thus we have arrive at the following definition.

Definition 2.14. Let $A \in \mathcal{G}(d)$ and

$$\mathbb{R}^d = E_1(\omega) \oplus \cdots \oplus E_k(\omega), \quad k \geqslant 2, \tag{5}$$

be an invariant splitting of A, i.e. for almost all $\omega \in \Omega$ and any $i = 1, \ldots, k$ we have $A(\omega)E_i(\omega) = E_i(\theta\omega)$. The splitting (5) is called *exponentially separated* if the following two conditions are satisfied:
(i) there exists numbers $K, \alpha > 0$ such that for any $n \in \mathbb{N}$, $\omega \in \Omega$ and any $i = 1, \ldots, k-1$ the inequality

$$\frac{\|\Phi_A(n,\omega)x\|}{\|x\|} \leqslant Ke^{-n\alpha} \cdot \frac{\|\Phi_A(n,\omega)y\|}{\|y\|}$$

holds for all $0 \neq x \in E_1(\omega) \oplus \cdots \oplus E_i(\omega)$ and $0 \neq y \in E_{i+1}(\omega) \oplus \cdots \oplus E_k(\omega)$;
(ii) for any $i = 1, \ldots, k-1$, the angle between $E_i(\omega)$ and $E_1(\omega) \oplus \cdots \oplus E_{i-1}(\omega) \oplus E_{i+1}(\omega) \oplus \cdots \oplus E_k(\omega)$ is separated from zero by a positive constant independent of $\omega \in \Omega$.

Note that due to Proposition 2.6, Definition 2.14 is equivalent to Definition 2.5 if $A \in \mathcal{G}_\infty(d)$. If $A \in \mathcal{G}_{IC}(d)$ has an exponentially separated splitting then it has at least two different Lyapunov exponents and its Oseledets splitting is nontrivial.

The following proposition shows that the condition (ii) of Definition 2.14 is crucial for the important property of robustness of exponentially separated splitting.

Proposition 2.15. *If in Definition 2.14 we drop the angle condition (ii), then there exists $A \in \mathcal{G}_{IC}(2)$ such that A has an exponentially separated splitting but in any neighborhood of A there is a cocycle B which has no exponentially separated splitting.*

Proof. By Lemma 2.11 we can find a measurable set $U \subset \Omega$ such that

$$U = \bigcup_{k=4}^{\infty} \bigcup_{j=0}^{2k} \theta^j U_k, \tag{6}$$

where the sets $\theta^j U_k$, $j = 0, 1 \ldots, 2k$, $k \geqslant 4$, are measurable, disjoint, have positive measure and $\mathbb{P}(U_k) \leqslant \frac{1}{k^4}$ for all $k \geqslant 4$. Moreover, we can choose U such that for any $k \geqslant 4$ the sets U_k are not coboundaries, i.e. they cannot be represented in the form $U_k = V_k \triangle \theta V_k$ with $V_k \in \mathcal{F}$ (see Knill [12, Corollary 3.5]).

We will construct a cocycle A satisfying the assertion of the proposition together with the Oseledets splitting of A. Let $\{e_1, e_2\}$ denote the standard

Euclidean basis of $\mathbb{R}^2$. We construct a random basis $\{f_1(\omega), f_2(\omega)\}$ of $\mathbb{R}^2$ by setting $f_1(\omega) \equiv e_1$, and for any $k \geqslant 4$

$$f_2(\omega) = \begin{cases} \cos(\frac{\pi}{2^{i+1}})e_1 + \sin(\frac{\pi}{2^{i+1}})e_2 & \text{if} \quad \omega \in \theta^i U_k,\ i = 0, 1, \dots k-1, \\ \cos(\frac{\pi}{2^k})e_1 + \sin(\frac{\pi}{2^k})e_2 & \text{if} \quad \omega \in \theta^k U_k, \\ \cos(\frac{\pi}{2^{2k-i}})e_1 + \sin(\frac{\pi}{2^{2k-i}})e_2 & \text{if} \quad \omega \in \theta^i U_k,\ i = k+1, k+2, \dots 2k-1, \\ e_2 & \text{if} \quad \omega = \theta^{2k}\omega_0,\ \omega_0 \in U_k, \end{cases}$$

and $f_2(\omega) = e_2$ for $\omega \in \Omega \setminus U$. For definition of $A \in \mathcal{G}(2)$ we set $A(\omega) = \mathrm{id}$ on $\bigcup_{k=4}^{\infty} \theta^{k-1} U_k \cup \theta^{2k-1} U_k$ and

$$A(\omega) f_1(\omega) = f_1(\theta\omega), \quad A(\omega) f_2(\omega) = \frac{1}{2} f_2(\theta\omega)$$

on $\Omega \setminus \bigcup_{k=4}^{\infty} \theta^{k-1} U_k \cup \theta^{2k-1} U_k$. By construction, $A \in \mathcal{G}_{IC}(2)$ and

$$\mathbb{R}^2 = E_1(\omega) \oplus E_2(\omega), \tag{7}$$

where $E_1(\omega), E_2(\omega)$ are the subspaces spanned by vectors $f_1(\omega), f_2(\omega)$ respectively, is an exponentially separated splitting of A.

Now, let $0 < \varepsilon < 1$ be arbitrary. We will show that there exists $B \in \mathcal{G}_{IC}(2)$ such that $\rho(A, B) < \varepsilon$ and B has no exponentially separated splitting.

Choose and fix $n \in \mathbb{N}$ such that $2^{n-3}\varepsilon > 1$. We define B by setting

(i) $B(\omega) = \begin{pmatrix} \cos(\frac{\pi}{2^n}) & -\sin(\frac{\pi}{2^n}) \\ \sin(\frac{\pi}{2^n}) & \cos(\frac{\pi}{2^n}) \end{pmatrix} A(\omega)$ for $\omega \in \theta^{n-1} U_n$;

(ii) $B(\omega) = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} A(\omega)$ for $\omega \in \theta^{2n-1} U_n$, where $b = \frac{\cos \frac{\pi}{2^n}}{2^{n-2}}$;

(iii) $B(\omega) = A(\omega)$ for $\omega \in \Omega \setminus \big(\theta^{n-1} U_n \cup \theta^{2n-1} U_n\big)$.

By construction, $\rho(A, B) < \varepsilon$ and for any $\omega \in U_n$ we have

$$\Phi_B(2n, \omega) f_1(\omega) \text{ is collinear with } e_2,$$
$$\Phi_B(2n, \omega) f_2(\omega) \text{ is collinear with } e_1.$$

The set U_n is not a coboundary, $\mathbb{P}(U_n) > 0$, and the sets $U_n, \theta U_n, \dots, \theta^{2n} U_n$ are disjoint. Furthermore, for all $\omega \in U_n$ we have

$$\begin{aligned} \Phi_B(2n, \omega) E_1(\omega) &= E_2(\theta^{2n}\omega), \\ \Phi_B(2n, \omega) E_2(\omega) &= E_1(\theta^{2n}\omega). \end{aligned}$$

and on $\Omega \setminus \big(\cup_{i=0}^{2n} \theta^i U_n \big)$ we have $B(\omega) = A(\omega)$. Therefore, by a version of Lemma 4.4 of Knill [12] B has one Lyapunov exponent with multiplicity 2 (see also Bochi [5]), hence the Oseledets splitting of B is trivial and so B has no exponentially separated splitting. □

The following theorem shows that for unbounded cocycles the exponential separation property is also robust, thus gives us another justification for the angle condition in the Definition 2.14.

Theorem 2.16. *If $A \in \mathcal{G}(d)$ has an exponentially separated splitting then there exists $\varepsilon > 0$ such that any cocycle $B \in \mathcal{G}(d)$ satisfying $\rho(A, B) < \varepsilon$ also has an exponentially separated splitting.*

Proof. By Lemma 2.11 of Nguyen Dinh Cong [16], if A has an exponentially separated splitting then it is cohomologous to a block-diagonal cocycle by a cohomology which is bounded together with its inverse. Therefore, we may assume that A has block diagonal form. First, we give a proof for the two-dimensional case. Suppose we have a two-dimensional cocycle

$$A(\omega) = \begin{pmatrix} a_1(\omega) & 0 \\ 0 & a_2(\omega) \end{pmatrix}$$

with the exponentially separated splitting $\mathbb{R}^2 = \mathbb{R}e_1 \oplus \mathbb{R}e_2$. Then there are positive constants K, α such that

$$\prod_{k=0}^{n-1} \|a_2(\theta^k\omega)\| \geqslant K e^{\alpha n} \prod_{k=0}^{n-1} \|a_1(\theta^k\omega)\|, \tag{8}$$

for all $n \in \mathbb{N}$ and $\omega \in \Omega$. We can assume that $K < 1$. Put $\beta := \frac{\alpha}{2}$. We construct a diagonal cocycle $\tilde{A}(\omega) = \begin{pmatrix} \tilde{a}_1(\omega) & 0 \\ 0 & \tilde{a}_2(\omega) \end{pmatrix}$ by setting $\tilde{A}(\omega) = A(\omega)$ in case $|a_2(\omega)| \geqslant \frac{1}{K} e^{\beta}$ and $|a_1(\omega)| \leqslant K e^{-\beta}$, and

$$\tilde{a}_1(\omega) = \frac{e^{\beta} a_1(\omega)}{|a_2(\omega)|}, \quad \tilde{a}_2(\omega) = \frac{e^{\beta} a_2(\omega)}{|a_2(\omega)|} = e^{\beta}, \tag{9}$$

otherwise. We show that $\tilde{A}$ exhibits an exponential dichotomy. Let $n \in \mathbb{N}$ and $\omega \in \Omega$ be arbitrary. We estimate the product

$$\|\Phi_{\tilde{A}}(n, \omega) e_1\| = \prod_{k=0}^{n-1} \|\tilde{a}_1(\theta^k\omega)\| \tag{10}$$

in three basic cases:
(i) **Case 1,** $\tilde{a}_1(\omega) = a_1(\omega)$ **and** $\tilde{a}_1(\theta^i\omega) = \frac{e^{\beta} a_1(\theta^i\omega)}{|a_2(\theta^i\omega)|}$ **for all** $i = 1, \ldots, n-1$**:** In this case, by (8) and (9) we have $\tilde{a}_1(\omega) = a_1(\omega) \leqslant K e^{-\beta}$ and

$$\|\Phi_{\tilde{A}}(n, \omega) e_1\| = \prod_{k=0}^{n-1} \|\tilde{a}_1(\theta^k\omega)\| \leqslant K e^{-\beta} \frac{1}{K e^{(n-1)\beta}} = e^{-n\beta};$$

(ii) **Case 2,** $\tilde{a}_1(\theta^i\omega) = a_1(\theta^i\omega)$ **for all** $i = 0, \ldots, n-1$**:** In this case, by (9) we have $a_1(\theta^i\omega) \leqslant K e^{-\beta}$ for all $i = 0, \ldots, n-1$, hence

$$\|\Phi_{\tilde{A}}(n, \omega) e_1\| = \prod_{k=0}^{n-1} \|\tilde{a}_1(\theta^k\omega)\| \leqslant K^n e^{-n\beta} \leqslant e^{-n\beta};$$

(iii) **Case 3,** $\tilde{a}_1(\theta^i\omega) = \frac{e^{\beta} a_1(\theta^i\omega)}{|a_2(\theta^i\omega)|}$ **for all** $i = 0, \ldots, n-1$**:** In this case, by (8) and (9) we have

$$\|\Phi_{\tilde{A}}(n, \omega) e_1\| = \prod_{k=0}^{n-1} \|\tilde{a}_1(\theta^k\omega)\| \leqslant \frac{1}{K} e^{-n\beta}.$$

Note that for arbitrary $n \in \mathbb{N}$ and $\omega \in \Omega$ the product (10) can be decomposed into product of terms of three above basic types with the case 3 occurs possibly only once. Thus, we always have

$$\|\Phi_{\tilde{A}}(n,\omega)e_1\| = \prod_{k=0}^{n-1} \|\tilde{a}_1(\theta^k\omega)\| \leqslant \frac{1}{K}e^{-n\beta}.$$

By construction and by (9) we always have $|\tilde{a}_2(\omega)| \geqslant e^{\beta}$, hence

$$\prod_{k=0}^{n-1} \|\Phi_{\tilde{A}}(n,\omega)e_2\| \geqslant e^{\beta n}$$

for arbitrary $n \in \mathbb{N}$ and $\omega \in \Omega$. Therefore, $\tilde{A}$ exhibits an exponential dichotomy. Consequently, by Theorem 2.3 there exists $0 < \delta_1 < \frac{e^{\beta}}{K}$ such that any cocycle A' satisfying $\operatorname{ess\,sup}_{\omega\in\Omega} \|A'(\omega) - A(\omega)\| < \delta_1$ also exhibits an exponential dichotomy.

Choose and fix a number $0 < \delta < \frac{1}{2}$ which satisfies the following inequalities

$$\delta \leqslant \frac{\delta_1}{2}, \quad \delta < \frac{K^2}{2e^{\beta}} \quad \text{and} \quad \delta \leqslant \frac{K^3\delta_1}{2(e^{2\beta} + K\delta_1 e^{\beta})}.$$

We show that any cocycle B satisfying $\rho(A,B) < \delta$, will have an exponentially separated splitting. For, let us construct a cocycle $\tilde{B}$ by setting $\tilde{B}(\omega) = B(\omega)$ in case $|a_2(\omega)| \geqslant \frac{1}{K}e^{\beta}$ and $|a_1(\omega)| \leqslant Ke^{-\beta}$, and

$$\tilde{B}(\omega) = \frac{e^{\beta}B(\omega)}{|a_2(\omega)|}$$

otherwise. We will estimate $\|\tilde{B}(\omega) - \tilde{A}(\omega)\|$. There are two cases.
(i) **First case,** $|a_2(\omega)| \geqslant \frac{1}{K}e^{\beta}$**:** If $|a_1(\omega)| \leqslant Ke^{-\beta}$ we have

$$\|\tilde{B}(\omega) - \tilde{A}(\omega)\| = \|B(\omega) - A(\omega)\| \leqslant \delta \leqslant \delta_1.$$

If $|a_1(\omega)| \geqslant Ke^{-\beta}$, then from the definition of $\tilde{A}$ and $\tilde{B}$ we have

$$\|\tilde{B}(\omega) - \tilde{A}(\omega)\| = \frac{e^{\beta}}{|a_2(\omega)|}\|B(\omega) - A(\omega)\| \leqslant K\delta \leqslant \delta_1.$$

(ii) **Second case,** $|a_2(\omega)| < \frac{1}{K}e^{\beta}$**:** From (8) we get $|a_1(\omega)| \leqslant \frac{1}{K}|a_2(\omega)|$, hence $\|A(\omega)\| \leqslant \frac{1}{K}|a_2(\omega)| < \frac{e^{\beta}}{K^2}$. From the definition of $\tilde{B}$ and $\tilde{A}$ we have

$$\|\tilde{B}(\omega) - \tilde{A}(\omega)\| = \frac{e^{\beta}}{|a_2(\omega)|}\|B(\omega) - A(\omega)\|.$$

Setting $C(\omega) = A^{-1}(\omega) - B^{-1}(\omega)$, we see that $\|C(\omega)\| \leqslant 2\rho(A,B) < 2\delta$, and

$$B^{-1}(\omega) = A^{-1}(\omega)(I - A(\omega)C(\omega)).$$

Since $\|A(\omega)C(\omega)\| < \frac{e^{\beta}}{K^2} \cdot 2\delta := \delta_2 < 1$, the matrix $I - A(\omega)C(\omega)$ is invertible and $B(\omega) = (I - A(\omega)C(\omega))^{-1}A(\omega)$. Put $D(\omega) := A(\omega)C(\omega)$. We have $B(\omega) = (I + D(\omega) + \cdots)A(\omega)$, hence $B(\omega) - A(\omega) = (D(\omega) + D(\omega)^2 + \cdots)A(\omega)$. Therefore,

$$\|B(\omega) - A(\omega)\| \leqslant (\delta_2 + \delta_2^2 + \cdots)\|A(\omega)\| \leqslant \frac{\delta_2}{1-\delta_2} \cdot \frac{1}{K}|a_2(\omega)|.$$

Hence

$$\|\tilde{B}(\omega) - \tilde{A}(\omega)\| \leqslant \frac{e^{\beta}}{K} \cdot \frac{\delta_2}{1-\delta_2} \leqslant \delta_1.$$

Thus, in any case, by the choice of δ_1, the cocycle $\tilde{B}$ exhibits an exponential dichotomy, hence has an exponentially separated splitting. Since B differs from $\tilde{B}$ only by a scalar multiplier this implies that B has the same exponentially separated splitting as $\tilde{B}$. The theorem is proved in this two-dimensional case.

The general d-dimensional case is similar to the two-dimensional case treated above. We list here the changes necessary for transition from the two-dimensional to the d-dimensional case with the splitting consisting of two subspaces: instead of scalars (one-dimensional matrices) $a_1(\omega)$, $a_2(\omega)$ we have to deal with matrices $a_1(\omega)$, $a_2(\omega)$ (of higher order, in general); the absolute values $|a_1(\omega)|$ should be changed to the matrix norm $\|a_1(\omega)\|$ and the absolute value $|a_2(\omega)|$ should be changed to the matrix co-norm $m(a_2(\omega)) := \|a_2(\omega)^{-1}\|^{-1}$ (cf. Bochi and Viana [8]); the product $\prod_{j=0}^{n-1}|a_1(\theta^j\omega)|$ should be changed to the norm $\|\prod_{j=0}^{n-1}a_1(\theta^j\omega)\|$ and the product $\prod_{j=0}^{n-1}|a_2(\theta^j\omega)|$ should be changed to the co-norm $m(\prod_{j=0}^{n-1}a_2(\theta^j\omega))$. The case of the splitting consisting of more than two subspaces can be easily deduced from the case of two subspaces. □

From the proof of Theorem 2.16 above we can see that for small ε the exponentially separated splitting of B is close to that of A (exponentially separated splitting varies continuously in $(\mathcal{G}(d), \rho)$). Although in $\mathcal{G}_\infty(d)$ the exponential separation is robust in an weaker than ρ metric as stated in Theorem 2.3, in the unbounded case this is no longer true (this can be seen from the essential use of the smallness of $\|A^{-1} - B^{-1}\|$ in the proof of Theorem 2.16).

Proposition 2.17. *There exists $A \in \mathcal{G}_{IC}(2)$ with exponentially separated splitting such that for any $\varepsilon > 0$ one can find a cocycle $B \in \mathcal{G}(2)$ which has no exponentially separated splitting and satisfies $\operatorname*{ess\,sup}_{\omega\in\Omega}\|A(\omega) - B(\omega)\| < \varepsilon$.*

Proof. By Lemma 2.11 we can find a measurable set F which can be represented in the form $F = \bigcup_{k=4}^{\infty}\bigcup_{j=0}^{k-1}\theta^j U_k$, where the sets $\theta^j U_k$, $k \geqslant 4$, $0 \leqslant j \leqslant k-1$, are pairwise disjoint and all they are of positive measure. We can assume additionally that the sets U_k satisfy the inequality $\sum_{k=4}^{\infty}k^2\mathbb{P}(U_k) \leqslant 1$. Define a cocycle $A \in \mathcal{G}(2)$ by

$$A(\omega) = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} & \text{for} \quad \omega \in \Omega \setminus F, \\ \\ \begin{pmatrix} \frac{1}{k} & 0 \\ 0 & \frac{1}{2k} \end{pmatrix} & \text{for} \quad \omega \in \bigcup_{j=0}^{k-1}\theta^j U_k, \quad k \geqslant 4. \end{cases}$$

It is easily seen that $A \in \mathcal{G}_{IC}(2)$. For any $\varepsilon > 0$ choose and fix $n \in \mathbb{N}$ such that $n \geqslant 4\varepsilon^{-1}$. Define $B \in \mathcal{G}_{IC}(2)$ by setting

$$B(\omega) = \begin{pmatrix} \frac{1}{2k} & 0 \\ 0 & \frac{1}{2k} \end{pmatrix}$$

for $\omega \in \cup_{j=0}^{k-1}\theta^j U_k$, $k \geqslant n$, and $B(\omega) = A(\omega)$ for other $\omega \in \Omega$. It is easily seen that B furnishes the assertions of the proposition. □

3 An open set of cocycles with simple Lyapunov spectrum but having no exponentially separated splitting

In this section we will construct an open set of cocycles such that any cocycle from it has simple Lyapunov spectrum but has no exponentially separated splitting. Moreover, the Lyapunov exponents considered as function of cocycles are continuous in this set; this is a distinguished feature of unbounded cocycles since in the bounded case continuity of all Lyapunov exponents implies exponential separation of the Oseledets splitting (see J. Bochi [6] and J. Bochi and M. Viana [7, 8]).

We will construct a cocycle $A_0 \in \mathcal{G}_{IC}(2)$ such that a neighborhood of it will have the properties claimed in the title of the section. First, by Lemma 2.11 we can find a measurable set $U = \cup_{k=4}^{\infty} \cup_{j=0}^{3k-1} \theta^j U_k$ such that the sets $\theta^j U_k$, $0 \leqslant j \leqslant 3k-1$, $k \geqslant 4$, are pairwise disjoint, measurable and of positive measure. Denote $\mathbb{P}(U_k)$ by x_k for all $k \geqslant 4$. We can assume additionally that the sets U_k satisfy also the condition

$$\sum_{k=4}^{\infty} k^2 x_k \leqslant \frac{1}{4}. \tag{11}$$

Set $F := \bigcup_{k=4}^{\infty} \bigcup_{j=k}^{2k-1} \theta^j U_k$, we get $\mathbb{P}(F) = \sum_{k=4}^{\infty} k x_k \leqslant \frac{1}{16}$. Now, the cocycle $A_0 \in \mathcal{G}(2)$ is constructed as follows:

$$A_0(\omega) = \begin{cases} \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} & \text{for} \quad \omega \in \Omega \setminus F, \\ \\ \begin{pmatrix} k+1 & 0 \\ 0 & k \end{pmatrix} & \text{for} \quad \omega \in \bigcup_{j=k}^{2k-1} \theta^j U_k, \quad k \geqslant 4. \end{cases}$$

From (11) it follows that $A_0 \in \mathcal{G}_{IC}(2)$. Clearly, $A_0 \notin \mathcal{G}_{\infty}(2)$. Denote by $\{e_1, e_2\}$ the standard Euclidean basis of $\mathbb{R}^2$. It is easily to see that

$$\mathbb{R}^2 = \mathbb{R}e_1 \oplus \mathbb{R}e_2 \tag{12}$$

is the Oseledets splitting of the cocycle A_0. Using the Birkhoff's theorem we can compute the Lyapunov exponents of A_0, and they are

$$\begin{aligned} \lambda_1(A_0) &= \int_{\Omega} \log \|A_0(\omega)e_1\| d\mathbb{P}(\omega) = \sum_{k=4}^{\infty} k x_k \log(k+1) + (1 - \mathbb{P}(F)) \log 2 > 0, \\ \lambda_2(A_0) &= \int_{\Omega} \log \|A_0(\omega)e_2\| d\mathbb{P}(\omega) = \sum_{k=4}^{\infty} k x_k \log k - (1 - \mathbb{P}(F)) \log 2 \\ &\qquad < \frac{1}{4} - \frac{3\log 2}{4} < 0. \end{aligned}$$

Hence the cocycle A_0 is hyperbolic and has simple Lyapunov spectrum. However, A_0 has no exponentially separated splitting because otherwise the exponentially separated splitting must be the Oseledets splitting (12) but for any fixed positive numbers α, K we can find $n > 4$ such that $(n+1)^n < K e^{\alpha n} n^n$.

We also define a cocycle $\hat{A}$ by setting

$$\hat{A}(\omega) := \begin{cases} \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} & \text{for} \quad \omega \in \Omega \setminus F, \\ \\ \begin{pmatrix} 1+\frac{1}{k} & 0 \\ 0 & 1 \end{pmatrix} & \text{for} \quad \omega \in \bigcup_{j=k}^{2k-1} \theta^j U_k, \quad k \geqslant 4. \end{cases}$$

Proposition 3.1. *There exists a positive number δ such that any cocycle $B \in \mathcal{G}_{IC}(2)$ satisfying $\rho(A_0, B) < \delta$ will have simple Lyapunov spectrum. Moreover, the functions $\lambda_i(\cdot) : \mathcal{G}_{IC}(2) \to \mathbb{R}$, $B \mapsto \lambda_i(B)$, $i = 1, 2$, are continuous on the ball centered at A_0 with radius δ in $(\mathcal{G}_{IC}(2), \rho)$.*

Proof. We choose and fix a positive number δ satisfying $\delta < \frac{1}{40}$ and $\delta < \frac{1}{3}\sum_{k=4}^{\infty} k x_k \log(1+\frac{1}{k})$. Let $B \in \mathcal{G}_{IC}(2)$ be an arbitrary cocycle satisfying $\rho(A_0, B) < \delta/2$. Setting

$$\hat{B}(\omega) := \begin{cases} B(\omega) & \text{for} \quad \omega \in \Omega \setminus F, \\ \frac{1}{k} B(\omega) & \text{for} \quad \omega \in \bigcup_{j=k}^{2k-1} \theta^j U_k, \quad k \geqslant 4, \end{cases}$$

and $\hat{C}(\omega) := \hat{B}(\omega) - \hat{A}(\omega)$, we have

$$\begin{aligned} \|\hat{C}(\omega)\| &\leqslant \delta \quad \text{if} \quad \omega \in \Omega \setminus F, \\ \|\hat{C}(\omega)\| &\leqslant \frac{1}{k}\delta \quad \text{if} \quad \omega \in \bigcup_{j=k}^{2k-1} \theta^j U_k, \quad k \geqslant 4. \end{aligned}$$

Define a random projector P_ω of $\mathbb{R}^2$ by setting $P_\omega = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ for all $\omega \in \Omega$. We make the following claim.

Claim. *For any $\omega \in \Omega$ and any $n \in \mathbb{N}$ the following inequalities hold*

$$\sum_{k=0}^{n-1} \|\Phi_{\hat{A}}(n-k-1, \theta^{k+1}\omega) P_{\theta^{k+1}\omega} \hat{C}(\theta^k\omega)\| \leqslant (7+\sqrt{2})\delta, \tag{13}$$

$$\sum_{k=n}^{\infty} \|\Phi_{\hat{A}}^{-1}(k+1-n, \theta^n\omega)(I - P_{\theta^{k+1}\omega}) \hat{C}(\theta^k\omega)\| \leqslant (7+\sqrt{2})\delta. \tag{14}$$

To prove the claim we set $E_m := U_m \cup \theta U_m \cdots \cup \theta^{3m-2} U_m$ and $E := \cup_{m=4}^{\infty} E_m$ and notice that $\|\Phi_{\hat{A}}(n-k-1, \theta^{k+1}\omega) P_{\theta^{k+1}\omega}\| = \prod_{j=k+1}^{n-1} |\hat{a}_2(\theta^j\omega)|$, where

$$\hat{a}_2(\omega) := \begin{cases} \frac{1}{2} & \text{for} \quad \omega \in \Omega \setminus F, \\ 1 & \text{for} \quad \omega \in \bigcup_{j=k}^{2k-1} \theta^j U_k, \quad k \geqslant 4. \end{cases}$$

Therefore, by the construction of U, E it follows that if $\theta^{n-1}\omega \notin E$ then

$$\|\Phi_{\hat{A}}(n-k-1, \theta^{k+1}\omega) P_{\theta^{k+1}\omega}\| \leqslant (\frac{1}{\sqrt{2}})^{n-k-1} \tag{15}$$

for all integers $0 \leqslant k \leqslant n-1$. Now, back to (13) we see that there are two cases: either $\theta^{n-1}\omega \in E$ or not. If $\theta^{n-1}\omega \notin E$, then by (15) we have

$$\sum_{k=0}^{n-1} \|\Phi_{\hat{A}}(n-k-1, \theta^{k+1}\omega) P_{\theta^{k+1}\omega} \hat{C}(\theta^k\omega)\| \leqslant \delta \cdot \sum_{k=0}^{n-1} (\frac{1}{\sqrt{2}})^{n-k-1} \leqslant (2+\sqrt{2})\delta, \tag{16}$$

which furnishes (13). If $\theta^{n-1}\omega \in E$ then $\theta^{n-1}\omega \in \theta^h U_m$ for some $m \geqslant 4$ and $0 \leqslant h < 3m-1$. In this case $\theta^{n-1-h-1}\omega \notin E$, hence using (16) and the fact that $\|\Phi_{\hat{A}}(r,\omega)P_\omega\| \leqslant 1$ for all $r \geqslant 0$ and all $\omega \in \Omega$, we have

$$\sum_{k=0}^{n-h-2} \|\Phi_{\hat{A}}(n-k-1,\theta^{k+1}\omega)P_{\theta^{k+1}\omega}\hat{C}(\theta^k\omega)\| =$$
$$\sum_{k=0}^{n-h-2} \|\Phi_{\hat{A}}(h+1,\theta^{n-h-1}\omega)\Phi_{\hat{A}}(n-h-k-2,\theta^{k+1}\omega)P_{\theta^{k+1}\omega}\hat{C}(\theta^k\omega)\| \leqslant$$
$$\sum_{k=0}^{n-h-2} \|\Phi_{\hat{A}}(n-h-k-2,\theta^{k+1}\omega)P_{\theta^{k+1}\omega}\hat{C}(\theta^k\omega)\| \leqslant (2+\sqrt{2})\delta. \quad (17)$$

To estimate the term $\sum_{k=n-h-1}^{n-1} \|\Phi_{\hat{A}}(n-k-1,\theta^{k+1}\omega)P_{\theta^{k+1}\omega}\hat{C}(\theta^k\omega)\|$ we notice that

$$\sum_{k=n-h-1}^{n-1} \|\Phi_{\hat{A}}(n-k-1,\theta^{k+1}\omega)P_{\theta^{k+1}\omega}\hat{C}(\theta^k\omega)\| \leqslant \sum_{k=n-h-1}^{n-1} \|\hat{C}(\theta^k\omega)\| \prod_{j=k+1}^{n-1} |\hat{a}_2(\theta^j\omega)|.$$

By considering three possible cases $0 \leqslant h \leqslant m-1$, $m \leqslant h \leqslant 2m-1$ and $2m \leqslant h < 3m-1$, using the definition of $\hat{A}$ and $\hat{C}$ (remember that on $\theta^h U_m$ with $m \leqslant h \leqslant 2m-1$ we have $\|\hat{C}(\omega)\| \leqslant \delta/m$) one can show that

$$\sum_{k=n-h-1}^{n-1} \|\Phi_{\hat{A}}(n-k-1,\theta^{k+1}\omega)P_{\theta^{k+1}\omega}\hat{C}(\theta^k\omega)\| \leqslant 5\delta. \quad (18)$$

From (17)–(18) it follows (13). The inequality (14) can be proved in a similar way. Thus, the claim is proved.

Now, let $\mathbf{B}$ denote the Banach space of all bounded matrix-valued function $f : \mathbb{N} \longrightarrow Mat(2,\mathbb{R})$, where $Mat(2,\mathbb{R})$ is the space of all two-by-two matrices with matrix norm, with the norm

$$\|f\|_{\mathbf{B}} = \sup_{n\in\mathbb{N}} \|f(n)\|.$$

For every $\omega \in \Omega$ we define a mapping $T_\omega : \mathbf{B} \to \mathbf{B}$ by

$$(T_\omega f)(n) = \Phi_{\hat{A}}(n,\omega)P_\omega + \sum_{k=0}^{n-1} \Phi_{\hat{A}}(n,\omega)P_\omega\Phi_{\hat{A}}^{-1}(k+1,\omega)\hat{C}(\theta^k\omega)f(k)$$
$$- \sum_{k=n}^{\infty} \Phi_{\hat{A}}(n,\omega)(I-P_\omega)\Phi_{\hat{A}}^{-1}(k+1,\omega)\hat{C}(\theta^k\omega)f(k).$$

By the definition of $\hat{A}$ and P_ω we have $\|\Phi_{\hat{A}}(n,\omega)P_\omega\| \leqslant 1$ for all $n \in \mathbb{N}$, $\omega \in \Omega$. Therefore, due to (13)–(14) the mapping T_ω is well-defined and depends measurably on $\omega \in \Omega$. Moreover, for every $f_1, f_2 \in \mathbf{B}$ we have

$$\|T_\omega f_1 - T_\omega f_2\|_{\mathbf{B}} \leqslant (14+2\sqrt{2})\delta\|f_1-f_2\|_{\mathbf{B}} < \frac{1}{2}\|f_1-f_2\|_{\mathbf{B}},$$

hence T_ω is a contraction mapping for all $\omega \in \Omega$. By the contraction principle, the mapping T_ω has a unique fixed point which depends measurably on $\omega \in \Omega$, too. Denoting this point by Y_ω we have

$$\begin{aligned} Y_\omega(n) \quad &= \quad \Phi_{\hat{A}}(n,\omega)P_\omega + \sum_{k=0}^{n-1} \Phi_{\hat{A}}(n,\omega)P_\omega \Phi_{\hat{A}}^{-1}(k+1,\omega)\hat{C}(\theta^k\omega)Y_\omega(k) \\ &\quad - \sum_{k=n}^{\infty} \Phi_{\hat{A}}(n,\omega)(I-P_\omega)\Phi_{\hat{A}}^{-1}(k+1,\omega)\hat{C}(\theta^k\omega)Y_\omega(k). \end{aligned} \tag{19}$$

From this formula we derive $Y_\omega(n+1) = \hat{B}(\theta^n\omega)Y_\omega(n)$. Since $Y_\omega(n)P_\omega$ is also a fixed point of T_ω we get $Y_\omega(n)P_\omega = Y_\omega(n)$. Put $Q_\omega := Y_\omega(0)$ then $Q_\omega P_\omega = Q_\omega$. It is easily seen that $Y_\omega Q_\omega$ satisfies (19), hence it is also a fixed point of T_ω. Consequently, $Y_\omega Q_\omega = Y_\omega$, so $Q_\omega^2 = Q_\omega$ and Q_ω is a random projector. Set

$$M_\omega := \|Y_\omega\|_{\mathbf{B}} = \sup_{n\in\mathbb{N}} \|Y_\omega(n)\| = \sup_{n\in\mathbb{N}} \|\Phi_{\hat{B}}(n,\omega)Q_\omega\|.$$

For any $n \in \mathbb{N}$ we have

$$\begin{aligned} \|Y_\omega(n)\| \leqslant \|\Phi_{\hat{A}}(n,\omega)P_\omega\| &+ \sum_{k=0}^{n-1} \|\Phi_{\hat{A}}(n,\omega)P_\omega\Phi_{\hat{A}}^{-1}(k+1,\omega)\hat{C}(\theta^k\omega)Y_\omega(k)\| \\ &+ \sum_{k=n}^{\infty} \|\Phi_{\hat{A}}(n,\omega)(I-P_\omega)\Phi_{\hat{A}}^{-1}(k+1,\omega)\hat{C}(\theta^k\omega)Y_\omega(k)\| \\ &\leqslant 1 + (14+2\sqrt{2})\delta M_\omega, \end{aligned}$$

hence $M_\omega \leqslant 1 + (14+2\sqrt{2})\delta M_\omega$ which implies $M_\omega \leqslant (1-(14+2\sqrt{2})\delta)^{-1} < 2$.

Now we will show that the cocycle $\hat{B}$ has simple Lyapunov spectrum. We will argue by contradiction. Assume that $\hat{B}$ has one point Lyapunov exponent denoted by λ, then for any vector $0 \neq x \in \mathbb{R}^2$ we have

$$\lim_{n\to\pm\infty} n^{-1} \log \|\Phi_{\hat{B}}(n,\omega)x\| = \lambda. \tag{20}$$

Let f be a unit vector in the space $\operatorname{Im} Q_\omega$ (it exists because $Q_\omega \neq 0$), then

$$\frac{1}{n}\log\|\Phi_{\hat{B}}(n,\omega)f\| = \frac{1}{n}\log\|\Phi_{\hat{B}}(n,\omega)Q_\omega f\| = \frac{1}{n}\log\|Y_\omega(n)f\| \leqslant \frac{\log M_\omega}{n} \leqslant \frac{\log 2}{n}$$

for all $n \in \mathbb{N}$. Therefore, $\lambda \leqslant 0$, hence

$$\int_\Omega \log\det \hat{B}(\omega)d\mathbb{P}(\omega) = 2\lambda \leqslant 0. \tag{21}$$

By the construction of cocycle $\hat{A}$ we have

$$\det \hat{A}(\omega) = \begin{cases} 1 & \text{if} \quad \omega \in \Omega \setminus F, \\ 1+\frac{1}{k} & \text{if} \quad \omega \in \bigcup_{j=k}^{2k-1} \theta^j U_k, \quad k \geqslant 4. \end{cases}$$

Hence, $\int_\Omega \log\det \hat{A}(\omega)d\mathbb{P}(\omega) = \sum_{k=4}^{\infty} kx_k \log(1+\frac{1}{k})$. Since $\|\hat{B}(\omega)-\hat{A}(\omega)\| \leqslant \delta$, by the construction of $\hat{A}$ we have $|\log\det\hat{B}(\omega) - \log\det\hat{A}(\omega)| \leqslant 3\delta$. Therefore,

$$\int_\Omega \log\det \hat{B}(\omega)\mathbb{P}d\omega \geqslant \sum_{k=4}^{\infty} kx_k \log(1+\frac{1}{k}) - 3\delta > 0,$$

which contradicts (21). Thus, $\hat{B}$ has simple Lyapunov. Since B differs from $\hat{B}$ only by a scalar multiplier (scalar function) this implies that B also has simple Lyapunov spectrum, $\lambda_1(B) - \lambda_2(B) = \lambda_1(\hat{B}) - \lambda_2(\hat{B})$, and has the same Oseledets splitting as $\hat{B}$.

Next we estimate the difference $\|Q_\omega - P_\omega\|$. Setting $n = 0$ in (19) we get

$$Q_\omega = P_\omega - \sum_{k=0}^{\infty} (I - P_\omega)\Phi_{\hat{A}}(k+1,\omega)\hat{C}(\theta^k\omega)Y_\omega(k).$$

Hence, by (14),

$$\|Q_\omega - P_\omega\| \leqslant M_\omega \sum_{k=0}^{\infty} \|(I - P_\omega)\Phi_{\hat{A}}(k+1,\omega)\hat{C}(\theta^k\omega)\| \leqslant M_\omega(7+\sqrt{2})\delta < 17\delta.$$

Now, since B has simple Lyapunov exponent, we denote by $\lambda_1(B) > \lambda_2(B)$ the Lyapunov exponents and by $\mathbb{R}^2 = E_1(\omega) \oplus E_2(\omega)$ the Oseledets splitting of B. It is easily seen that $E_2(\omega) = \operatorname{Im} Q_\omega$. Choose measurably unit vector $f_2(\omega)$ in the space $E_2(\omega)$ then

$$\lim_{n\to\infty} \frac{1}{n} \log \|\Phi_B(n,\omega) f_2(\omega)\| = \lambda_2(B).$$

In fact we can choose $f_2(\omega) = \alpha(\omega)e_1 + \beta(\omega)e_2$ with measurable functions $\alpha : \Omega \to \mathbb{R}$, $\beta : \Omega \to [0,\infty)$ and $\alpha(\omega)^2 + \beta(\omega)^2 = 1$. Set $x(\omega) := \|A_0(\omega)e_2\|$ and $y(\omega) := \|B(\omega)f_2(\omega)\|$. Using the Birkhoff's theorem we have

$$\lambda_2(A_0) = \int_\Omega \log x(\omega) d\mathbb{P}(\omega), \quad \lambda_2(B) = \int_\Omega \log y(\omega) d\mathbb{P}(\omega).$$

By the construction of A_0 we have

$$x(\omega) = \begin{cases} \frac{1}{2} & \text{if} \quad \omega \in \Omega \setminus F, \\ k & \text{if} \quad \omega \in \bigcup_{j=k}^{2k-1} \theta^j U_k, \quad k \geqslant 4. \end{cases}$$

Therefore,

$$\lambda_2(A_0) = -(1 - \mathbb{P}(F)) \log 2 + \sum_{n=4}^{\infty} n x_n \log n.$$

By definition of P_ω, since $\|Q_\omega - P_\omega\| \leqslant 17\delta$ we have $|\alpha(\omega)| \leqslant \|P_\omega - Q_\omega\| \leqslant 17\delta$ and $0 < \beta(\omega) \leqslant 1 - 17\delta$. Therefore, $\|f_2(\omega) - e_2\| \leqslant \sqrt{2} \cdot 17\delta < 30\delta$ (note that if P is not orthogonal then instead of e_1 we should use an unit vector from $\ker P_\omega$, and we still have $|\alpha(\omega)| \leqslant 17\delta$, $|1 - \beta(\omega)| \leqslant 17\delta$ but $\|f_2(\omega) - e_2\| \leqslant 2 \cdot 17\delta$). Since $|x(\omega) - y(\omega)| \leqslant \|A_0(\omega)e_2 - B(\omega)f_2(\omega)\|$, for every $\omega \in \Omega$ we have

$$\begin{aligned} |x(\omega) - y(\omega)| &\leqslant \|A_0(\omega)e_2 - A_0(\omega)f_2(\omega)\| + \|A_0(\omega)f_2(\omega) - B(\omega)f_2(\omega)\| \\ &\leqslant \|A_0(\omega)\|\|e_2 - f_2(\omega)\| + \delta \\ &\leqslant \|A_0(\omega)\| \cdot 30\delta + \delta = (30\|A_0(\omega)\| + 1)\delta. \end{aligned}$$

Consequently,

1. For $\omega \in \Omega \setminus F$ we have $\|A_0(\omega)\| = 2$, $|x(\omega)| = \frac{1}{2}$, and
$$\frac{1}{2} - 60\delta - \delta \leqslant y(\omega) \leqslant \frac{1}{2} + 60\delta + \delta.$$
2. For $\omega \in \bigcup_{j=k}^{2k-1} \theta^j U_k$, $k \geqslant 4$, we have $\|A_0(\omega)\| = k+1$, $|x(\omega)| = k$, and
$$k - 30(k+1)\delta - \delta \leqslant y(\omega) \leqslant k + 30(k+1)\delta + \delta.$$

Therefore, we obtain
$$\lambda_2(B) \geqslant (1 - \mathbb{P}(F)) \log(\frac{1}{2} - 61\delta) + \sum_{n=4}^{\infty} n x_n \log(n - (30n+31)\delta),$$
and
$$\lambda_2(B) \leqslant (1 - \mathbb{P}(F)) \log(\frac{1}{2} + 61\delta) + \sum_{n=4}^{\infty} n x_n \log(n + (30n+31)\delta).$$
From these inequalities, using the fact that for any $a > 0$, $0 < x < a/4$ the inequalities $\log(a+x) < \log a + x/a$ and $\log(a-x) > \log a - 2x/a$ hold, we get
$$\lambda_2(A_0) - 2(122+3)\delta \leqslant \lambda_2(B) \leqslant \lambda_2(A_0) + (122+3)\delta.$$
It implies that the Lyapunov exponent $\lambda_2(\cdot)$ is continuous at A_0.

Now, note that if we have another cocycle B' which also satisfies $\rho(A_0, B') < \delta$, and B' is close to B, then B' also has simple Lyapunov spectrum and the corresponding random projector Q'_ω of B' (onto its stable subspace) is close to the above random projector Q_ω of B. By the same arguments as that for proving $\lambda_2(B)$ is close to $\lambda_2(A_0)$ above, we can show that $\lambda_2(B')$ is close to $\lambda_2(B)$, hence $\lambda_2(\cdot)$ is continuous at B.

The continuity of $\lambda_2(\cdot)$ implies the continuity of $\lambda_1(\cdot)$ because they add up to the exponent of the determinant, but the exponent of the determinant is easily seen continuous in $(\mathcal{G}_{IC}(2), \rho)$. □

Theorem 3.2. *There exist $A \in \mathcal{G}_{IC}(2)$ and $\varepsilon > 0$ such that every cocycle $B \in \mathcal{G}_{IC}(2)$ satisfying $\rho(A, B) < \varepsilon$ has simple Lyapunov spectrum but has no exponentially separated splitting. Moreover, the functions $\lambda_i(\cdot) : \mathcal{G}_{IC}(2) \to \mathbb{R}$, $B \mapsto \lambda_i(B)$, $i = 1, 2$, are continuous on the ball centered at A with radius ε in $(\mathcal{G}_{IC}(2), \rho)$.*

Proof. Take $A = A_0$ and $\varepsilon = \delta$ with A_0 and δ provided by Proposition 3.1. Thanks to Proposition 3.1, it remains only to prove that any $B \in \mathcal{G}_{IC}(2)$ satisfying $\rho(A, B) < \varepsilon$ has no exponentially separated splitting. Indeed, for any fixed positive numbers α, K we can find $n > 4$ such that $(n+2)^n < Ke^{\alpha n}(n-1)^n$ and, like the proof of no exponential separation of A_0 above, we see that B has no exponentially separated splitting. □

Note that Theorem 3.2 can be easily generalized to the d-dimensional case. We conclude this paper by mention an open problem: Is simple Lyapunov spectrum generic in $(\mathcal{G}_{IC}(d), \rho)$?

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