On the stability of solutions of Ito differential equations

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Abstract

In this paper, we discuss the relationship between different types of stability of stochastic differential equations. We prove that for linear Ito stochastic differential equations, stablility in probability is equivalent to almost sure stability. Almost sure stability does not depend on initial time. In the case of 1-dimensional Ito stochastic differential equation with constant coefficients satisfying nondegeneracy condition, we show that stability in probability is equivalent to weak stability in probability.

Key words: Nonautonomous stochastic differential equation, two-parameter stochastic flow, stability, Lyapunov stability.
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1 Introduction

Most differential equations, deterministic or stochastic, can not be solved explicitly. Nevertheless we often can deduce a lot of useful information, usually qualitative, about the behaviour of their solutions from the functional form of their coefficients. Of particular interests in applications are the long term asymptotic behaviour and sensitivity of the solutions to small changes, for example, measurement errors in initial values. It is known that under some usual conditions, the solutions of a differential equation depend continuously on their initial values over a finite time interval. Extension of this

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idea to an infinite time interval leads to the concept of stability. For ordinary differential equations, we know concepts of stability, asymptotical stability, exponential stability, uniform stability and global stability. In view of the variety of convergences for stochastic processes, there are many different ways of defining stability for stochastic differential equations (see Khasminskii [3]). In this paper, we aim to make a clear relationship between some of these notions of stability. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. We consider a linear d-dimensional Ito stochastic differential equations.

$$dX_t = F_0(t)X_t dt + \sum_{j=1}^{m} F_j(t)X_t dW_t^j, \tag{1}$$
$$X(t_0) = x_0,$$

where $F_j(t) : \mathbb{R}^+ \to \mathbb{R}^{d\times d}(j = \overline{0,m})$ are continuous matrix-valued functions bounded by a constant K, x_0 is a non-random initial value, $W_t^j(j = \overline{1,m})$ are independent 1-dimensional standard Wiener processes.
It is known that with the above assumption, the Cauchy problem of (1) has a unique solution (see Khasminskii [3]).

By Kunita [4], the linear Ito stochastic differential equations (1) generates two-parameter stochastic flow $\Phi_{s,t}(\omega)$ of linear operators of $\mathbb{R}^d$. Each solution of (1) satisfying initial value condition $X(t_0, \omega) = x_0$, is a stochastic process given by formula $X(t, \omega) = \Phi_{t_0,t}(\omega)x_0$.

Recall from Kunita [4] that, a two-parameter stochastic flow of diffeomorphisms of $\mathbb{R}^d$ is a family of continuous maps (map-valued random variables) $\{\phi_{s,t}(\omega) : \omega \in \Omega,\ \ s, t \in \mathbb{R}^+\}$ which satisfies the following conditions for any ω from a subset $\Omega' \subset \Omega$ of full $\mathbb{P}$-measure:

(i) $\phi_{s,t}(\omega) = \phi_{u,t}(\omega) \circ \phi_{s,u}(\omega)$ holds for all $s, t, u \in \mathbb{R}^+$, where $\circ$ denotes the composition of maps;

(ii) $\phi_{s,s}(\omega)$ is the identity map for all $s \in \mathbb{R}^+$;

(iii) the map $\phi_{s,t}(\omega) : \mathbb{R}^d \to \mathbb{R}^d$ is an onto homeomorphism for all $s, t \in \mathbb{R}^+$;

(iv) $\phi_{s,t}(\omega)x$ is differentiable with respect to $x \in \mathbb{R}^d$ for all $s, t \in \mathbb{R}^+$ and derivative is continuous in s, t, x.

A family of continuous maps (map-valued random variables) $\Phi_{s,t}(\omega)$ of $\mathbb{R}^d$ is called two-parameter stochastic flow of linear operators of $\mathbb{R}^d$, if it is a two-parameter flow of diffeomorphisms and, additionally, $\Phi_{s,t}(\omega)$ is a linear operator.

Note that fixing an $\omega \in \Omega$, the two-parameter flow $\Phi_{s,t}(\omega)$ of linear operators of $\mathbb{R}^d$ is an analogue of the Cauchy operator of a linear system of differential equations.

The paper is organized as follows. In Sec. 2, we present the definitions of stability of the trivial solution of linear Ito stochastic differential equations. It is known that the investigation of stability of solutions of differential equations can be reduced to the investigation of stability of the trivial solution by a change of variables. Therefore, we will give only the definition of stability of the trivial solution. In Sec. 3, we discuss the relationship between different kinds of stochastic stability. In the theory of probability, a property being true almost surely is also true in probability, but the inverse is not true. Here, for the linear Ito stochastic differential equations, we prove that stability in probability is equivalent to almost sure stability. It is well known that, for deterministic ordinary differential equations stability does not depend on initial time, we prove that this is also true for almost sure stability. From the definitions of stability, we see that stability in probability is stronger than weak stability. Finally, we show that for 1-dimensional linear Ito stochastic differential equation with constant coefficients satisfying nondegeneracy condition, we have stability in probability equivalent to the weak stability.

2 Definitions of stability of trivial solution of Ito stochastic differential equations

Let us consider again equations (1) and the two-parameter flow $\Phi_{t_0,t}(\omega)$ generated by (1). It is easily seen that $X(t,\omega) \equiv 0$ is the trival solution of (1). Following Khasminskii [3], we give here different kinds of stability of the trival solution $X(t,\omega) \equiv 0$ of the Ito stochastic differential equations (1).

Definition 2.1 (Weak stability in probability) *The trivial solution $X(t,\omega) \equiv 0$ of (1) is called weakly stable in probability (for $t \geqslant t_0$ or on $[t_0,+\infty)$) if for every $\varepsilon > 0$ and $\delta > 0$ there exists an $r > 0$ such that if $t \geqslant t_0$ and $\|x_0\| < r$, then*

$$\mathbb{P}\left\{\omega : \|\Phi_{t_0,t}(\omega)\, x_0\| > \varepsilon\right\} < \delta. \tag{2}$$

Definition 2.2 (Stability in probability for $t \geqslant 0$) *The trivial solution $X(t,\omega) \equiv 0$ of (1) is called stable in probability for $t \geqslant 0$ if for any $t_0 \geqslant 0$ and $\varepsilon > 0$*

$$\lim_{x_0 \to 0} \mathbb{P}\left\{\omega : \sup_{t > t_0} \|\Phi_{t_0,t}(\omega)\, x_0\| > \varepsilon\right\} = 0. \tag{3}$$

Definition 2.3 (Almost sure stability for $t \geqslant t_0$) *The trivial solution $X(t,\omega) \equiv 0$ of (1) is called stable almost surely for $t \geqslant t_0$ if the set Ω' of those $\omega \in \Omega$ such that $X(t,\omega) \equiv 0$ is stable for $t \geqslant t_0$ has probability 1.*

Definition 2.4 (Asymptotical stability in probability) *The trivial solution $X(t,\omega) \equiv 0$ of (1) is called asymptotically stable in probability if*

$$X(t,\omega) \equiv 0 \text{ is stable in probability and} \tag{4}$$

$$\lim_{x_0 \to 0} \mathbb{P}\left\{\lim_{t \to +\infty} \Phi_{t_0,t}(\omega)\, x_0 = 0\right\} = 1. \tag{5}$$

Definition 2.5 (Asymptotic stability in probability in the large) *The trivial solution $X(t,\omega) \equiv 0$ of (1) is called asymptotic stability in probability in the large if*

$$X(t,\omega) \equiv 0 \text{ is weakly stable in probability and} \tag{6}$$

$$\text{for every } \varepsilon > 0 \text{ and } x_0 \text{ we have } \lim_{t \to +\infty} \mathbb{P}\{\omega : \|\Phi_{0,t}(\omega)\, x_0\| > \varepsilon\} = 0. \tag{7}$$

The above definitions are applicable to nonliear systems without modification. The stability of non-trivial solutions can be defined similarly. For more details on various kinds of stability of solutions of Ito stochastic differential equations we refer to Khasminskii [3].

3 Results

First of all, we recall an assertion which was stated by Khasminskii [3, p. 194] but was not proved there. We give here a detailed proof of it.

Propotion 3.1 *If the trivial solution $X(t,\omega) \equiv 0$ of (1) is asymtotically stable in probability, then it is asymptotically stable in probability in the large.*

Proof. Suppose $X(t) \equiv 0$ of (1) is asymtotically stable in probability. Since stability in probability clearly implies weak stability in probability, we only have to prove that the condition (7) holds. Note that (7) is equivalent to the following statement: For every $\varepsilon > 0, \delta > 0$ and x_0, there exists $T(\delta, \varepsilon, x_0)$ such that for $t > T(\delta, \varepsilon, x_0)$ we have

$$\mathbb{P}\left\{\omega : \|\Phi_{0,t}(\omega)\, x_0\| > \varepsilon\right\} < \delta. \tag{8}$$

Given x_0, T, for every $t > T$ we have

$$\begin{aligned}\{\, \omega :\ \|\Phi_{0,t}(\omega)x_0\| > \varepsilon\} &\subset \{\, \omega :\ \|\Phi_{0,t}(\omega)x_0\| \geqslant \varepsilon\} \\ &\subset \left\{ \omega :\ \sup_{t>T} \|\Phi_{0,t}(\omega)x_0\| \geqslant \varepsilon \right\}.\end{aligned}$$

Since $X(t) \equiv 0$ of (1) is asymtotically stable in probability, for every $\varepsilon > 0, \delta > 0$ there exists $r(\delta) > 0$ such that for any x_0 sastifying $\|x_0\| < r(\delta)$ there exists $T(\delta, \varepsilon, x_0)$ such that for $t > T(\delta, \varepsilon, x_0)$, we have

$$\mathbb{P}\{\, \omega :\ \|\Phi_{0,t}(\omega)x_0\| > \varepsilon\} \leqslant \mathbb{P}\left\{ \omega :\ \sup_{t>T(\delta,\varepsilon,x_0)} \|\Phi_{0,t}(\omega)x_0\| \geqslant \varepsilon \right\} < \delta. \tag{9}$$

Thus for every given $\varepsilon,\ \delta$ then (8) is true for every x_0 satisfying $\|x_0\| < r(\delta)$. For arbitrary x_1 satisfying $\|x_1\| \geqslant r(\delta)$, denote $x^* = \frac{r(\delta).x_1}{\|x_1\|+r(\delta)}$. Then we have $\|x^*\| = r(\delta)\frac{\|x_1\|}{\|x_1\|+r(\delta)} < r(\delta)$. For $\varepsilon > 0$ be given, choose $\varepsilon_1 = \frac{\varepsilon.r}{\|x_1\|+r} > 0$. Then by (9) there exists $0 < T(\varepsilon, \delta, x^*) =: T(\varepsilon_1)$ such that

$$\mathbb{P}\left\{\omega : \sup_{t>T(\varepsilon_1)} \|\Phi_{0,t}(\omega)\, x^*\| \geqslant \varepsilon_1\right\} < \delta.$$

We have

$$\begin{aligned}&\left\{\omega : \sup_{t>T(\varepsilon_1)} \|\Phi_{0,t}(\omega)x^*\| \geqslant \varepsilon_1\right\} \\ &\qquad = \left\{\omega : \sup_{t>T(\varepsilon_1)} \left\|\Phi_{0,t}(\omega)\frac{r(\delta).x_1}{\|x_1\|+r(\delta)}\right\| \geqslant \frac{\varepsilon.r(\delta)}{\|x_1\|+r(\delta)}\right\} \\ &\qquad = \left\{\omega : \sup_{t>T(\varepsilon_1)} \frac{r(\delta)}{\|x_1\|+r(\delta)} \|\Phi_{0,t}(\omega)x_1\| \geqslant \frac{\varepsilon.r(\delta)}{\|x_1\|+r(\delta)}\right\} \\ &\qquad = \left\{\omega : \sup_{t>T(\varepsilon_1)} \|\Phi_{0,t}(\omega)\, x_1\| \geqslant \varepsilon\right\}.\end{aligned}$$

So, for every $\varepsilon > 0, \delta > 0$ any x_1 sastifying $\|x_1\| \geqslant r(\delta)$, there exists $T(\delta, \varepsilon, x_1) := T(\varepsilon_1)$ such that for $t > T(\delta, \varepsilon, x_1)$, we have

$$\mathbb{P}\{\,\omega : \|\Phi_{0,t}(\omega)x_1\| > \varepsilon\} \leqslant \mathbb{P}\left\{\omega : \sup_{t>(T(\varepsilon,\delta,x_1)} \|\Phi_{0,t}(\omega)x_1\| \geqslant \varepsilon\right\} < \delta.$$

Thus, we have prove (8), hence the proposition.

Theorem 3.2 *Let $t_0, t^* \in \mathbb{R}^+$ be arbitrary fixed non-negative numbers. If the trivial solution $X(t,\omega) \equiv 0$ of (1) is almost surely stable for $t \geqslant t_0$ then it is almost surely stable for $t \geqslant t^*$.*

Proof. Suppose the trivial solution $X(t,\omega) \equiv 0$ of (1) is almost surely stable for $t \geqslant t_0$ for some fixed $t_0 \in \mathbb{R}^+$. We show that for any fixed $t^* \geqslant 0$, the set Ω_{t^*} of those $\omega \in \Omega$ such that for every $\varepsilon > 0$, there exists $r_{(\varepsilon,t^*,\omega)} > 0$, which depends measureably on ω, such that for any x_0 satisfying $\|x_0\| < r_{(\varepsilon,t^*,\omega)}$ we have

$$\sup_{t>t^*} \|\,\Phi_{t^*,t}(\omega)x_0\| < \varepsilon \text{ has probility 1.}$$

Since $X(t,\omega) \equiv 0$ is almost surely stable for $t \geqslant t_0$, we found a set Ω_{t_0} of full measure $\mathbb{P}(\Omega_{t_0}) = 1$ which has a property that for any $\varepsilon > 0$ there is a function $r_{(\varepsilon,t_0,.)} : \Omega_{t_0} \to \mathbb{R}^+$ such that for any $x_0 \in \mathbb{R}^d$ satisfying $\|x_0\| < r_{(\varepsilon,t_0,\omega)}$

$$\sup_{t\geqslant t_0} \|\,\Phi_{t_0,t}(\omega)x_0\| < \varepsilon\ .$$

Note that $\Phi_{t_*,t}(\omega) = \Phi_{t_0,t}(\omega) \circ \Phi_{t^*,t_0}(\omega)$ for all $t \in \mathbb{R}^+$. There are two cases: $t^* \leqslant t_0$ or $t^* > t_0$. First, we consider the case $t^* \leqslant t_0$. Put $h(\omega) := \sup\limits_{t^*\leqslant s\leqslant t_0} \|\Phi_{t^*,s}(\omega)\|$, $r_{(\varepsilon,t^*,\omega)} := \min\left(\frac{r_{(\varepsilon,t_0,\omega)}}{h(\omega)}, \frac{\varepsilon}{h(\omega)}\right)$. Take an $\|x_1\| < r_{(\varepsilon,t^*,\omega)}$ arbitrarity.
If $t^* \leqslant t \leqslant t_0$ then

$$\|\Phi_{t^*,t}(\omega)x_1\| \leqslant \|\Phi_{t^*,t}(\omega)\|\,\|x_1\| \leqslant h(\omega)\,\|x_1\| \leqslant h(\omega) \times \frac{\varepsilon}{h(\omega)} = \varepsilon.$$

If $t > t_0$ then we have

$$\Phi_{t^*,t}(\omega)x_1 = \Phi_{t_0,t}(\omega) \circ \Phi_{t^*,t_0}(\omega)x_1 = \Phi_{t_0,t}(\omega)\left[\Phi_{t^*,t_0}(\omega)x_1\right].$$

Note that

$$\|\Phi_{t^*,t_0}(\omega)x_1\| \leqslant \|\Phi_{t^*,t_0}(\omega)\| \, \|x_1\| \leqslant h(\omega)\, \|x_1\| \leqslant h(\omega) \times r_{(\varepsilon,t^*,\omega)} < r_{(\varepsilon,t_0,\omega)}.$$

Therefore $\|\Phi_{t^*,t}(\omega)x_1\| = \|\Phi_{t_0,t}(\omega)\,[\Phi_{t^*,t_0}(\omega)x_1]\| < \varepsilon$. Thus, $\|\Phi_{t^*,t}(\omega)x_1\| < \varepsilon$ for $t \geqslant t^*$. Set $\Omega_{t^*} = \Omega_{t_0}$ we found the set Ω_{t^*} with the required property and $\mathbb{P}(\Omega_{t^*}) = 1$. The case $t^* > t_0$ can be treated similarly. The theorem is proved.

Theorem 3.3 *If the trivial solution $X(t,\omega) \equiv 0$ of (1) is stable almost surely for $t \geqslant 0$ then it is stable in probability for $t \geqslant 0$ and vice versa.*

Proof. *(i)* Suppose that $X(t,\omega) \equiv 0$ is stable almost surely for $t \geqslant 0$. By theorem 3.2, for any t_0, $X(t,\omega) \equiv 0$ is stable almost surely for $t \geqslant t_0$. Fix $t_0 \geqslant 0$, denote Ω' the set of those $\omega \in \Omega$ such that for every $\varepsilon > 0$, there exists a measurable $r_{(\varepsilon,\omega)} > 0$ such that for any x_0 satisfying $\|x_0\| < r_{(\varepsilon,\omega)}$ we have $\| \Phi_{t_0,t}(\omega)x_0 \| < \varepsilon$ for all $t > t_0$ has probability 1. Thus $\mathbb{P}(\Omega') = 1$ and for $\omega \in \Omega'$ and x_0 satisfying $\|x_0\| < r_{(\varepsilon,\omega)}$ we have

$$\sup_{t>t_0} \| \Phi_{t_0,t}(\omega)x_0\| < \varepsilon \ . \tag{10}$$

We prove first part of the theorem by arguing from contradition. Supopse that $X(t,\omega) \equiv 0$ is not stable in probability for $t \geqslant 0$, i.e. there exists $\varepsilon_1 > 0, t_1 \geqslant 0$ and $\delta_1 > 0$ such that for any $r > 0$ the set A of those $\omega \in \Omega$ such that there is $x_{(\varepsilon_1,t_1,\delta_1,\omega)}$ satisfying $\| x_{(\varepsilon_1,t_1,\delta_1,\omega)} \| < r$ and

$$\sup_{t>t_1} \left\|\Phi_{t_1,t}(\omega)x_{(\varepsilon_1,t_1,\delta_1,\omega)}\right\| > \varepsilon_1 \text{ have probability } \mathbb{P}(A) \geqslant \delta_1. \tag{11}$$

In (10), choose $\varepsilon = \varepsilon_1$, $t_0 = t_1$, then there exists a measureable $r_{(\varepsilon,\omega)} > 0$ such that for any x_0 satisfying $\|x_0\| < r_{(\varepsilon,\omega)}$ then $\sup\limits_{t>t_1} \|\Phi_{t_1,t}(\omega)x_0\| < \varepsilon_1$. Since $r_{(\varepsilon,t_0,\omega)} > 0$, there exists $r_2 > 0$, such that for the set

$$B := \left\{\omega : r_{(\varepsilon,t_1,\omega)} < r_2\right\} \text{ we have } \mathbb{P}(B) < \delta_1. \tag{12}$$

In (11), we choose $r = r_2$, from definition of Ω' and (11), (12) we have $\mathbb{P}(B^c \cap A \cap \Omega') > 0$, hence there exists $\omega^o \in B^c \cap A \cap \Omega'$. Since $\omega^o \in B^c$

$$r_{(\varepsilon_1,t_0,\omega)} \geqslant r_2. \tag{13}$$

Since $\omega^o \in A$ there is $x_{(\varepsilon_1,t_1,\delta_1,\omega^o)}$ satisfying $\| x_{(\varepsilon_1,t_1,\delta_1,\omega^o)} \| < r_2$ such that

$$\sup_{t>t_1} \left\| \Phi_{t_1,t}(\omega^o) x_{(\varepsilon_1,t_1,\delta_1,\omega^o)} \right\| > \varepsilon_1. \tag{14}$$

Since $\omega^o \in \Omega'$, $\varepsilon = \varepsilon_1$, $t_0 = t_1$ and since $\left\| x_{(\varepsilon_1,t_1,\delta_1,\omega^o)} \right\| < r_2 \leqslant r_{(\varepsilon_1,t_1,\omega^o)}$ we have

$$\sup_{t>t_1} \left\| \Phi_{t_1,t}(\omega^o) x_{(\varepsilon_1,t_1,\delta_1,\omega^o)} \right\| < \varepsilon_1, \tag{15}$$

which contracdicts (14). Thus, the first part of the theorem is proved.
(ii) Now we turn to the second part of the theorem. Suppose that the trivial solution $X(t,\omega) \equiv 0$ of (1) is stable in probability for $t \geqslant 0$. We have to show that it is stable almost surely. We again argue from contradition.
Assume that $X(t,\omega) \equiv 0$ is not stable almost surely. Then the set

$$C := \{\omega : X(t,\omega) \equiv 0 \ \text{ is not stable for } \ t \geqslant 0\}$$

has probability $\mathbb{P}(C) = \delta^* > 0$. Since equations (1) is linear, by the same argument as for deterministic linear differential equations (see Demidovich, §7 chap 2, p.81 [1]) we have

$$C = \left\{ \omega : \sup_{t>0} \|\Phi_{0,t}(\omega)\| = +\infty \right\}.$$

Let $\{e_1, e_2, ..., e_n\}$ denotes the standard basis in $\mathbb{R}^n$. Let M be an arbitrary linear operator in $\mathbb{R}^n$, a $x \in \mathbb{R}^n$ be an arbitrary unit vector then we can write $x = \alpha_1 e_1 + \alpha_2 e_2 + ... + \alpha_n e_n$, $\alpha_1, \alpha_2, ..., \alpha_n$ are real numbers, $\alpha_1^2 + \alpha_2^2 + ... + \alpha_n^2 = 1$. We have

$$\|Mx\| = \left\| M(\sum_{i=1}^n \alpha_i e_i) \right\| = \left\| \sum_{i=1}^n \alpha_i M e_i \right\| \leqslant \sum_{i=1}^n |\alpha_i| \, \|Me_i\| \leqslant \sum_{i=1}^n \|Me_i\| .$$

Let $\omega_0 \in C$ be arbitrary. Then

$$\sup_{t>0} \|\Phi_{0,t}(\omega_0)\| = +\infty.$$

Therefore, $\sup\limits_{t>0} \sum\limits_{i=1}^n \|\Phi_{0,t}(\omega_0) e_i\| = +\infty$. Hence there exists an index $i_{\omega_0} \in \{1, 2, ..., n\}$ such that

$$\sup_{t>0} \left\| \Phi_{0,t}(\omega_0) e_{i_{\omega_0}} \right\| = +\infty. \tag{16}$$

Now, since $X(t,\omega) \equiv 0$ is stable in probability for $t \geqslant 0$, for any $\varepsilon > 0$ and $\delta > 0$ there exists $r_{(\delta,\varepsilon)} > 0$ such that for any x_0 satisfying $\|x_0\| < r_{(\delta,\varepsilon)}$

$$\mathbb{P}\left\{\omega : \sup_{t>0} \|\Phi_{0,t}(\omega)x_0\| > \varepsilon\right\} < \delta.$$

Put $D_{(\epsilon,\delta,r,x_0)} = \left\{\omega : \sup_{t>0} \|\Phi_{0,t}(\omega)x_0\| > \varepsilon\right\}$ then $\mathbb{P}(D) < \delta$.
Fix $\epsilon > 0$ and choose $\delta = \frac{\delta^*}{2n}$, we find $r_{(\frac{\delta^*}{2n},\varepsilon)} > 0$ such that for any x_0 satisfying $\|x_0\| < r_{(\frac{\delta^*}{2n},\varepsilon)}$ then $D_{(\epsilon,\frac{\delta^*}{2n},r,x_0)}$ has probability less than $\frac{\delta^*}{2n}$.
Consider n vectors

$$x_i = \frac{r_{(\frac{\delta^*}{2n},\varepsilon)}}{2} e_i, \quad i = 1, 2, ..., n.$$

Then $\|x_i\| = \frac{r_{(\frac{\delta^*}{2n},\varepsilon)}}{2} < r_{(\frac{\delta^*}{2n},\varepsilon)}$, hence $D_{(\epsilon,\frac{\delta^*}{2n},r,x_i)}$ has probability less than $\frac{\delta^*}{2n}$ for all $i = 1, 2, ..., n$.
Let $\omega_0 \in C$ be arbitrary. By (16) there is $i_{\omega_0} \in \{1,\, 2,\, ...,\, n\}$ such that

$$\sup_{t>0} \left\|\Phi_{0,t}(\omega_0)x_{i_{\omega_0}}\right\| = \sup_{t>0} \left\|\Phi_{0,t}(\omega_0)\frac{\delta^*}{2n}e_{i_{\omega_0}}\right\| = +\infty > \epsilon.$$

Therefore,

$$C \subset \bigcup_{i=1}^{n} D_{(\epsilon,\frac{\delta^*}{2n},r_{(\frac{\delta^*}{2n},\varepsilon)},x_i)},$$

which implies

$$\mathbb{P}(C) \leqslant \sum_{i=1}^{n} \mathbb{P}(D_{(\epsilon,\frac{\delta^*}{2n},r_{(\frac{\delta^*}{2n},\varepsilon)},x_i)}) < \frac{\delta^*}{2}$$

This contradicts assumtion that $\mathbb{P}(C) = \delta^* > 0$. Thus the second part of the theorem is proved, hence the theorem. □

Theorem 3.4 *Let the* 1*-dimensional linear Ito stochastic differential equation with constant coefficients*

$$dX_t = BX_tdt + \sum_{j=1}^{m} \sigma_j X_t dW_t^j \tag{17}$$

satisfy a nondegeneracy condition

$$(A(x)\alpha,\alpha)=\sum_{j=1}^{m}(\sigma_j x,\alpha)^2 \geqslant K\,\|x\|^2\,\|\alpha\|^2$$

where K is a positive constant. Then the stability in probability and the weak stability in probability of its trivial solution $X(t,\omega)\equiv 0$ are equivalent.

Proof. Since (17) is 1-dimensional linear Ito stochastic differential equation with constant coefficients we can solve it explicitly as follows (see Khasminskii [3]).
First note that (17) is equivalent to the following Stratonovich diffirential equation

$$dX_t=\left(B-\frac{1}{2}\sum_{j=1}^{m}\sigma_j^2\right)X_t dt+\sum_{j=1}^{m}\sigma_j X_t\circ dW_t^j$$

or equivalent to

$$\frac{dX_t}{X_t}=\left(B-\frac{1}{2}\sum_{j=1}^{m}\sigma_j^2\right)dt+\sum_{j=1}^{m}\sigma_j\circ dW_t^j.$$

Therefore,

$$\ln|X_t|-\ln|X_{t_0}|=\left(B-\frac{1}{2}\sum_{j=1}^{m}\sigma_j^2\right)(t-t_0)+\sum_{j=1}^{m}\sigma_j\left(W_t^j-W_{t_0}^j\right).$$

Consequently,

$$\rho(t):=\ln|X_t|=\ln|X_{t_0}|+\left(B-\frac{1}{2}\sum_{j=1}^{m}\sigma_j^2\right)(t-t_0)+\sum_{j=1}^{m}\sigma_j\left(W_t^j-W_{t_0}^j\right).$$

Computing the variance of $\rho(t)$ we have $D\rho(t)=\sum_{j=1}^{m}\sigma_j^2\,(t-t_0)$ hence $\lim_{t\to+\infty}D\rho(t)=+\infty$. Thus for 1-dimensional linear Ito stochastic differential equation with constant coefficients (17), the condition $D\rho(t)\to+\infty$ when $t\to+\infty$ is satisfied. Now, by Theorem 11.1 of Khasminskii [3, p. 243] the stability in probability of trivial solution of (17) is equivalent to the weak stability in probability.

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