Global infimum of strictly convex quadratic functions with bounded perturbation

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Abstract The problem of minimizing $\tilde{f} = f + p$ over some convex subset of a Euclidean space is investigated, where $f(x) = x^T Ax + b^T x$ is a strictly convex quadratic function and $|p|$ is only assumed to be bounded by some positive number s. It is shown that the function $\tilde{f}$ is strictly outer γ-convex for any $\gamma > \gamma^*$, where γ^* is determined by s and the smallest eigenvalue of A. As consequence, a γ^*-local minimal solution of $\tilde{f}$ is its global minimal solution and the diameter of the set of global minimal solutions of $\tilde{f}$ is less than or equal to γ^*. Especially, the distance between the global minimal solution of f and any global minimal solution of $\tilde{f}$ is less than or equal to $\gamma^*/2$. This property is used to prove the rough support property of $\tilde{f}$ and some generalized optimality conditions.

Keywords Quadratic function · Convexity modulus · Generalized convexity · Outer γ-convexity · Bounded perturbation · Global minimizer · Support property · Optimality condition

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1 Introduction

Throughout this paper, $(\mathbb{R}^n, \|\cdot\|)$ is the n-dimensional Euclidean space and D is a convex subset of $\mathbb{R}^n$, which is not necessarily closed.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a strictly convex quadratic function defined by

$$f(x) := x^T Ax + b^T x, \tag{1.1}$$

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where A is a symmetric positive definite n-by-n matrix and $b \in \mathbb{R}^n$. Traditionally, the quadratic convex programming problem

$$(P) \qquad \text{minimize } f(x) \quad \text{subject to } x \in D,$$

was often investigated, in particular, when D is a polyhedral set. In this paper, we are interested in the problem

$$(\tilde{P}) \qquad \text{minimize } \tilde{f}(x) := f(x) + p(x) \quad \text{subject to } x \in D,$$

where $p : D \to \mathbb{R}$ is only assumed to be bounded by some finite number s, i.e.,

$$\sup_{x \in D} |p(x)| \leq s < +\infty. \tag{1.2}$$

Why do we study this problem? Obviously, the class of $(\tilde{P})$ is essentially larger and formally contains the classical class of (P), while corresponding properties seem to be quite different. This is an ordinary reason to do research, at least from theoretical point of view. But it is not the only incentive. Let us explain some practical reasons.

A familiar scenario is that f is some original objective function and p is some perturbation. p may comprise additional (deterministic or random) influences to the objective function and errors caused by modeling, measurement, calculation, etc. The particular point is that we restrict ourself to consider only bounded perturbation. This restriction is weak enough to be satisfied by several relevant problems as illustrated by the following two examples.

One of the most prominent applications of quadratic programming is portfolio selection, which was formulated firstly by Markowitz (1952, 1959). A possible problem statement is to allocate capital over n available assets in order to minimize the risk and to maximize the return, i.e., to find a ratio vector $x \in \{x = (x_1, x_2, \ldots, x_n) \mid \sum_{i=1}^{n} x_i = 1\}$ which minimizes $f(x) = \varpi x^T \Sigma x - \rho^T x$, where ϖ is the risk aversion parameter, Σ is the n-by-n covariance matrix, and $\rho \in \mathbb{R}^n$ is the expected return. Since Σ and ρ are usually not determined exactly but only approximated by $\tilde{\Sigma}$ and $\tilde{\rho}$, one has to minimize

$$\tilde{f}(x) = \varpi x^T \tilde{\Sigma} x - \tilde{\rho}^T x = f(x) + p(x), \quad \text{where } p(x) = \varpi x^T (\tilde{\Sigma} - \Sigma) x - (\tilde{\rho} - \rho)^T x.$$

Assume that short sales are not allowed, i.e., $x_i \geq 0$ for $i = 1, 2, \ldots, n$, then the feasible set D is bounded and, therefore, perturbation p is bounded on D, too. In general, the boundedness of perturbation p is guaranteed whenever D is bounded and p is continuous, that is valid for many practical problems.

A further example for bounded perturbation appears always when solving any optimization problem (P) by computers. Since most of real numbers cannot be exactly represented and computed by computers, for most of $x \in D$ the value $f(x) = x^T Ax + b^T x$ cannot be exactly computed but only approximated by some floating-point number $\tilde{f}(x)$, and the corresponding function $\tilde{f}$ is neither convex nor quadratic, even not continuous. Then the function $p(x) = \tilde{f}(x) - f(x)$ describes the computing errors, which may be rather wild, but it is reasonable to assume that these computing errors are bounded by some upper bound s, which can be estimated. Moreover, by using longer floating-point numbers and/or better algorithms, one can reduce the upper bound s.

Another significant and opposite scenario is that $\tilde{f}$ is the proper objective function and f is the idealized or substituted objective function. In fact, many functions expressing some practical goals are assumed to be convex or quadratic, or to have some favorable properties which were already intensively investigated or which are more easily to study, but it is actually not the case. This fact was discussed in Phu and Bock and Pickenhain (2000). In such a situation, $p(x) = \tilde{f}(x) - f(x)$ is the correcting term, which may be assumed to be bounded (at least on the feasible set) by some sufficiently small positive number s, since if $|p(x)|$ is too large, then the substitution may fail to be suitable.

To explain it here, let us mention the often studied problem of economic power dispatch, whose task is to allocate amounts of electric power to thermal units of generation such that the total generation costs are minimal, while an electric power demand is met and restrictions for the power output of each unit are satisfied. It is often assumed that the total cost function is quadratic and strictly convex, even if this objective function contains only fuel cost, or also other costs like load-following cost, spinning-reserve cost and supplemental-reserve cost, or if emission and transmission loss are taken into account (see for instance van den Bosch and Lootsma (1987), Danaraj and Gajendran (2005), Guddat and Röhmisch and Schultz (1992), Zhu and Tomsovic (2007)). Of course, this assumption is a strong idealization. The real costs may be neither absolutely quadratic and nor strictly convex, i.e., some bounded correcting term p is actually necessary. In particular, if valve-point effect is considered, the quadratic cost function must be rectified by the sum of finite number of sinusoidal functions (see for instance Al-Othman and Al-Sumait and Sykulski (2007), Walters and Sheble (1993), Wood and Wollenberg (1984)). This correcting term is obviously bounded.

For the sake of shortness, we often call p as perturbation (although it is not its only possible role as explained above), $\tilde{f}$ as perturbed function and $(\tilde{P})$ as perturbed problem. Indeed, they are just lent terminology and do not always have the exact meaning as usual.

What could be essentially new in researching $(\tilde{P})$? This question is eligible because there were already many contributions to the area of sensitivity analysis, both to convex and quadratic programming problems. A common characteristic of many earlier works is that perturbations do not change typical properties of original problems. For instance, perturbed convex programs remain convex as in Canovas et al. (2008), Klatte (1979, 1997), Kummer (1984), Schultz (1984, 1988), Singer (1981), Trudzik (1985), Zlobec (1983), Zlobec et al. (1982), and quadratic programs remain quadratic as in Daniel (1973), Lee et al. (2005, 2006), Mirnia and Ghaffari-Hadigheh (2007), Phu (2007b), Phu and Yen (2001). In contrast, in this paper the perturbed function $\tilde{f}$ is neither convex nor quadratic, although the unperturbed function f is strictly convex and quadratic. Moreover, because of the merely bounded perturbation p, $\tilde{f}$ might be nowhere continuous. It seems to be that nothing could be done then. Our aim is to show the contrary.

In Section 2 we show that the convexity of f is not completely lost due to the bounded perturbation p, but some trace still remains in form of rough convexity of $\tilde{f}$. Using the quantity

$$\gamma^* := 2\sqrt{2s/\lambda_{\min}}, \tag{1.3}$$

where $\lambda_{\min}$ is the smallest eigenvalue of A, Proposition 2.2 points out that $\tilde{f} = f + p$ is outer γ-convex for any $\gamma \geq \gamma^*$ and strictly outer γ-convex for any $\gamma > \gamma^*$.

Section 3 is devoted to some resulting properties. Similarly to the property of convex functions that a local minimizer is global minimizer, Proposition 3.1 says that a γ^*-

minimizer $x^* \in D$ of $(\tilde{P})$ defined by

$$\tilde{f}(x^*) = \inf_{x \in \bar{B}(x^*, \gamma^*) \cap D} \tilde{f}(x)$$

is a *global minimizer*, i.e.,

$$\tilde{f}(x^*) = \inf_{x \in D} \tilde{f}(x),$$

and a *local* γ^**-infimizer* $x^* \in D$ defined by

$$\liminf{}_{x \in D,\, x \to x^*} \tilde{f}(x) = \inf_{x \in B(x^*, \gamma^* + \varepsilon) \cap D} \tilde{f}(x) \quad \text{for some } \varepsilon > 0$$

is a *global infimizer*, i.e.,

$$\liminf{}_{x \in D,\, x \to x^*} \tilde{f}(x) = \inf_{x \in D} \tilde{f}(x).$$

Note that there is little hope for $(\tilde{P})$ to have minimizers, since p is only assumed to be bounded. Therefore, we take infimizers into consideration in addition to minimizers. Obviously, each minimizer is a infimizer.

Correspondingly to the uniqueness of global minimizer of strictly convex functions, Propositions 3.2–3.3 say that the diameter of the set of global minimizers or of global infimizers of $(\tilde{P})$ cannot be greater that γ^*.

Proposition 3.4 points out that the distance between the global minimizer of (P) and any global infimizer of $(\tilde{P})$ cannot exceed $\gamma^*/2$. Due to (1.3), this distance converges to zero if s tends to zero.

In Section 4, Proposition 3.4 is applied to prove the rough support property of $\tilde{f}$ (Proposition 4.1) and two optimality conditions for Problem $(\tilde{P})$ (Propositions 4.2–4.3).

It is worth emphasizing that, beside studying the distance between minimal solutions of $(\tilde{P})$ and the minimal solution of (P) in Proposition 3.4, as usually done in traditional sensitivity analysis, the rest of this paper is for investigating other properties of Problem $(\tilde{P})$ as goal object by using Problem (P) as auxiliary means. This task is essential for the scenario where $\tilde{f}$ is the proper objective function and f is the substituted function and p is the correcting term.

2 Outer γ-convexity of perturbed function

In this section we show that, despite the unruly perturbation p, the perturbed function $\tilde{f}$ is still roughly convex in the following sense. $\tilde{f} : D \to \mathbb{R}$ is said to be *outer γ-convex* (resp. *strictly outer γ-convex*) w.r.t. roughness degree $\gamma > 0$ if for all $x_0, x_1 \in D$ satisfying $\|x_0 - x_1\| > \gamma$ there is a closed subset $\Lambda \subset [0, 1]$ containing $\{0, 1\}$ such that

$$[x_0, x_1] \subset \{(1-\lambda)x_0 + \lambda x_1 \mid \lambda \in \Lambda\} + \bar{B}(0, \gamma/2) \tag{2.1}$$

and

$$\forall \lambda \in \Lambda : \tilde{f}((1-\lambda)x_0 + \lambda x_1) \leq (1-\lambda)\tilde{f}(x_0) + \lambda \tilde{f}(x_1) \tag{2.2}$$

resp.

$$\forall \lambda \in \Lambda \setminus \{0, 1\} : \tilde{f}((1-\lambda)x_0 + \lambda x_1) < (1-\lambda)\tilde{f}(x_0) + \lambda \tilde{f}(x_1). \tag{2.3}$$

As usual, $B(x, r)$ and $\bar{B}(x, r)$ denote the open ball or the closed ball, resp., with radius r centered at $x \in \mathbb{R}^n$, and

$$[x_0, x_1] := \{(1-\lambda)x_0 + \lambda x_1 \mid \lambda \in [0, 1]\}.$$

Note that the above kinds of rough convexity were introduced in Phu and An (1999) and in Phu (2003, 2008).

The key of our investigation is the convexity modulus of f defined by

$$h_1(\gamma) := \inf_{x_0,\, x_1 \in \mathbb{R}^n,\, \|x_0 - x_1\| = \gamma} \left(\frac{1}{2}\big(f(x_0) + f(x_1)\big) - f\big(\frac{1}{2}(x_0 + x_1)\big) \right), \tag{2.4}$$

where $\gamma \in \mathbb{R}_+$. This modulus was introduced in Phu (2007a) for general convex functions, where the infimum was subject only to $x_0,\ x_1 \in D$. In contrary, here we take all $x_0,\ x_1 \in \mathbb{R}^n$ into account. As an advantage, h_1 is easier to determine, independently on the form of D. Applying definition to quadratic functions, we get the following.

Proposition 2.1 *Let $\lambda_{\min}$ be the smallest eigenvalue of the symmetric positive definite matrix A and $f : \mathbb{R}^n \to \mathbb{R}$ be defined by* (1.1)*. Then*

$$h_1(\gamma) = \frac{1}{4}\, \lambda_{\min}\, \gamma^2 \quad for \quad \gamma \in \mathbb{R}_+. \tag{2.5}$$

Proof (2.4) implies immediately $h_1(0) = 0$. Hence, we only have to consider an arbitrary $\gamma > 0$. Let x_0 and x_1 be any vectors in $\mathbb{R}^n$ satisfying $\|x_0 - x_1\| = \gamma$. By the spectral theorem, there is an orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ of $\mathbb{R}^n$ consisting of unit eigenvectors of A, i.e.,

$$Ae_i = \lambda_i e_i, \ \|e_i\| = 1 \quad \text{for } i = 1, 2, \ldots, n,$$
$$e_i^T e_j = 0 \quad \text{for } i \neq j,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the corresponding real eigenvalues. Let

$$x_0 = \sum_{i=1}^{n} \mu_{0i}\, e_i, \quad x_1 = \sum_{i=1}^{n} \mu_{1i}\, e_i.$$

Then $x_0 - x_1 = \sum_{i=1}^{n} (\mu_{0i} - \mu_{1i})\, e_i$ and

$$\gamma^2 = \|x_0 - x_1\|^2 = \sum_{i=1}^{n} (\mu_{0i} - \mu_{1i})^2$$

and

$$\begin{aligned}\frac{1}{2}\big(f(x_0) + f(x_1)\big) - f\big(\frac{1}{2}(x_0 + x_1)\big) &= \frac{1}{2}(x_0^T A x_0 + x_1^T A x_1) - \frac{1}{4}(x_0 + x_1)^T A (x_0 + x_1) \\ &= \frac{1}{4}(x_0^T A x_0 + x_1^T A x_1 - 2x_0^T A x_1) \\ &= \frac{1}{4}(x_0 - x_1)^T A (x_0 - x_1) \\ &= \frac{1}{4}\sum_{i=1}^{n} (\mu_{0i} - \mu_{1i})^2 \lambda_i.\end{aligned}$$

Note that the linear component $b^T x$ vanishes in the above calculation since

$$\frac{1}{2}\big(b^T x_0 + b^T x_1\big) - b^T\big(\frac{1}{2}(x_0 + x_1)\big) = 0.$$

For $\lambda_{\min} = \min\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, we have

$$\frac{1}{2}\big(f(x_0) + f(x_1)\big) - f\big(\frac{1}{2}(x_0 + x_1)\big) \geq \frac{1}{4}\lambda_{\min} \sum_{i=1}^{n} (\mu_{0i} - \mu_{1i})^2 = \frac{1}{4}\lambda_{\min}\gamma^2.$$

In particular, if $x_0 - x_1$ is equal to the unit eigenvector corresponding to the eigenvalue $\lambda_{\min}$ then

$$\frac{1}{2}\big(f(x_0) + f(x_1)\big) - f\big(\frac{1}{2}(x_0 + x_1)\big) = \frac{1}{4}\lambda_{\min}\gamma^2.$$

Hence,

$$h_1(\gamma) = \inf_{x_0,\, x_1 \in \mathbb{R}^n,\, \|x_0 - x_1\| = \gamma} \left(\frac{1}{2}\big(f(x_0) + f(x_1)\big) - f\big(\frac{1}{2}(x_0 + x_1)\big)\right) = \frac{1}{4}\,\lambda_{\min}\gamma^2,$$

i.e., (2.5) is true. ☐

For instance, let $A = I$, where I is the unit matrix, then all eigenvalues are equal to 1. Hence, $h_1(\gamma) = \frac{1}{4}\gamma^2$. This result coincides with the one in Phu (2007a), where $f(x) = \|x\|^2$.

Since A is positive definite, we have $\lambda_{\min} > 0$. Therefore, $h_1(\gamma) > 0$ for $\gamma > 0$.

We will use the convexity modulus h_1 in the proof of the next statement .

Proposition 2.2 *Let $\gamma^* = 2(2s/\lambda_{\min})^{1/2}$. Then $\tilde{f} = f + p$ is outer γ-convex for any $\gamma \geq \gamma^*$ and strictly outer γ-convex for any $\gamma > \gamma^*$.*

Proof Due to Proposition 8 in Phu (2007a), $\tilde{f}$ is outer γ-convex if $|p(x)| \leq s \leq h_1(\gamma)/2$ for all $x \in D$. By Proposition 2.1, this sufficient condition means

$$2\,s \leq h_1(\gamma) = \frac{1}{4}\,\lambda_{\min}\gamma^2,$$

i.e., $\gamma \geq 2(2s/\lambda_{\min})^{1/2} = \gamma^*$. Furthermore, by Proposition 9 in Phu (2007a), $\tilde{f}$ is strictly outer γ-convex if $|p(x)| \leq s < h_1(\gamma)/2$ for all $x \in D$. This sufficient condition is fulfilled if $\gamma > \gamma^*$. ☐

Note that, for function f considered in this paper, $\tilde{f}$ is strictly outer γ-convex for any $\gamma \geq \gamma^*$. But, for shortness, we apply directly Proposition 9 in Phu (2007a) to get a weaker statement, which, nevertheless, does not affect our main results formulated later.

$\tilde{f} = f + p$ may be no more outer γ-convex for $\gamma < \gamma^*$. To see it, for an arbitrary pair of fixed positive numbers γ and γ^* satisfying $\gamma < \gamma^*$, consider the example

$$\begin{aligned} &f(x) = ax^2,\ x \in \mathbb{R}, \\ &p(x) = \begin{cases} s & \text{for } |x| \leq \gamma/2, \\ -s & \text{for } |x| > \gamma/2, \end{cases} \end{aligned} \tag{2.6}$$

and $D = \mathbb{R}$, where $a > 0$ and $s = a\gamma^{*2}/8$. Obviously, $\lambda_{\min} = a$, $\sup_{x\in\mathbb{R}} |p(x)| = s$ and $\gamma^* = 2\sqrt{2s/a}$, i.e., the given value γ^* fulfills (1.3). Let $x_0 = -x_1 = (\gamma + \gamma^*)/4$. Then $|x_0 - x_1| = (\gamma + \gamma^*)/2 > \gamma$ and, for all $\lambda \in [0, 1]$, there holds

$$(1 - \lambda)\tilde{f}(x_0) + \lambda\tilde{f}(x_1) = \tilde{f}(x_0) = a(\gamma + \gamma^*)^2/16 - s < a\gamma^{*2}/4 - s = s.$$

On the other hand,

$$\tilde{f}((1 - \lambda)x_0 + \lambda x_1) \geq s \ \text{ whenever } (1 - \lambda)x_0 + \lambda x_1 \in [-\gamma/2, \gamma/2].$$

Hence, there exists no closed subset $\Lambda \subset [0,1]$ containing $\{0,1\}$ such that (2.1)–(2.2) are fulfilled, i.e., $\tilde{f}$ is not outer γ-convex.

An important property of convex functions is that their lower level sets are convex. Similarly, each lower level set $L_\alpha(\tilde{f}) = \{x \in D \mid \tilde{f}(x) \le \alpha\}$ of $\tilde{f}$ is outer γ-convex for any $\gamma \ge \gamma^*$, i.e., for all x_0, $x_1 \in L_\alpha(\tilde{f})$ there exists a closed subset $\Lambda \subset [0,1]$ containing $\{0,1\}$ such that $\{x_\lambda \mid \lambda \in \Lambda\} \subset L_\alpha(\tilde{f})$ and $[x_0, x_1] \subset \{x_\lambda \mid \lambda \in \Lambda\} + \bar{B}(0, \gamma/2)$. This fact follows from Proposition 3.2 from Phu and An (1999) and Proposition 2.2.

3 Global infimizers

One of the most important properties of convex functions is that local minimum is global minimum. For outer γ-convex functions, this property is modified as follows.

Proposition 3.1

(a) *If* $\tilde{f}(x^*) = \inf_{x \in \bar{B}(x^*, \gamma^*) \cap D} \tilde{f}(x)$ *then* $\tilde{f}(x^*) = \inf_{x \in D} \tilde{f}(x)$. *(This property says that a* γ^**-minimum is a global minimum.)*

(b) *If there exists an* $\varepsilon > 0$ *such that* $\liminf_{x \to x^*} \tilde{f}(x) = \inf_{x \in B(x^*, \gamma^* + \varepsilon) \cap D} \tilde{f}(x)$ *then* $\liminf_{x \to x^*} \tilde{f}(x) = \inf_{x \in D} \tilde{f}(x)$. *(This property says that a local* γ^**-infimum is a global infimum.)*

Proposition 3.1 follows immediately from Proposition 4.2 of Phu and An (1999) and Proposition 2.2.

Note that if $\tilde{f}(x^*) = \inf_{x \in \bar{B}(x^*, \gamma) \cap D} \tilde{f}(x)$ holds only for some $\gamma < \gamma^*$ then $\tilde{f}(x^*) > \inf_{x \in D} \tilde{f}(x)$ is possible. To see it, for an arbitrary pair of fixed positive numbers γ and γ^* satisfying $\gamma < \gamma^*$, let us modify example (2.6) as follows:

$$f(x) = ax^2, \ x \in \mathbb{R},$$
$$p(x) = \begin{cases} -s & \text{for } x \le -\gamma^*/2, \\ s & \text{for } -\gamma^*/2 < x < \gamma/2, \\ -s & \text{for } x \ge \gamma/2, \end{cases}$$

and $D = \mathbb{R}$, where $a > 0$ and $s = a{\gamma^*}^2/8$. There hold $\lambda_{\min} = a$, $\sup_{x \in \mathbb{R}} |p(x)| = s$ and $\gamma^* = 2\sqrt{2s/a}$, i.e., the given value γ^* fulfills (1.3). Moreover, we have

$$\tilde{f}(-\gamma^*/2) = a{\gamma^*}^2/4 - a{\gamma^*}^2/8 = s$$

and

$$\tilde{f}(x) \geq s \quad \text{for all } x \in]-\infty, \gamma/2[$$

while

$$\tilde{f}(\gamma/2) = a\gamma^2/4 - a{\gamma^*}^2/8 < s,$$

which yields

$$\tilde{f}(-\gamma^*/2) = \inf_{x \in \bar{B}(-\gamma^*/2,\gamma)\cap D} \tilde{f}(x) = s > \tilde{f}(\gamma/2) \geq \inf_{x\in D} \tilde{f}(x).$$

A crucial property of strictly convex functions is that they have at most one minimizer, which plays an important role for proving the continuous dependence of minimal solutions on problem parameter or the continuity of optimal control functions. On contrary, strictly outer γ-convex functions may have a lot of global minimizers, i.e., the uniqueness of their minimizer fails. However it does not disappear completely but is still maintained in the following form.

Proposition 3.2 *Let x_0^* and x_1^* be two arbitrary global minimizers of Problem $(\tilde{P})$. Then $\|x_0^* - x_1^*\| \leq \gamma^*$.*

Proof Due to Proposition 2.2, $\tilde{f}$ is strictly outer γ-convex for any $\gamma > \gamma^*$. By definition, a strictly outer γ-convex function is strictly γ-convexlike, i.e., for all $x_0, x_1 \in D$ satisfying $\|x_0 - x_1\| > \gamma$ there exists $\lambda \in]0,1[$ such that

$$\tilde{f}((1-\lambda)x_0 + \lambda x_1) < (1-\lambda)\tilde{f}(x_0) + \lambda \tilde{f}(x_1).$$

Therefore, due to Proposition 2.2 from Phu (2003), $\|x_0^* - x_1^*\| \leq \gamma$ holds for any pair x_0^* and x_1^* of global minimizers of Problem $(\tilde{P})$ and any $\gamma > \gamma^*$, which yields immediately $\|x_0^* - x_1^*\| \leq \gamma^*$. □

Due to Proposition 3.2, the diameter of the set of global minimizers of $(\tilde{P})$ is less than or equal to γ^*. This result is sharp, as shown by the example

$$\begin{aligned} f(x) &= ax^2, \ x \in \mathbb{R}, \\ p(x) &= \begin{cases} s & \text{for } |x| < \gamma^*/2, \\ -s & \text{for } |x| \geq \gamma^*/2, \end{cases} \end{aligned} \tag{3.1}$$

and $D = \mathbb{R}$, where $a > 0$. There hold $\gamma^* = 2\sqrt{2s/a}$ and

$$\tilde{f}(-\gamma^*/2) = \tilde{f}(\gamma^*/2) = s \leq \tilde{f}(x) \text{ for all } x \in D,$$

i.e., $x_0^* = -\gamma^*/2$ and $x_1^* = \gamma^*/2$ are two global minimizers of $(\tilde{P})$ while $\|x_0^* - x_1^*\| = \gamma^*$.

Since the perturbation p is only assumed to be bounded without any other analytical properties, it is very likely that $\tilde{f} = f + p$ has no global minimizer but only global infimizers. Therefore, it is worth having a similar bound for the distance between global infimizers of $\tilde{f}$. Normally, the strict outer γ-convexity is not enough for our purpose, since the diameter of the set of global infimizers of a strictly outer γ-convex function may be infinity. But, for the considered $\tilde{f}$, we can prove the following.

Proposition 3.3 *Let x_0^* and x_1^* be two arbitrary global infimizers of Problem $(\tilde{P})$. Then $\|x_0^* - x_1^*\| \leq \gamma^*$.*

Proof Let

$$\bar{p}(x) := \liminf{}_{x' \in D,\, x' \to x} p(x') \tag{3.2}$$

denote the lower semicontinuous hull of p and

$$\sigma := \inf_{x \in D} \big(f(x) + p(x)\big) - \inf_{x \in D} \big(f(x) + \bar{p}(x)\big). \tag{3.3}$$

Since $p(x) \geq \bar{p}(x)$ for all $x \in D$, there holds $\sigma \geq 0$. To show $\sigma = 0$, assume the contrary that $\sigma > 0$. Then (3.3) implies that there is a $y \in D$ satisfying

$$\begin{aligned} f(y) + \bar{p}(y) &< \inf_{x \in D} \big(f(x) + \bar{p}(x)\big) + \sigma/2 \\ &= \inf_{x \in D} \big(f(x) + p(x)\big) - \sigma/2. \end{aligned} \tag{3.4}$$

Since f is continuous, we can choose a neighborhood $U(y)$ of y such that $f(x) < f(y) + \sigma/4$ for all $x \in U(y) \cap D$. Due to (3.2), we can find a $z \in U(y) \cap D$ with $p(z) < \bar{p}(y) + \sigma/4$. By adding together, we get

$$f(z) + p(z) < f(y) + \bar{p}(y) + \sigma/2.$$

Combining with (3.4) yields

$$f(z) + p(z) < \inf_{x \in D} \big(f(x) + p(x)\big),$$

in contradiction. Hence, σ must be equal to 0, i.e.,

$$\inf_{x \in D} \big(f(x) + p(x)\big) = \inf_{x \in D} \big(f(x) + \bar{p}(x)\big). \tag{3.5}$$

Let x^* be a global infimizer of $\tilde{f} = f + p$. Since f is continuous, it follows from (3.2) that

$$\begin{aligned} \inf_{x \in D} \tilde{f}(x) &= \liminf_{x \in D,\, x \to x^*} \big(f(x) + p(x)\big) \\ &= f(x^*) + \liminf_{x \in D,\, x \to x^*} p(x) \\ &= f(x^*) + \bar{p}(x^*). \end{aligned}$$

Combining with (3.5), we get

$$\inf_{x \in D} \big(f(x) + \bar{p}(x)\big) = f(x^*) + \bar{p}(x^*).$$

Hence, each global infimizer of $\tilde{f}$ is a global minimizer of $f + \bar{p}$, i.e., x_0^* and x_1^* are global minimizers of $f + \bar{p}$ subject to $x \in D$. Since

$$\sup_{x \in D} |\bar{p}(x)| \leq \sup_{x \in D} |p(x)| \leq s,$$

applying Proposition 3.2 to $f + \bar{p}$ we get $\|x_0^* - x_1^*\| \leq \gamma^*$. □

Due to Proposition 3.3, the diameter of the set of global infimizers of $(\tilde{P})$ is less than or equal to γ^*. Since each global minimizer is a global infimizer, example (3.1) shows that this estimation is sharp.

It is a crucial question to ask after the distance between an arbitrary global infimizer of $(\tilde{P})$ and the unique global minimizer of (P) (if any). The answer can be found in the following.

Proposition 3.4 *Let x^* be the minimizer of Problem (P) and $\tilde{x}^*$ be any global infimizer of Problem $(\tilde{P})$. Then $\|x^* - \tilde{x}^*\| \leq \gamma^*/2$.*

Proof Let $e \in \mathbb{R}^n$, $\|e\| = 1$, $x^* + te \in D$ for all $t \in [0, \varepsilon[$ and some $\varepsilon > 0$. Then

$$f(x^* + te) = e^T A e t^2 + (2Ax^* + b)^T e t + x^{*T} A x^* + b^T x^*.$$

Since $f(x^*) = \inf_{x \in D} f(x)$, there holds

$$\frac{d}{dt} f(x^* + te)|_{t=0} = (2Ax^* + b)^T e \geq 0.$$

Consequently, for any $\tau > 0$, we have

$$
\begin{aligned}
f(x^* + \tau e) + f(x^* - \tau e) &= \big(e^T A e \tau^2 + (2Ax^* + b)^T e\tau + x^{*T} A x^* + b^T x^*\big) \\
&\quad + \big(e^T A e \tau^2 - (2Ax^* + b)^T e\tau + x^{*T} A x^* + b^T x^*\big) \\
&= 2\big(e^T A e \tau^2 + x^{*T} A x^* + b^T x^*\big) \\
&= 2\big(f(x^* + \tau e) - (2Ax^* + b)^T e\tau\big) \\
&\leq 2\, f(x^* + \tau e),
\end{aligned}
$$

which implies by (2.4) that

$$
\begin{aligned}
f(x^* + \tau e) - f(x^*) &\geq \frac{1}{2}\big(f(x^* + \tau e) + f(x^* - \tau e)\big) - f(x^*) \\
&\geq \inf_{x_0,\, x_1 \in \mathbb{R}^n,\, \|x_0 - x_1\| = 2\tau} \left(\frac{1}{2}\big(f(x_0) + f(x_1)\big) - f\big(\frac{1}{2}(x_0 + x_1)\big)\right) \\
&= h_1(2\tau).
\end{aligned}
$$

(Note that if the infimum in (2.4) were subject only to $x_0,\, x_1 \in D$ like in Phu (2007a) then the above derivation were not correct since maybe $x^* - \tau e \notin D$ even for $x^*,\, x^* + \tau e \in D$.) Hence, if $x^* + \tau e \in D$ then (1.2)–(1.3) and (2.5) yield

$$
\begin{aligned}
\tilde{f}(x^* + \tau e) - \tilde{f}(x^*) &= f(x^* + \tau e) + p(x^* + \tau e) - f(x^*) - p(x^*) \\
&\geq h_1(2\tau) - 2\, s \\
&= \lambda_{\min}\tau^2 - \lambda_{\min}\gamma^{*2}/4 \\
&= \lambda_{\min}\big(\tau^2 - \gamma^{*2}/4\big).
\end{aligned}
\tag{3.6}
$$

Now assume by contradiction that $\omega := \|x^* - \tilde{x}^*\| - \gamma^*/2 > 0$, then for all $x \in D \cap B(\tilde{x}^*, \omega/2)$ we have

$$
\begin{aligned}
\|x - x^*\| &\geq \|x^* - \tilde{x}^*\| - \|x - \tilde{x}^*\| \\
&> \|x^* - \tilde{x}^*\| - \omega/2 \\
&= \frac{1}{2}(\omega + \gamma^*).
\end{aligned}
$$

Consequently, by applying (3.6) for $e = \|x - x^*\|^{-1}(x - x^*)$ and $\tau = \|x - x^*\|$, we get $x^* + \tau e = x$ and

$$
\begin{aligned}
\tilde{f}(x) - \tilde{f}(x^*) &\geq \lambda_{\min}\big(\|x - x^*\|^2 - \gamma^{*2}/4\big) \\
&> \lambda_{\min}\big(\frac{1}{4}\,(\omega + \gamma^*)^2 - \gamma^{*2}/4\big) \\
&= \frac{1}{4}\,\lambda_{\min}\,\omega(\omega + 2\gamma^*)
\end{aligned}
$$

for all $x \in D \cap B(\tilde{x}^*, \omega/2)$. This implies immediately

$$\liminf_{x\in D,\, x\to\tilde{x}^*} \tilde{f}(x) \geq \tilde{f}(x^*) + \frac{1}{4}\,\lambda_{\min}\,\omega(\omega + 2\gamma^*) > \tilde{f}(x^*),$$

which conflicts with $\liminf_{x\in D,\, x\to\tilde{x}^*} \tilde{f}(x) = \inf_{x\in D} \tilde{f}(x)$. Thus, $\|x^* - \tilde{x}^*\| \leq \gamma^*/2$ must be true. □

The estimation $\|x^* - \tilde{x}^*\| \leq \gamma^*/2$ in Proposition 3.4 is sharp. To see it, just consider (3.1) once again. For this example, $x^* = 0$ is the only global minimizer of $f(x) = ax^2$, $\tilde{x}_0^* = -\gamma^*/2$ and $\tilde{x}_1^* = \gamma^*/2$ are two global minimizers (that means also global infimizers) of $\tilde{f} = f + p$ over $D = \mathbb{R}$, and there holds $\|x^* - \tilde{x}_0^*\| = \|x^* - \tilde{x}_1^*\| = \gamma^*/2$.

Since $\gamma^* = 2(2s/\lambda_{\min})^{1/2}$ and $\lambda_{\min} > 0$, $\|x^* - \tilde{x}^*\| \leq \gamma^*/2$ implies that the distance $\|x^* - \tilde{x}^*\|$ between the minimizer x^* of Problem (P) and any global infimizer $\tilde{x}^*$ of Problem $(\tilde{P})$ converges to zero if s tends to zero. Even if s does not tend to zero, e.g. in case $\tilde{f}$ is the proper objective function as explained in the introduction (Section 1), the estimation $\|x^* - \tilde{x}^*\| \leq \gamma^*/2$ in Proposition 3.4 is nevertheless of interest and useful as shown in the next section.

4 Support property and optimality condition

A special feature of convex functions is the following support property: For any convex function $f : \mathbb{R}^n \to \mathbb{R}$ and any $x^* \in \mathbb{R}^n$, there exists a so-called subgradient $\eta \in \mathbb{R}^n$ satisfying

$$f(x) \geq f(x^*) + \eta^T(x - x^*) \quad \text{for all} \quad x \in \mathbb{R}^n, \tag{4.1}$$

i.e., there exists a linear supporting function $g(x) = f(x^*) + \eta^T(x - x^*)$ such that $f(x^*) = g(x^*)$ and $f(x) \geq g(x)$ for all $x \in \mathbb{R}$ (see, e.g., Rockafellar 1970). For our considered convex function $f(x) = x^T Ax + b^T$, the only supporting linear function is determined by its ordinary gradient $\eta = \nabla f(x^*) = 2Ax^* + b$.

Because of p, which is only assumed to be bounded by some $s < +\infty$, there is no hope for $\tilde{f} = f + p$ to have this support property. But we show next that the roughly convex function $\tilde{f}$ also has a rough support property. To this aim, we rewrite (4.1) as

$$f(x^*) - \eta^T x^* \leq f(x) - \eta^T x \quad \text{for all} \quad x \in \mathbb{R}^n. \tag{4.2}$$

Since such a relation rarely appears for $\tilde{f}$, we replace the term on the left of (4.2) by

$$\inf_{x' \in B(x^*,r)} (\tilde{f}(x') - \eta^T x') \quad \text{or} \quad \min_{x' \in \bar{B}(x^*,r)} (\tilde{f}(x') - \eta^T x')$$

for some suitable $r > 0$ and the term on the right of (4.2) by $\tilde{f}(x) - \eta^T x$ to get

$$\inf_{x' \in B(x^*,r)} (\tilde{f}(x') - \eta^T x') \leq \tilde{f}(x) - \eta^T x \quad \text{for all} \;\; x \in \mathbb{R}^n$$

or

$$\min_{x' \in \bar{B}(x^*,r)} (\tilde{f}(x') - \eta^T x') \leq \tilde{f}(x) - \eta^T x \quad \text{for all} \;\; x \in \mathbb{R}^n,$$

which describes the rough support property of $\tilde{f}$. This was already done for general outer γ-convex functions in Phu (2008). By applying Proposition 3.4, we prove in the following a better result for the particular function $\tilde{f}(x) = x^T Ax + b^T x + p(x)$, where the radius r is essentially smaller and $\eta \in \mathbb{R}^n$ is a concrete vector, namely $\eta = 2Ax^* + b$.

Proposition 4.1 *Suppose $D = \mathbb{R}^n$. Then, for any $x^* \in \mathbb{R}^n$ and $\varepsilon > 0$, there holds*

$$\inf_{x' \in B(x^*,\gamma^*/2+\varepsilon)} \left(\tilde{f}(x') - (2Ax^* + b)^T x'\right) \leq \tilde{f}(x) - (2Ax^* + b)^T x \quad \textit{for all} \;\; x \in \mathbb{R}^n. \quad (4.3)$$

In particular, if p is lower semicontinuous, then there holds for any $x^* \in \mathbb{R}^n$ that

$$\min_{x' \in \bar{B}(x^*,\gamma^*/2)} \left(\tilde{f}(x') - (2Ax^* + b)^T x'\right) \leq \tilde{f}(x) - (2Ax^* + b)^T x \quad \textit{for all} \;\; x \in \mathbb{R}^n. \quad (4.4)$$

Note that (4.4) differs from (4.3) not only in replacing "inf" by "min", but the minimum in (4.4) is taken over the smaller closed ball $\bar{B}(x^*, \gamma^*/2)$ while the infimum in (4.3) is taken over the larger open ball $B(x^*, \gamma^*/2 + \varepsilon)$.

Proof Let

$$\bar{f}(x) := f(x) - (2Ax^* + b)^T x = x^T Ax - 2\, x^{*T} Ax.$$

Then $\nabla \bar{f}(x^*) = 0 \in \mathbb{R}^n$. Since $\bar{f}$ is strictly convex, x^* is the only minimizer of $\bar{f}$. For $s \geq \sup_{x \in \mathbb{R}^n} |p(x)|$,

$$x^* \in \{x \in \mathbb{R}^n \mid \bar{f}(x) + p(x) \leq s\} \subset \{x \in \mathbb{R}^n \mid \bar{f}(x) \leq 2s\},$$

because $\bar{f}(x^*) = -x^{*T}Ax^* \le 0$. Since $\{x \in \mathbb{R}^n \mid \bar{f}(x) \le 2s\}$ is bounded, $\{x \in \mathbb{R}^n \mid \bar{f}(x) + p(x) \le s\}$ is also bounded and nonempty. Hence, $\bar{f} + p$ possesses at least one global infimizer $\bar{x}^* \in \mathbb{R}^n$, i.e.,

$$\liminf_{x' \to \bar{x}^*} \big(\bar{f}(x') + p(x')\big) = \inf_{x' \in \mathbb{R}^n} \big(\bar{f}(x') + p(x')\big). \tag{4.5}$$

Due to (1.2)–(1.3), the values of γ^* corresponding to $\bar{f}$ and to f are the same, namely $\gamma^* = 2(2s/\lambda_{\min})^{1/2}$. Applying Proposition 3.4 to the function pair $\bar{f}$ and $\bar{f} + p$, we get $\|x^* - \bar{x}^*\| \le \gamma^*/2$. Therefore,

$$\inf_{x' \in B(x^*, \gamma^*/2+\varepsilon)} \big(\bar{f}(x') + p(x')\big) \le \liminf_{x' \to \bar{x}^*} \big(\bar{f}(x') + p(x')\big), \tag{4.6}$$

which along with (4.5) yields

$$\inf_{x' \in B(x^*, \gamma^*/2+\varepsilon)} \big(\bar{f}(x') + p(x')\big) = \inf_{x' \in \mathbb{R}^n} \big(\bar{f}(x') + p(x')\big),$$

i.e.,

$$\inf_{x' \in B(x^*, \gamma^*/2+\varepsilon)} \big(\bar{f}(x') + p(x')\big) \le \bar{f}(x) + p(x) \quad \text{for all} \quad x \in \mathbb{R}^n. \tag{4.7}$$

By inserting

$$\bar{f}(x) + p(x) = f(x) - (2Ax^* + b)^T x + p(x) = \tilde{f}(x) - (2Ax^* + b)^T x, \tag{4.8}$$

in (4.7) we get (4.3) at once.

If p is lower semicontinuous, so is $\bar{f} + p$. Therefore, (4.5)–(4.6) can be replaced by

$$\bar{f}(\bar{x}^*) + p(\bar{x}^*) = \inf_{x' \in \mathbb{R}^n} \big(\bar{f}(x') + p(x')\big)$$

and

$$\min_{x' \in \bar{B}(x^*, \gamma^*/2)} \big(\bar{f}(x') + p(x')\big) \le \bar{f}(\bar{x}^*) + p(\bar{x}^*),$$

which yield

$$\min_{x' \in \bar{B}(x^*, \gamma^*/2)} \big(\bar{f}(x') + p(x')\big) = \inf_{x' \in \mathbb{R}^n} \big(\bar{f}(x') + p(x')\big),$$

i.e.,

$$\min_{x' \in \bar{B}(x^*, \gamma^*/2)} \big(\bar{f}(x') + p(x')\big) \le \bar{f}(x) + p(x) \quad \text{for all} \quad x \in \mathbb{R}^n.$$

Combining with (4.8), we obtain (4.4). □

For lower semicontinuous p we can write (4.4) as follows to get a form more similar to (4.1): There exists $\tilde{x}^* \in \bar{B}(x^*, \gamma^*/2)$ such that

$$\tilde{f}(x) \geq \tilde{f}(\tilde{x}^*) + (2Ax^* + b)^T(x - \tilde{x}^*) \quad \text{for all} \quad x \in \mathbb{R}^n.$$

Next, let us apply Proposition 3.4 to prove two optimality conditions for Problem $(\tilde{P})$, where D is defined by

$$D := \{x \in \mathbb{R}^n \mid c_i^T x \leq d_i \text{ for } 1 \leq i \leq m\}, \tag{4.9}$$

where $c_i \in \mathbb{R}^n$ and $d_i \in \mathbb{R}$ for $1 \leq i \leq m$. Firstly, we have the following sufficient condition.

Proposition 4.2 *Suppose that D is defined by* (4.9)*, $x^* \in D$ and there exist Lagrange multipliers $\nu_1 \geq 0, \ldots, \nu_m \geq 0$, such that*

$$\begin{aligned} & 2Ax^* + b + \sum_{i=1}^{m} \nu_i c_i = 0, \\ & \nu_i(c_i^T x^* - d_i) = 0 \quad \text{for } 1 \leq i \leq m. \end{aligned} \tag{4.10}$$

Then there is $\tilde{x}^* \in \bar{B}(x^*, \gamma^*/2) \cap D$ which is a global infimizer of Problem $(\tilde{P})$.

Proof Due to (4.10), x^* is the only global minimizer of Problem (P) (see, e.g., Cottle and Pang and Stone, 1992). Since $f(x) = x^T Ax + b^T x$, A is positive definite, $\sup_{x \in D} |p(x)| \leq s < +\infty$, and D is closed, Problem $(\tilde{P})$ has at least a global infimizer, say $\tilde{x}^*$. Proposition 3.4 yields that $\tilde{x}^* \in \bar{B}(x^*, \gamma^*/2) \cap D$. □

Secondly, let us state a necessary optimality condition for global infimizers of Problem $(\tilde{P})$.

Proposition 4.3 *Suppose that D is defined by* (4.9) *and $\tilde{x}^* \in D$ is a global infimizer of Problem $(\tilde{P})$. Then there exist Lagrange multipliers $\nu_1 \geq 0, \ldots, \nu_m \geq 0$, such that*

$$\|2A\tilde{x}^* + b + \sum_{i=1}^{m} \nu_i c_i\| \leq 2\,\lambda_{\max}(2s/\lambda_{\min})^{\frac{1}{2}} = \lambda_{\max}\,\gamma^* \tag{4.11}$$

and, for $1 \leq i \leq m$,

$$\nu_i = 0 \quad if \quad c_i^T \tilde{x}^* < d_i - (2s/\lambda_{\min})^{\frac{1}{2}}\,\|c_i\|. \tag{4.12}$$

Proof Since $f(x) = x^T Ax + b^T x$ and A is positive definite, Problem (P) possesses exactly one global minimizer, say x^*. Proposition 3.4 implies $x^* \in \bar{B}(\tilde{x}^*, \gamma^*/2) \cap D$, i.e.,

$$\|x^* - \tilde{x}^*\| \leq \gamma^*/2 = (2s/\lambda_{\min})^{\frac{1}{2}}. \tag{4.13}$$

Moreover, by Karush-Kuhn-Tucker condition, there exist Lagrange multipliers $\nu_1 \geq 0$, ..., $\nu_m \geq 0$ satisfying (4.10). Condition $2Ax^* + b + \sum_{i=1}^m \nu_i c_i = 0$ yields that

$$\begin{aligned} \|2A\tilde{x}^* + b + \sum_{i=1}^m \nu_i c_i\| &= \|2Ax^* + b + \sum_{i=1}^m \nu_i c_i - 2A(x^* - \tilde{x}^*)\| \\ &= 2\,\|A(x^* - \tilde{x}^*)\|. \end{aligned} \tag{4.14}$$

By the spectral theorem, there is an orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ of $\mathbb{R}^n$ consisting of unit eigenvectors of A such that $x^* - \tilde{x}^* = \sum_{i=1}^n \mu_i\, e_i$ for suitable $\mu_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the corresponding real eigenvalues of A, then

$$A(x^* - \tilde{x}^*) = \sum_{i=1}^n \mu_i A e_i = \sum_{i=1}^n \mu_i \lambda_i e_i$$

implies

$$\|A(x^* - \tilde{x}^*)\|^2 = \sum_{i=1}^n \mu_i^2 \lambda_i^2 \leq \lambda_{\max}^2 \sum_{i=1}^n \mu_i^2 = \lambda_{\max}^2\, \|x^* - \tilde{x}^*\|^2.$$

Combining with (4.13) and (4.14), we get

$$\|2A\tilde{x}^* + b + \sum_{i=1}^m \nu_i c_i\| \leq 2\,\lambda_{\max}(2s/\lambda_{\min})^{\frac{1}{2}} = \lambda_{\max}\,\gamma^*,$$

i.e., (4.11).

Assume now $c_i^T \tilde{x}^* < d_i - (2s/\lambda_{\min})^{\frac{1}{2}}\, \|c_i\|$ for some $i \in \{1, 2, \ldots, m\}$. Then (4.13) yields

$$\begin{aligned} c_i^T x^* &= c_i^T \tilde{x}^* + c_i^T (x^* - \tilde{x}^*) \\ &< d_i - (2s/\lambda_{\min})^{\frac{1}{2}}\, \|c_i\| + \|c_i\|\, \|x^* - \tilde{x}^*\| \\ &\leq d_i, \end{aligned}$$

i.e., $c_i^T x^* - d_i < 0$ holds for this i. Hence, $\nu_i = 0$ follows from condition $\nu_i(c_i^T x^* - d_i) = 0$ in (4.10), i.e., (4.12) holds true. □

5 Concluding remarks

The crucial quantity used in this paper is $\gamma^* = 2(2s/\lambda_{\min})^{1/2}$, which depends on an upper bound s of function p and the smallest eigenvalue $\lambda_{\min}$ of the positive definite symmetric matrix A. The upper bound s is assumed to be known. Methods for computing the smallest eigenvalue $\lambda_{\min}$ can be read for instance in Crouzeix et al. (1994), Cullum (1978), Davidson (1975), Jansen et al. (1996), Lanczos (1950), Ma and Zarowski (1995), Sleijpen and van der Vorst (1996), and Voss (2001). Even for unknown $\lambda_{\min}$, it is valuable enough to know that the distance between the minimizer x^* of Problem (P) and any global infimizer $\tilde{x}^*$ of Problem $(\tilde{P})$ converges to zero if s tends to zero, as implied by Proposition 3.4.

Similar research is done for minimizing nonquadratic convex functions with bounded perturbations in Phu (2009). In addition, we also investigate the problem of minimizing nonconvex quadratic functions with bounded perturbations.

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