

H_∞ control problem of linear time-varying systems via controllability approach

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Abstract

This paper considers a H_∞ control problem of a class of linear uncertain time-varying systems via controllability approach. We show that the solution to this problem can be verified by the global null-controllability of a linear control system. The feedback stabilizing controller for the problem is constructed via the solution of a Riccati differential equation.

Key words. Time-varying system, H_∞ control, stabilization, controllability, Riccati equation.

1 Introduction

In the last decade, the H_∞ control problem for linear uncertain systems has attracted increasing attention (see, for instance, [2, 9, 17, 19] and the references therein), and still many questions remain unsolved. The standard H_∞ control problem is to find conditions that guarantee the existence of a feedback controller stabilizing given system and satisfies a prescribed γ -suboptimal level on perturbations/uncertainties. There are some solution techniques have been proposed for the problem. In the H_∞ control problem for linear autonomous systems, the appropriate methods make use of Lyapunov-Krasovskii function approach and the sufficient conditions are obtained via solving either linear matrix inequalities (LMIs) [5, 11], or algebraic Riccati-type equations [17, 19]. However, this approach may not be readily applied to time-varying systems due to the fact that the solution of depending on the infinite time LMIs in general can not be solved. For linear time-varying systems the investigation of the

stability and control problem becomes more complicated. For instance, in contrast to linear autonomous systems, the stability (and instability) of linear time-varying systems may not be determined from the spectral property of their system matrix $A(t)$. It was shown that the real parts of eigenvalues of system matrix $A(t)$ for every t are negative does not imply the asymptotic stability, and there is a linear time-varying system stable with positive eigenvalues. In spite of this, for linear time-varying systems, in the literature to date, there are several papers dealing with stability and control problem. Stabilization problem of linear time-varying systems has been investigated in [10, 12, 14] via Riccati differential equations. In order to find H_∞ controller for linear time-varying systems, the state-space approach is used in [16]. The paper [4] proposed H_∞ control conditions in terms of the solution of two couple Riccati differential equations. However, the problem of existence of solutions of Riccati differential equations is still under active investigations. One of the efficient approaches to this problem is the controllability approach introduced by Kalman [6]. Some sufficient conditions for global stabilization of linear time-varying systems using controllability assumption are given in [13, 15, 18]. The controllability approach was also used in [3, 10] as a systematic method for solving the solution of partial differential equations. To the best of our knowledge, surprisingly few conditions have so far been established for the H_∞ control of uncertain linear time-varying systems. Therefore, finding new conditions for the problem is of interest.

In this paper, we propose a controllability approach for studying the H_∞ control problem of linear uncertain time-varying systems. The feature of our paper is twofold. Firstly, we show that the solution to this problem can be verified by the global null-controllability of a linear control system. Then, we construct feedback stabilizing controllers for the problem via the solution of a Riccati differential equation (RDE).

The paper is organized as follows. Section 2 introduces the main notations, definitions and some auxiliary propositions needed for the proofs. The main result and an illustrated example of the result are given in Section 3. The paper ends with conclusions and cited references.

2 Preliminaries

The following notations will be used throughout this paper. R^+ denotes the set of all non-negative real numbers; R^n denotes a n -dimensional Euclidean space with the norm $\|\cdot\|$ and the inner product $\langle \cdot, \cdot \rangle$; $L_2([t, \infty), R^n)$ denotes the set of all strongly measurable L_2 -integrable R^n -valued functions on $[t, \infty)$; I denotes the identity matrix. A matrix $Q \in M^{n \times n}$ is called non-negative definite ($Q \geq 0$) if $\langle Qx, x \rangle \geq 0$, for all $x \in R^n$. If for some $c > 0$, $\langle Qx, x \rangle \geq c\|x\|^2$ for all $x \in R^n$, then Q is called positive definite ($Q > 0$). $A \geq B$ means $A - B \geq 0$; A matrix function $Q(t)$ is uniformly positive definite ($Q(t) \gg 0$) if

$$\exists c > 0 : \quad \langle Q(t)x, x \rangle \geq c\|x\|^2, \quad \forall (t, x) \in R^+ \times R^n.$$

Matrix A is called symmetric if $A = A^T$. It is well known that, if the matrix A is symmetric positive definite, then there is a matrix B such that $A = B^2$ and the matrix B is usually defined by $B = A^{\frac{1}{2}}$. $BM^+(0, \infty)$ denotes the set of all symmetric non-negative definite matrix functions, which are continuous and bounded on R^+ ; $BMU^+(0, \infty)$ denotes the set of all symmetric uniformly positive definite matrix functions, which are continuous and bounded in $t \in R^+$.

Consider the following uncertain linear time-varying system

$$\begin{aligned}\dot{x}(t) &= A(t)x(t) + B(t)u(t) + B_1(t)w(t), & t \in R^+, \\ z(t) &= C(t)x(t) + D(t)u(t), & x(0) = x_0,\end{aligned}\tag{2.1}$$

where $x \in R^n$ is the state; $u \in R^m$ is the control; $w \in R^p$ is the uncertain input, $z \in R^q$ is the observation output; $A(t), B(t), B_1(t), C(t), D(t)$ are given matrix functions continuous and bounded on R^+ . In the sequel, we say that the control $u(t)$ is admissible if $u(t) \in L_2(0, t), R^m$ for every $t \geq 0$, and the uncertainty $w(\cdot)$ is admissible if $w(\cdot) \in L_2([0, \infty), R^p)$. For every initial state $x_0 \in R^n$, for every admissible control $u(t)$, and admissible uncertainties $w(t)$ linear control system (2.1) has a solution given by

$$x(t) = U(t, 0)x_0 + \int_0^t U(t, s)[B(s)u(s) + B_1(s)w(s)]ds,$$

where $U(t, s)$ is the fundamental matrix solution of the linear time-varying system

$$\dot{x}(t) = A(t)x(t), \quad x(0) = x_0.$$

Let us first introduce basic theorems of asymptotic stability and stabilization from [20, 21]. Consider a non-autonomous differential equation of the form

$$\dot{x}(t) = f(t, x(t)), \quad t \in R^+, \tag{2.2}$$

where $f(t, x) : R^+ \times R^n \rightarrow R^n, f(t, 0) = 0, t \geq 0$, is a given function such that the system has a unique solution $x(t, x_0)$ for every initial condition $x(0) = x_0$.

Proposition 2.1. The zero solution of system (2.2) is uniformly asymptotically stable if there exist numbers $\lambda_1, \lambda_2, \lambda_3 > 0$ and a Lyapunov function $V(t, x) : R^+ \times R^n \rightarrow R^+$, such that

- (i) $\lambda_1 \|x\|^2 \leq V(t, x) \leq \lambda_2 \|x\|^2, \quad \forall (t, x) \in R^+ \times R^n$.
- (ii) $\dot{V}_f(t, x(t)) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x(t)) \leq -\lambda_3 \|x(t)\|^2$, for all solution $x(t)$ of the system.

Definition 2.1. Linear control system (2.1), where $w(\cdot) = 0$, is stabilizable if there exist a feedback control $u(t) = h(x(t))$, where $h(\cdot) : R^n \rightarrow R^m$ is to be defined as a feedback control function, such that the zero solution of the closed-loop system

$$\dot{x}(t) = A(t)x(t) + B(t)h(x(t)), \tag{2.3}$$

is uniformly asymptotically stable.

The standard H_∞ control problem for the system (2.1) is concerned with a stabilization problem and the existence of a γ -suboptimal level problem. In this paper, we consider the following H_∞ control problem, which guarantees the existence of a γ -suboptimal controller under the nonzero initial condition.

Definition 2.2. Given $\gamma > 0$. The H_∞ control problem for the system (2.1) has a solution if there is a feedback control $u(t) = h(x(t))$ such that

(i) The closed-loop system (2.3) is uniformly asymptotically stable, i.e., the control system (2.1), where $w(t) = 0$, is stabilizable.

(ii) There is a number $c_0 > 0$ such that

$$\sup \frac{\int_0^\infty \|z(t)\|^2 dt}{c_0 \|x_0\|^2 + \int_0^\infty \|w(t)\|^2 dt} \leq \gamma, \quad (2.4)$$

where the supremum is taken over all initial states x_0 and non-zero admissible uncertainties $w(t)$. In this case we say that the feedback control $u(t) = h(x(t))$ stabilizes the system (2.1).

In the sequel, we recall the concept of global controllability, which is concerned with the possibility of steering any state to an another state of the system in finite time. Consider the following linear time-varying control system, briefly denoted by $[A(t), B(t)]$,

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad t \in R^+. \quad (2.5)$$

Definition 2.3. System $[A(t), B(t)]$ is globally null-controllable (GNC) in a finite time $T > 0$ if for every initial state x_0 , there is an admissible control $u(t)$ such that the solution $x(t)$ of the system satisfies:

$$x(0) = x_0, \quad x(T) = 0.$$

The following controllability criterion given in [7] will be used later.

Proposition 2.2. *Assume that the matrix functions $A(t), B(t)$ are analytic on R^+ . The system $[A(t), B(t)]$ is GNC in some finite time if*

$$\exists t_0 > 0 : \quad \text{rank}[M_1(t_0), M_2(t_0), \dots, M_n(t_0)] = n, \quad (2.6)$$

where

$$M_1(t) = B(t), \quad M_k(t) = -A(t) + \frac{d}{dt}M_{k-1}(t), \quad k = 2, \dots, n-1.$$

Associated with the control system (2.5), we consider the following RDE

$$\dot{P}(t) + A^T(t)P(t) + P(t)A(t) - P(t)B(t)B^T(t)P(t) + Q(t) = 0. \quad (2.7)$$

Proposition 2.3. [12] *If system $[A(t), B(t)]$ is globally null-controllable in some finite time, then for any matrix $Q \in BM^+(0, \infty)$, the RDE (2.7) has a solution $P \in BM^+(0, \infty)$.*

Proposition 2.4 For any matrix function $A(t)$ bounded on R^+ , there exists $Q \in BM^+(0, \infty)$ such that $Q(t) - A(t) \geq 0$.

Proof. By the same arguments used in the proof of Proposition 2.4 in [14], the matrix $Q(t)$ is chosen as

$$Q(t) = \text{diag}\{q_1(t), q_2(t), \dots, q_n(t)\},$$

where $q_i(t) \geq \max\{|q_i^0(t)|, 0\}$ and

$$q_i^0(t) = a_{ii}(t) + \frac{1}{4} \sum_{j \neq i}^n a_{ij}^2(t) + n - 1, i = 1, 2, \dots, n.$$

We conclude this section with the following well-known technical results for later use.

Proposition 2.5. Let Q, S are symmetric matrices of appropriate dimensions and $S > 0$. Then

$$2\langle Qy, x \rangle - \langle Sw, w \rangle \leq \langle QS^{-1}Q^T x, x \rangle, \quad \forall(x, y, w).$$

Proposition 2.6. The matrix $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$, where $a > 0$ or $c > 0$, is positive definite if $b^2 < ac$.

3 Main result

In this section, sometimes for the sake of brevity, we will omit the variable t of matrix functions, if it does not cause any confusion.

Consider linear control system (2.1). For the sake of technical simplification, without loss of generality, as in [11, 16, 19] we assume that

$$D^T(t)[D(t) \quad C(t)] = [I \quad 0], \quad t \geq 0. \quad (3.1)$$

Given $\gamma > 0$. Let us set

$$A_\gamma(t) = A(t) + \frac{1}{\gamma} B_1(t) B_1^T(t) - B(t) B^T(t), \quad B_\gamma(t) = \sqrt{B(t) B^T(t) - \frac{1}{\gamma} B_1(t) B_1^T(t)}.$$

The following assumption will be used in the proof of the main theorem.

A. $B(t) B^T(t) - \frac{1}{\gamma} B_1(t) B_1^T(t) > 0, \quad t \geq 0$.

Theorem 3.1. Assume that the condition A holds and linear control system $[A_\gamma(t), B_\gamma(t)]$ is globally null-controllable in some finite time. Then the H_∞ control problem for the system (2.1) has a solution. Moreover, the feedback stabilizing control is

$$u(t) = -B^T(t)[P(t) + I]x(t), \quad t \in R^+,$$

where $P \in BM^+(0, \infty)$ is a solution of RDE

$$\dot{P} + A_\gamma^T P + P A_\gamma - P B_\gamma B_\gamma^T P + Q = 0, t \in R^+, \quad (3.2)$$

and $Q(t) \geq 0$ is a matrix function satisfying

$$Q(t) \geq A(t) + A^T(t) + C^T(t)C(t) + \epsilon I, t \in R^+,$$

for some $\epsilon > 0$.

Proof. The proof of the theorem is based on the following lemma.

Lemma 3.1. *The H_∞ control problem for the system (2.1) has a solution if there exist matrix functions $X \in BMU^+(0, \infty)$, $R \in BMU^+(0, \infty)$, such that*

$$\dot{X} + A^T X + X A - X [B B^T - \frac{1}{\gamma} B_1 B_1^T] X + C^T C + R \leq 0, \quad t \geq 0. \quad (3.3)$$

The feedback control is defined as

$$u(t) = -B^T(t)X(t)x(t), \quad t \geq 0. \quad (3.4)$$

Proof. Using the feedback control (3.4), we consider the following Lyapunov function for the closed-loop system (2.3), where $w(t) = 0$, $h(x(t)) = -B^T(t)X(t)x(t)$:

$$V(t, x) = \langle X(t)x, x \rangle.$$

Since $X \in BMU^+(0, \infty)$, the condition (i) of Proposition 2.1 holds. To verify the condition (ii), taking the derivative of $V(t, \cdot)$ along the solution $x(t)$ of the closed-loop system, we have

$$\begin{aligned} \dot{V}(t, x(t)) &= \langle \dot{X}(t)x(t), x(t) \rangle + 2\langle X(t)\dot{x}(t), x(t) \rangle \\ &= \langle (\dot{X} + A^T X + X A)x(t), x(t) \rangle + 2\langle X B u(t), x(t) \rangle \\ &= \langle (\dot{X} + A^T X + X A)x(t), x(t) \rangle - 2\langle X B B^T X x(t), x(t) \rangle. \end{aligned}$$

Using RDE (3.3) we have

$$\begin{aligned} \dot{V}(t, x(t)) &\leq -\frac{1}{\gamma} \langle X B_1 B_1^T X x(t), x(t) \rangle - \langle X B B^T X x(t), x(t) \rangle \\ &\quad - \langle C^T C x(t), x(t) \rangle - \langle R x(t), x(t) \rangle. \end{aligned} \quad (3.5)$$

Since

$$\langle C^T C x(t), x(t) \rangle \geq 0, \quad \langle X B B^T X x(t), x(t) \rangle \geq 0, \quad \langle X B_1 B_1^T X x(t), x(t) \rangle \geq 0, \quad \forall t \in R^+,$$

and $R(t) \gg 0$, from (3.5) it follows that there is a number $\lambda > 0$ such that

$$\dot{V}(t, x(t)) \leq -\langle R(t)x(t), x(t) \rangle \leq -\lambda \|x(t)\|^2, \quad \forall t \in R^+,$$

which implies the second condition (ii) of Proposition 2.1, and hence the closed-loop system is uniformly asymptotically stable, i.e., the system is stabilizable. To complete the proof of the theorem, it remains to show the γ -suboptimal condition (2.4). For this, we consider the relation

$$\int_0^t [\|z(s)\|^2 - \gamma\|w(s)\|^2] ds = \int_0^t [\|z(s)\|^2 - \gamma\|w(s)\|^2 + \dot{V}(s, x(s))] ds - \int_0^t \dot{V}(s, x(s)) ds,$$

where $\dot{V}(t, x(t))$ is estimated as

$$\begin{aligned} \dot{V}(t, x(t)) &\leq -\lambda\|x(t)\|^2 - \langle C^T C x(t), x(t) \rangle - \langle X B B^T X x(t), x(t) \rangle \\ &\quad - \frac{1}{\gamma} \langle X B_1 B_1^T X x(t), x(t) \rangle + 2 \langle X B_1 w(t), x(t) \rangle. \end{aligned} \quad (3.6)$$

Since $V(t, x(t)) \geq 0$, we have

$$\int_0^t \dot{V}(s, x(s)) ds = V(t, x(t)) - V(0, x(0)) \geq -V(0, x(0)) := -\langle X(0)x_0, x_0 \rangle.$$

Therefore,

$$\int_0^t [\|z(s)\|^2 - \gamma\|w(s)\|^2] ds \leq \int_0^t [\|z(s)\|^2 - \gamma\|w(s)\|^2 + \dot{V}(s, x(s))] ds + \langle X(0)x_0, x_0 \rangle. \quad (3.7)$$

Taking the estimation of $\dot{V}(s, x(s))$ from (3.6) and putting

$$\|z(t)\|^2 = \langle [C^T(t)C(t) + X(t)B(t)B^T(t)X(t)]x(t), x(t) \rangle,$$

in the inequality (3.7.), we obtain that

$$\begin{aligned} \int_0^t [\|z(s)\|^2 - \gamma\|w(s)\|^2] ds &\leq \int_0^t \left[-\lambda\|x(s)\|^2 - \frac{1}{\gamma} \langle X B_1 B_1^T X x(s), x(s) \rangle \right. \\ &\quad \left. + 2 \langle X B_1 w(s), x(s) \rangle - \gamma \langle w(s), w(s) \rangle \right] ds + \langle X(0)x_0, x_0 \rangle. \end{aligned}$$

Using Proposition 2.5 gives

$$2 \langle X B_1 w, x \rangle - \gamma \langle w, w \rangle \leq \frac{1}{\gamma} \langle X B_1 B_1^T X x, x \rangle.$$

Then,

$$\int_0^t [\|z(s)\|^2 - \gamma\|w(s)\|^2] ds \leq -\lambda \int_0^t \|x(s)\|^2 ds + \langle X(0)x_0, x_0 \rangle \leq \langle X(0)x_0, x_0 \rangle, \forall t \in R^+.$$

Letting $t \rightarrow \infty$ we finally obtain that

$$\int_0^\infty [\|z(t)\|^2 - \gamma\|w(t)\|^2] dt \leq \langle X(0)x_0, x_0 \rangle,$$

and hence

$$\int_0^\infty [\|z(t)\|^2 dt] \leq \gamma \left\{ \int_0^\infty \|w(t)\|^2 dt + \frac{\|X(0)\|}{\gamma} \|x_0\|^2 \right\}.$$

Setting $c_0 = \frac{\|X(0)\|}{\gamma}$, from the last inequality we have

$$\frac{\int_0^\infty \|z(t)\|^2 dt}{c_0 \|x_0\|^2 + \int_0^\infty \|w(t)\|^2 dt} \leq \gamma,$$

for all x_0 and non-zero $w(t) \in L_2([0, \infty), W)$. This completes the proof of the theorem.

Proof of Theorem 3.1. Assume that the system $[A_\gamma(t), B_\gamma(t)]$ is GNC in some finite time $T > 0$. Using Proposition 2.4 define the matrix function $Q \in BM^+(0, \infty)$ such that

$$Q(t) \geq A(t) + A^T(t) + C^T(t)C(t) + \epsilon I, \quad t \geq 0, \quad (3.8)$$

for some $\epsilon > 0$. Then, the RDE

$$\dot{P} + A_\gamma^T P + P A_\gamma - P B_\gamma B_\gamma^T P + Q = 0, \quad (3.9)$$

by Proposition 2.3, has a solution $P \in BM^+(0, \infty)$. The RDE (3.9) can be reformulated as

$$\dot{P} + A^T(P+I) + (P+I)A - (P+I)[BB^T - \frac{1}{\gamma}B_1B_1^T](P+I) + Q - (A^T + A) + BB^T - \frac{1}{\gamma}B_1B_1^T = 0.$$

Therefore, taking (3.8) into account, we have

$$\dot{P} + A^T(P+I) + (P+I)A - (P+I)[BB^T - \frac{1}{\gamma}B_1B_1^T](P+I) + C^T C + (BB^T - \frac{1}{\gamma}B_1B_1^T) + \epsilon I \leq 0.$$

Putting

$$X(t) = P(t) + I, \quad R(t) = [B(t)B^T(t) - \frac{1}{\gamma}B_1(t)B_1^T(t)] + \epsilon I,$$

we see that the matrices $X(t) \gg 0$ and $R(t) \gg 0$, because of the assumption A, satisfy the RDE

$$\dot{X} + A^T X + X A - X[BB^T - \frac{1}{\gamma}B_1B_1^T]X + C^T C + R \leq 0,$$

and hence the proof of the theorem is completed by using Lemma 3.1.

Remark 3.1. Note that the problem of solving Riccati differential equations is in general still complicated, however some various efficient approaches to solving this problem can be found, for instance, in [1, 8].

The following simple procedure can be applied to solve the H_∞ control problem for the system (2.1).

Step 1. Given $\gamma > 0$, find the matrices $A_\gamma(t), B_\gamma(t)$.

Step 2. Check the condition (3.1) and the assumptions A.

Step 3. Check the global null-controllability of linear system $[A_\gamma(t), B_\gamma(t)]$ by Proposition 2.2.

Step 4. Find solution $P(t)$ of RDE (3.2) and the feedback stabilizing control is given by

$$u(t) = -B^T(t)[P(t) + I]x(t), \quad t \in R^+.$$

Example 3.1. Consider system (2.1), where

$$A(t) = \begin{bmatrix} -0.5e^{-2t} - 0.75 & 1 \\ -1 & -0.5e^{-2t} - 0.25 \end{bmatrix}, \quad B(t) = B_1(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$C(t) = \begin{bmatrix} 0.5e^{-2t} & -0.5e^{-2t} \\ -0.5e^{-2t} & 0.5e^{-2t} \end{bmatrix}, \quad D(t) = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}.$$

The assumption (3.1) holds:

$$D^T(t)D(t) = 1, \quad C^T(t)D(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Taking $\gamma = \frac{4}{3}$, we have

$$(i) \quad BB^T - 0.75B_1B_1^T = \begin{bmatrix} 0.25 & 0 \\ 0 & 0 \end{bmatrix} > 0.$$

$$(ii) \quad B_\gamma = \sqrt{BB^T - 0.75B_1B_1^T} = \begin{bmatrix} 0.5 & 0 \\ 0 & 0 \end{bmatrix}.$$

$$(iii) \quad A_\gamma = A - BB^T + 0.75B_1B_1^T = \begin{bmatrix} -0.5e^{-2t} - 1 & 1 \\ -1 & -0.5e^{-2t} - 0.25 \end{bmatrix}.$$

(iv) It is clear that both matrix functions $A_\gamma(t)$ and $B_\gamma(t)$ are analytic. Moreover,

$$M_1(t) = B_\gamma(t), \quad M_2(t) = -A_\gamma(t),$$

and

$$\text{rank}[M_1(t), M_2(t)] = \text{rank} \left[\begin{array}{cc|cc} 0.25 & 0 & 1 + 0.5e^{-2t} & -1 \\ 0 & 0 & 1 & 0.25 + 0.5e^{-2t} \end{array} \right]$$

and it's easy to verify the controllability condition (2.6): there exists $t_0 > 0$ so that $\text{rank}[B_\gamma(t_0), -A_\gamma(t_0)] = 2$. Thus, by Proposition 2.2 the system $[A_\gamma, B_\gamma]$ is *GNC* in some finite time. Taking $\epsilon = 0.5$ and

$$Q(t) = \begin{bmatrix} e^{-2t} + 2.25 & -0.5 \\ -0.5 & 0.5e^{-2t} + 0.25 \end{bmatrix},$$

it is easy to verify by Proposition 2.6 that $Q(t) \geq 0$ and

$$Q(t) - (A(t) + A^T(t) + C^T(t)C(t) + 0.5I) = \begin{bmatrix} 3.25 + 1.5e^{-2t} & -0.5 + 0.5e^{-2t} \\ -0.5 + 0.5e^{-2t} & 1 + e^{-2t} \end{bmatrix} \geq 0, t \geq 0.$$

Finally, it's straightforward to show that the RDE

$$\dot{P} + A_\gamma^T P + P A_\gamma - P B_\gamma B_\gamma^T P + Q = 0,$$

has a solution

$$P(t) = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} > 0.$$

Therefore, we conclude that the H_∞ control problem has a solution and the feedback stabilizing control is

$$u(t) = -B^T(t)[P(t) + I]x(t) = -[2 \quad 0]x(t),$$

where $x(t) = [x_1(t) \quad x_2(t)]$.

4 Conclusions

In this paper, we have shown that the H_∞ control conditions for linear time-varying systems has a solution if some appropriate linear control system is globally null-controllable. The feedback stabilizing controller is designed via the solution of a Riccati differential equation.

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