THE MIXED PROBLEMS FOR THE NAVIER- STOKES EQUATIONS

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Abstract. There are the papers devoted to the investigation of solutions of Navier-Stokes equations (see, eg., [1-3]). In this publication the mixed problems for the Navier- Stokes equations are considered.

1. Introduction

Let G be an open set in $\mathbb{R}^n$ with the boundary ∂G. The Navier-Stokes equations describe the motion of an incompressible fluid are given by (see [2], [4])

$$\frac{\partial u_k}{\partial t} + \sum_{j=1}^{n} u_j \frac{\partial u_k}{\partial x_j} - \nu \Delta u_k + \frac{\partial p}{\partial x_k} = f_k(x,t), (x,t) \in \Omega \tag{1}$$

$$divu = 0, (x,t) \in \Omega, \tag{2}$$

here $u(x,t) = (u_k(x,t))$ is the velocity vector and $p(x,t)$ is the pressure of the fluid, $f(x,t) = (f_k(x,t)), k = \overline{1,n}$, is the externally applied force, ν is viscosity coefficient, $\Omega \equiv G \times (0,\infty)$.

We will consider the following mixed problems: in addition to (1), (2) on Ω, one requires the initial conditions

$$u_k \mid_{t=0} = a_k^0 u^0(x), divu(x,0) = 0, x \in G, \tag{3}$$

and the boundary condition

$$u_k \mid_\Gamma = a_k^1 U^1(x,t), \tag{4}$$

or

$$\frac{\partial u_k}{\partial n}\,|_\Gamma = a_k^2 U^2(x,t), \tag{5}$$

here $\Gamma \equiv \partial G \times (0,\infty), a_k^l (l = 0,1,2; k = \overline{1,n})$ being the constants.

2. The solution of mixed problems

Consider now the generalized solutions of the problems (1), (2), (3), (4) and (1), (2), (3), (5). Let $\varphi(x,t) \in \mathscr{D}\ (\Omega)\,, \mathscr{D}\ (\Omega)$ being the Schwartz's space of test functions on Ω (see [5]). Multiplying equation (1) by φ, integrating the result over Ω and taking into account (2) we obtain

$$\int\limits_{\mathbb{R}^{n+1}} \left(\frac{\partial u_k^*}{\partial t} + \sum_{j=1}^{n} u_j^* \frac{\partial u_k^*}{\partial x_j} - \nu\Delta u_k^* + \frac{\partial p^*}{\partial x_k}\right)\varphi(x,t)dt =$$

$$\int\limits_{\mathbb{R}^{n+1}} f_k^*\varphi dxdt + \int\limits_G u_k(x,0)\varphi(x,0)dx + \int\limits_0^\infty\int\limits_{\partial G} \nu\frac{\partial u_k^*}{\partial n}\varphi dsdt - \int\limits_0^\infty\int\limits_{\partial G} \nu u_k^* \frac{\partial \varphi}{\partial n} dsdt. \tag{6}$$

Note that by [6- 8] the solutions $u_k(x,t)$ of the system (1), (2) satisfy the equalities (except for special cases which were discussed in [7], [8]):

$$\left\langle \sum_{j=1}^{n} u_j^* \frac{\partial u_k^*}{\partial x_j}, \varphi \right\rangle = 0, \varphi \in \mathscr{D}\,(\mathbb{R}^{n+1}).$$

The integrals on the right-hand side (6) can be written in the forms of simple layers

$$\langle u_k(x,0)\times\delta(t), \varphi(x,t)\rangle = \int\limits_G u_k(x,0)\varphi(x,0)dx$$

$$\left\langle \left(\nu\frac{\partial u_k^*}{\partial n}\right)\delta_\Gamma, \varphi \right\rangle = \int\limits_0^\infty\int\limits_{\partial G} \nu\frac{\partial u_k^*}{\partial n}\varphi(x,t)dsdt$$

and the double layer (with density $u_k^*(x,t)$)

$$\left\langle -\nu \frac{\partial}{\partial n}\left(u_k^* \delta_\Gamma\right), \varphi \right\rangle = \int\limits_0^\infty \int\limits_{\partial G} \nu u_k^*(x,t) \frac{\partial \varphi(x,t)}{\partial n} ds dt$$

here δ_Γ being the surface distribution (see [5]).

Thus from (6) one has:

$$\frac{\partial u_k^*}{\partial t} + \sum_{j=1}^n u_j^* \frac{\partial u_k^*}{\partial x_j} - \nu \Delta u_k^* = f_k^* - \frac{\partial p^*}{\partial x_k} + a_k^0 u_k^0(x) \times \delta(t) + \left(\nu \frac{\partial u_k^*}{\partial n}\right) \delta_\Gamma + \nu \frac{\partial}{\partial n}\left(u_k^* \delta_\Gamma\right)$$

where u_k^*, f_k^* being distributions extended from u_k, f_k by zero onto $\Omega^- = \mathbb{R}^4 \backslash \overline{\Omega}$.

Then, by the propositions 3 (sec. 3.2. [6]) and propos. 1 (sec. 2, [7]), we consider the potentials of a simple layer: $v_k^0 = a_k^0 E * \left[u^0(x) \times \delta(t)\right]$, $v^2 = -2E * \left(\nu \frac{\partial u^*}{\partial n}\right) \delta_\Gamma$, and the potential of a double layer: $v^1 = -2E * \nu \frac{\partial}{\partial n}\left(u^* \delta_\Gamma\right)$, where

$$E(x,t) = \frac{\theta(t)}{(4\pi\nu t)^{n/2}} \exp\left\{-\frac{|x|^2}{4\nu t}\right\}, \tag{7}$$

here $\theta(t)$ being the Heaviside (unit) function.

One has

$$v_k^0(x,t) = \int\limits_{\mathbb{R}^n} u_k^*(x,0) E(x-y,t) dy, \tag{8}$$

$$v_k^2(x,t) = -2 \int\limits_0^t \int\limits_{\partial G} \nu \frac{\partial u_k^*(y,\tau)}{\partial n_y} E(x-y,t-\tau) dS_y d\tau, \tag{9}$$

$$v_k^1(x,t) = -2 \int\limits_0^t \int\limits_{\partial G} \nu u_k^*(y,\tau) \frac{\partial E(x-y,t-\tau)}{\partial n_y} dS_y d\tau, \tag{10}$$

and

$$E(x,t) \longrightarrow \delta(x), t \longrightarrow +0, \text{ in } \mathscr{S}'(\mathbb{R}^n), \tag{11}$$

$$\int\limits_{\mathbb{R}^n} E(x,t)dx = 1, t > 0, \tag{12}$$

here $\mathscr{S}'(\mathbb{R}^n)$ being the space of temperate distributions.

Next we will show that using the potentials one can find the solutions of initial boundary problems.

For simplicity of presentation let $f_k = 0$, $p(x,t) = h(t)$ ($h(t)$ being an arbitrary function) and

$\Omega = R^3_+(x) \times R_+(t) = \{(x,t) : -\infty < x_i < +\infty, i = 1, 2, x_3 > 0, t > 0\}.$

2.1. The mixed problem (1)-(4)

In this case

$$\Gamma = \{(x,t) : -\infty < x_1, x_2 < +\infty, x_3 = 0, t > 0\}$$

then

$$v_k^1(x,t) = -2\int\limits_{\Gamma} \nu u_k(y,\tau)\frac{\partial E(x_1-y_1, x_2-y_2, x_3, t-\tau)}{\partial x_3}dy_1dy_2d\tau =$$

$$\frac{x_3 a_k^1}{(4\pi\nu)^{3/2}}\int\limits_0^t \frac{e^{-\frac{x_3^2}{4\nu(t-\tau)}}}{(t-\tau)^{5/2}}d\tau \int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} U^1(y_1,y_2,\tau)e^{-\frac{(x_1-y_1)^2+(x_2-y_2)^2}{4\nu(t-\tau)}}dy_1dy_2, \tag{13}$$

$$\frac{1}{(4\pi\nu)^{3/2}}\int\limits_0^\infty \frac{x_3 e^{-\frac{x_3^2}{4\nu\tau}}}{\tau^{5/2}}d\tau \int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} e^{-\frac{y_1^2+y_2^2}{4\nu\tau}}dy_1dy_2 = 1.$$

Hence,

$$\lim_{x_3 \longrightarrow 0} v_k^1(x,t) = a_k^1 U^1(x_1,x_2,t), \tag{14}$$

$$\lim_{t \longrightarrow 0, x \in \mathbb{R}_+^3} v_k^1(x,t) = 0. \tag{15}$$

Continue now the function u_k^0 onto

$$R_-^3(x) = \{x : -\infty < x_1, x_2 < +\infty, x_3 < 0\}$$

by putting

$$u^0(x_1,x_2,-x_3) = -u^0(x_1,x_2,x_3), x \in \mathbb{R}_+^3.$$

Then one can write the potential v^0 (8) in the form

$$v_k^0(x,t) = \frac{a_k^0}{(4\pi\nu t)^{3/2}} \int_{-\infty}^{+\infty} dy_1 \int_{-\infty}^{+\infty} dy_2 \int_0^{+\infty} u^0(y_1,y_2,y_3) \left[e^{-\frac{(x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2}{4\nu t}} - \right.$$
$$\left. -e^{-\frac{(x_1-y_1)^2+(x_2-y_2)^2+(x_3+y_3)^2}{4\nu t}} \right] dy_3 \tag{16}$$

By (11), (12) and (16) one has

$$\lim_{t \longrightarrow 0} v_k^0(x,t) = a_k^0 u^0(x), x \in \mathbb{R}_+^3 \tag{17}$$

$$\lim_{x_3 \longrightarrow 0} v_k^0(x,t) = 0. \tag{18}$$

Consider now the functions

$$u_k(x,t) \equiv \sum_{l=0}^{1} a_k^l V^l(x,t) = \sum_{l=0}^{N} a_k^l G_l(x,t), k = 1,2,3, \tag{19}$$

where $G_l(x,t)$ being the functions defined by (4) in [8],

$$V^0(x,t) = \frac{1}{(4\pi\nu t)^{3/2}} \int_{-\infty}^{+\infty} dy_1 \int_{-\infty}^{+\infty} dy_2 \int_0^{+\infty} u^0(y) \left[e^{-\frac{(x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2}{4\nu t}} - \right.$$
$$\left. -e^{-\frac{(x_1-y_1)^2+(x_2-y_2)^2+(x_3+y_3)^2}{4\nu t}} \right] dy_3$$

$$V^1(x,t) = \frac{x_3}{(4\pi\nu)^{3/2}} \int\limits_0^t \frac{e^{-\frac{x_3^2}{4\nu(t-\tau)}}}{(t-\tau)^{5/2}} d\tau \int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} U^1(y_1,y_2,\tau) e^{-\frac{(x_1-y_1)^2+(x_2-y_2)^2}{4\nu(t-\tau)}} dy_1 dy_2,$$

$$\sum_{k=1}^{n} a_k^l \frac{\partial G_l}{\partial x_k} = 0, \sum_{k=1}^{n} a_k^m \frac{\partial G_l}{\partial x_k} = 0; l,\ m = 0,1, l \neq m. \qquad (20)$$

It is easy to verify that the functions $u_k(x,t)$ defined by (19) and (20) satisfy the system (1), (2) and the conditions (3), (4).

Thus, we obtain the following.

Theorem 1. *Let $f(x,t) = 0$ and the given functions $u_k^0(x), U_k^1(x,t)$ satisfy the conditions (20). Then there exist the solutions (p,u) of the problems (1)-(4), where $p(x,t) = h(t)$ (h(t) being an arbitrary function) and the velocity vector $u(x,t)$ can be represented in the form (19) of a sum of two potentials (16) and (13).*

By this theorem and the proposition 2 in [8] (sec. 2) , we get immediately the following.

Corollary 1. *Let the external force $f(x,t)$ be identically zero and let the initial and boundary conditions ((3) and (4)) do not satisfy the conditions (20). Then there exist no solutions (p,u) of the problems (1)-(4) (except for special cases which were discussed in [6-8]).*

Remark

1^0. Let, in (20), $G_l(x) = G_l\left(\sum\limits_{j=1}^{n} b_j^l x_j\right)$. Then, the conditions (20) have the form

$$\sum_{k=1}^{n} a_k^l b_k^l = 0, \sum_{k=1}^{n} a_k^m b_k^l = 0, l, m = 0, 1, l \neq m. \qquad (21)$$

2^0. If the initial and boundary conditions have the form

$$u_k|_{t=0} = \sum_{i=1}^{N} a_{k0}^i U_i^0(x), \; u_k|_\Gamma = \sum_{j=1}^{M} a_{k1}^j U_j^1(x,t), k = \overline{1,n}, N, M \geq 2$$

and $p = p(x,t)$, the mixed problem is analogously considered.

Example 1.

Let

$$\Omega = \left\{ (x,t) : -\infty < x_j < +\infty, 0 \leq \sum_{j=1}^{3} b_j x_j < +\infty, t > 0 \right\}.$$

Initial condition:

$$u_k(x,t)|_{t=0} = a_k^0, -\infty < x_1, x_2, x_3 < +\infty, 0 < \sum_{j=1}^{3} b_j x_j < +\infty, \qquad (22)$$

and the boundary condition:

$$u_k \Bigg|_{\sum\limits_{j=1}^{3} b_j x_j = 0} = a_k^1, -\infty < x_1, x_2, x_3 < +\infty, t > 0 \qquad (23)$$

here $a_k^l, b_j, (k = 1,2,3, l = 0,1, j = 1,2,3)$ being the constants:

By (8), (10) and (19), the solution (p, u) has the form

$$p(x,t) = h(t),$$

$$u_k(x,t) = a_k^0 \phi \left(\frac{\sum\limits_{j=1}^{3} b_j x_j}{\left(4\nu \sum\limits_{j=1}^{3} b_j^2 t \right)^{1/2}} \right) + a_k^1 \left[1 - \phi \left(\frac{\sum\limits_{j=1}^{3} b_j x_j}{\left(4\nu \sum\limits_{j=1}^{3} b_j^2 t \right)^{1/2}} \right) \right]$$

where, by (21), the constants a_k^l and b_j ($b_{j0} = b_{j1} = b_j$) satisfy the conditions:

$$\sum_{k=1}^{n} a_k^l b_k = 0, l = 0, 1$$

and $\phi(z)$ being the error integral:

$$\phi(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-\alpha^2} d\alpha.$$

For example

$$a_1^0 = 1, a_2^0 = 1, a_3^0 = 2; a_1^1 = 2, a_2^1 = 2, a_3^1 = 4, b_1 = 1, b_2 = 1, b_3 = -1$$

Then, one has the solution:

$$p(x,t) = h(t),$$
$$u_1(x,t) = u_2(x,t) = 2 - \phi\left(\frac{x_1 + x_2 - x_3}{(12\nu t)^{1/2}}\right),$$
$$u_3(x,t) = 4 - 2\phi\left(\frac{x_1 + x_2 - x_3}{(12\nu t)^{1/2}}\right).$$

If the given constants in (22) and (23) do not satisfy the condition (20):

$$a_1^0 = 1, a_2^0 = 1, a_3^0 = 2; a_1^1 = 2, a_2^1 = 2, a_3^1 = 3, b_1 = 1, b_2 = 1, b_3 = -1$$

then, in this case, there exist no solutions (p, u), ($u \not\equiv constant$) of the problem (1)-(4).

2. 2. The mixed problem (1)-(3), (5)

Let

$$\Gamma = \{(x,t) : -\infty < x_1, x_2 < +\infty, x_3 = 0, t > 0\}.$$

Then, by (9) and (5) one has

$$v_k^2(x,t) = -2\nu a_k^2 \int\limits_{\Gamma} U^2(y_1,y_2,\tau)E(x_1-y_1,x_2-y_2,x_3,t-\tau)dy_1dy_2d\tau \tag{24}$$

$$\frac{\partial v_k^2}{\partial x_3} = \frac{a_k^2 x_3}{(4\pi\nu)^{3/2}} \int\limits_0^t \frac{e^{-\frac{x_3^2}{4\nu(t-\tau)}}}{(t-\tau)^{5/2}}d\tau \int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} U^2(y_1,y_2,\tau)e^{-\frac{(x_1-y_1)^2+(x_2-y_2)^2}{4\nu(t-\tau)}}dy_1dy_2$$

Hence,

$$\lim_{x_3\longrightarrow 0}\frac{\partial v_k^2}{\partial x_3} = a_k^2 U^2(x_1,x_2,t), k=1,2,3, \tag{25}$$

$$\lim_{t\longrightarrow 0} v_k^2(x,t) = 0, x\in\mathbb{R}^3_+. \tag{26}$$

Continue the function $u_k^0(x)$ onto $\mathbb{R}^3_-$ by putting

$$u_k^0(x_1,x_2,-x_3) = u_k^0(x_1,x_2,x_3), x\in\mathbb{R}^3_+.$$

Then one may write the potential $v_k^0(x,t)$ (8) in the form

$$v_k^0(x,t) = \frac{a_k^0}{(4\pi\nu t)^{3/2}} \int\limits_{-\infty}^{\infty} dy_1 \int\limits_{-\infty}^{\infty} dy_2 \int\limits_0^{+\infty} u^0(y_1,y_2,y_3)\times$$

$$\times\left[e^{-\frac{(x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2}{4\nu t}} + e^{-\frac{(x_1-y_1)^2+(x_2-y_2)^2+(x_3+y_3)^2}{4\nu t}}\right]dy_3 \tag{27}$$

By (8), (11) and (27) one has:

$$\lim_{t\longrightarrow 0} v_k^0(x,t) = a_k^0 u^0(x), x\in\mathbb{R}^3_+, \tag{28}$$

$$\lim_{x_3\longrightarrow 0}\frac{\partial v_k^0}{\partial x_3} = 0, -\infty < x_1,x_2 < +\infty, t>0. \tag{29}$$

Consider now the functions

$$u_k(x,t) \equiv v_k^0 + v_k^2 = \sum_{l=0,2} a_k^l V^l(x,t), k=1,2,3, \tag{30}$$

where

$$V^0(x,t) = \frac{1}{(4\pi\nu t)^{3/2}} \int\limits_{-\infty}^{\infty} dy_1 \int\limits_{-\infty}^{\infty} dy_2 \int\limits_{0}^{+\infty} u^0(y)\times$$

$$\times \left[e^{-\frac{(x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2}{4\nu t}} + e^{-\frac{(x_1-y_1)^2+(x_2-y_2)^2+(x_3+y_3)^2}{4\nu t}} \right] dy_3$$

$$V^2(x,t) = -\frac{2\nu}{(4\pi\nu)^{3/2}} \int\limits_{0}^{t} \frac{d\tau}{(t-\tau)^{3/2}} \int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} U^2(y_1,y_2,\tau) e^{-\frac{(x_1-y_1)^2+(x_2-y_2)^2+x_3^2}{4\nu(t-\tau)}} dy_1 dy_2,$$

$$\sum_{k=1}^{3} a_k^l \frac{\partial V^l}{\partial x_k} = 0, \sum_{k=1}^{3} a_k^m \frac{\partial V^l}{\partial x_k} = 0, l,m = 0,2, l \neq m. \qquad (31)$$

One can verify that the functions $u_k(x,t)$ defined by (30) and (31) satisfy the system (1), (2) and the conditions (3), (5).

Thus, we obtain the following.

Theorem 2. *Let $f(x,t) = 0$ and the given functions $u_k^0(x), U^2(x,t)$ satisfy the conditions (31). Then there exist the solutions (p,u) of the problem (1)-(3), (5), where $p = h(t)$ and the velocity $u_k(x,t)$ can be represented in the form (30) of a sum of two potentials (27) and (24).*

By this theorem and the proposition 2 in [8] (sec. 2), we get immediately the following.

Corollary 2. *Let $f(x,t) = 0$ and let the initial and boundary conditions ((3) and (5)) do not satisfy the conditions (31). Then there exists no solution (p,u) of the problem (1)-(3), (5) (except for special cases which were discussed in [6-8]) .*

Remark. Let, in (31), $V^l(x) = V^l\left(\sum\limits_{j=1}^{n} b_j^l x_j\right)$. Then the conditions (31) have form:

$$\sum_{k=1}^{3} a_k^l b_k^l = 0, \sum_{k=1}^{3} a_k^m b_k^l = 0, l, m = 0, 2, l \neq m. \tag{32}$$

Example 2.

Initial condition:

$$u_k(x,t)\,|_{t=0} = a_k^0, -\infty < x_1, x_2 < +\infty, x_3 > 0, k = 1, 2, 3$$

and the boundary condition

$$\left.\frac{\partial u_k}{\partial x_3}\right|_{x_3=0} = -a_k^2, -\infty < x_1, x_2 < +\infty, t > 0.$$

By (30), (27) and (24), the solution (p, u) has the form ($b_j^l = b_j, l = 0, 2, b_1 = b_2 = 0, b_3 = 1$):

$$p = h(t)$$

$$u_k = a_k^0 + a_k^2 \left[\left(\frac{4\nu t}{\pi}\right)^{1/2} e^{-\frac{x_3^2}{4\nu t}} - x_3 + x_3 \phi\left(\frac{x_3}{(4\nu t)^{1/2}}\right)\right], x_3, t \geq 0$$

where a_k^0 and a_k^2 satisfy the conditions (32).

References

[1] L.Caffarelli, R. Kohn and L. Nirenberg, *Partial regularity of suitable weak solutions of the Navier - Stokes equations*, Comm. Pure & Appl. Math., 35 (1982), 771-831.

[2] O. A. Ladyzhenskaya, *The mathematical theory of viscous incompressible flows*, Gordon and Breach, NewYork, 1969.

[3] P. Lions, *Mathematical topics in fluid mechanics*, Clarendon Press, Oxford, V.1, 1996 and V.2, 1998.

[4] C.Fefferman, *Existence and Smoothness of the Navier-Stokes equations*, Preprint of Clay Math. Inst., Announces, May 24, 2000.

[5] Hoàng Đình Dung, *Mở đầu về giải tích của phương trình đạo hàm riêng*, Giáo trình cao học Viện Toán, Hà nội, 2003.

[6] Hoàng Đình Dung, *On solutions of the Navier - Stokes equations*, 1, Preprint of Hanoi Math.Inst., N^0 06-02, 2006.

[7] Hoàng Đình Dung, *On solutions of the Navier - Stokes equations*, 2, Preprint of Hanoi Math. Inst, N^0 06/06, 2006.

[8] Hoàng Đình Dung, *On solutions of the Navier - Stokes equations*, 3, Preprint of Hanoi Math. Inst, N^0 07/22, 2007.