Lyapunov exponents and central exponents of linear Ito stochastic diferential equations

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Abstract

We study Lyapunov, central and auxiliary exponents of linear Ito stochastic equations. We show that the central exponents are nonrandom like Lyapunov exponents, the nonrandomness of which was proved in [8]. We prove that under a nondegeneracy condition the central exponents Θ_k of a linear Ito stochastic differential equation coincide with its auxiliary exponents γ_k, and, moreover, all the first exponents coincide: $\Theta_1 = \lambda_1 = \Omega_1 = \gamma_1$.

Key words: Lyapunov exponents, central exponents, Lyapunov spectrum, nonautonomous stochastic differential equation, two-parameter stochastic flow.
MSC2000: Primary 60H10, 37H10, 34D08; secondary 60G17, 34F05, 93E15.

1 Introduction

We consider a linear n-dimensional Ito stochastic differential equation

$$
\begin{aligned}
dX(t) &= F_0(t)X(t)dt + \sum_{k=1}^{m} F_k(t)X(t)dW_t^k, \qquad (1)\\
X(t_0) &= x_0,
\end{aligned}
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where $F_k(t) = \left(f_{ik}^j\right)_{n\times n}$ $(k \in \{0,1,2,...,m\})$ are continuous matrix-valued functions bounded by a constant K, x_0 is a non-random initial value, W_t^j $(j \in \{1,2,...,m\})$ are independent 1-dimensional standard Wiener processes on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. It is known that with the above assumption the Cauchy problem of (1) has unique solution (see Khasminskii [11, Theorem 3.2 page 79]). The linear Ito stochastic differential equation (1) generates a two-parameter stochastic flow $\Phi_{t_0,t}(\omega)$ of linear operators of $\mathbb{R}^n$ (see Kunita [13, page 116 and Theorem 4.5.1 page 155]). The solution of (1) satisfying initial value condition $X(t_0) = x_0$, is a stochastic process given by the formula $X(t) = \Phi_{t_0,t}(\omega)x_0$. Note that fixing an $\omega \in \Omega$ the two-parameter flow $\Phi_{t_0,t}(\omega)$ is an analogue of the Cauchy operator of a linear system of differential equations.

We denote by G_r the Grassmannian manifold of all r-dimentional subspaces in $\mathbb{R}^n$. For a linear subspace U of $\mathbb{R}^n$ we denote by U_* the subset of all nonvanishing vectors of U. For any nondegenerate $n \times n$ matrix X let us denote by X^* the transposed matrix of X and by $d_1(X) \geq d_2(X) \geq ... \geq d_n(X)$ the singular numbers of X, i.e. they are the positive square roots of the eigenvalues of the matrix X^*X. Clearly, for any $k \in \{1,2,...,n\}$ we have

$$d_k(X) = \inf_{U\in G_{n-k+1}} \sup_{x\in U_*} \frac{\|Xx\|}{\|x\|} = \sup_{V\in G_k} \inf_{x\in V_*} \frac{\|Xx\|}{\|x\|}.$$

Definition 1.1 The random variables $\lambda_k(\omega)$, $\Omega_k(\omega)$, $\Theta_k(\omega)$ $(k \in \{1,2,...,n\})$ defined by

$$\lambda_k(\omega) := \min_{U\in G_{n-k+1}} \max_{x\in U} \limsup_{t\to+\infty} \frac{1}{t} \ln \left\|\Phi_{0,t}(\omega)x\right\|, \tag{2}$$

$$\Theta_k(\omega) := \sup_{V\in G_k} \sup_{T\in\mathbb{R}^+} \limsup_{m\to+\infty} \frac{1}{mT} \sum_{i=0}^{m-1} \ln \left\| \Phi_{(i+1)T,iT}(\omega)\big|_{\Phi_{0,(i+1)T}(\omega)V} \right\|^{-1}, \tag{3}$$

$$\Omega_k(\omega) := \inf_{U\in G_{n-k+1}} \inf_{T\in\mathbb{R}^+} \limsup_{m\to+\infty} \frac{1}{mT} \sum_{i=0}^{m-1} \ln \left\| \Phi_{iT,(i+1)T}(\omega)\big|_{\Phi_{0,iT}(\omega)U} \right\|, \tag{4}$$

where $\Phi|_S$ denotes the restriction of the operator Φ on to S, are respectively called *Lyapunov exponents* and *central exponents* of the equation (1).

It will be shown in the proof of Theorem 2.6 that for any $V \in G_k$ and $T \in \mathbb{R}^+$

$$\limsup_{m\to+\infty} \frac{1}{mT} \sum_{i=0}^{m-1} \ln \left\| \Phi_{(i+1)T,iT}(\omega)\big|_{\Phi_{0,(i+1)T}(\omega)V} \right\|^{-1} = \limsup_{m\to+\infty} \frac{1}{2mT} \sum_{i=0}^{2m-1} \ln \left\| \Phi_{(i+1)T,iT}(\omega)\big|_{\Phi_{0,(i+1)T}(\omega)V} \right\|^{-1}$$

and

$$\frac{1}{2mT} \sum_{i=0}^{2m-1} \ln \left\| \Phi_{(i+1)T,iT}(\omega)\big|_{\Phi_{0,(i+1)T}(\omega)V} \right\|^{-1} \leqslant \frac{1}{m(2T)} \sum_{i=0}^{m-1} \ln \left\| \Phi_{(i+1)2T,i2T}(\omega)\big|_{\Phi_{0,(i+1)2T}(\omega)V} \right\|^{-1}.$$

Therefore, formula (3) is equivalent to the following formula which can serve as a definition of $\Theta_k(\omega)$ as well

$$\Theta_k(\omega) = \sup_{V\in G_k} \sup_{T>1} \limsup_{m\to+\infty} \frac{1}{mT} \sum_{i=0}^{m-1} \ln \left\| \Phi_{(i+1)T,iT}(\omega)\big|_{\Phi_{0,(i+1)T}(\omega)V} \right\|^{-1}. \qquad (5)$$

By the same argument we have the following equivalent definition of $\Omega_k(\omega)$

$$\Omega_k(\omega) = \inf_{U\in G_{n-k+1}} \inf_{T>1} \limsup_{m\to+\infty} \frac{1}{mT} \sum_{i=0}^{m-1} \ln \left\| \Phi_{iT,(i+1)T}(\omega)\big|_{\Phi_{0,iT}(\omega)U} \right\|. \qquad (6)$$

Definition 1.2 The random variables $\gamma_k(\omega)$ defined by

$$\gamma_k(\omega) := \limsup_{T\to+\infty} \limsup_{m\to+\infty} \frac{1}{mT} \sum_{i=0}^{m-1} \ln d_k\left[\Phi_{iT,(i+1)T}(\omega)\right], \quad k \in \{1,2,...,n\}, \qquad (7)$$

are called *auxiliary exponents* of the equation (1).

The function $\gamma_k(T)$ defined by

$$\gamma_k(T) := \limsup_{m\to+\infty} \frac{1}{mT} \sum_{i=0}^{m-1} \mathrm{E} \ln d_k\left[\Phi_{iT,(i+1)T}(\omega)\right], \quad k \in \{1,2,...,n\}, T \in \mathbb{R}^+, \qquad (8)$$

where $\mathrm{E}\xi(\omega)$ denotes the expectation of a random variable $\xi(\omega)$, are called *auxiliary functions* of the equation (1).

The above definitions of Lyapunov exponents, central exponents and auxiliary exponents for the stochastic differential equations have been introduced by Millionshchikov (see [14, 15]). Millionshchikov considered equation $\dot{u} = [B(t) + C(t, \omega)]u$, where $B(t)$ is a bounded continuous matrix-valued funtion and $C(t, \omega)$ is a piecewise-constant random matrix-valued process with independent values. Using Kolmogorov one-zero law Millionshchikov proved that Lyapunov exponents of such an equation do not depend on ω. Note that the equations Millionshchikov considered can be solved pathwisely, without the need of the Ito calculus. For the linear Ito stochastic differential equation

$$dX(t) = F_0(t)X(t)dt + \sum_{k=1}^{m} F_k X(t) dW_t^k, \tag{9}$$

where F_k, $(k \in \{1, 2, ..., m\})$ are constant matrix and $F_0(t)$ is continuous matrix-valued funtion, Nguyen Dinh Cong [7] noticed that Lyapunov exponents, central exponents, auxiliary exponents do not depend on ω. He gave in [8] a proof of independence of Lyapunov exponents of (1) (which is an equation of a more general type than (9)) on ω, i.e. the Lyapunov exponents are nonrandom.

For deriving the main results of the paper presented in Section 3 we will need the following *nondegeneracy condition* of the random part of the equation (1):

There are positive numbers μ_1, μ_2 such that for any $x, y \in \mathbb{R}^n$ and $t \in \mathbb{R}^+$

$$\mu_1 \|x\|^2 \|y\|^2 \leqslant \big\langle D(t,x)y, y \big\rangle \leqslant \mu_2 \|x\|^2 \|y\|^2 , \tag{10}$$

where $\big\langle y_1, y_2 \big\rangle$ denotes the scalar product of two vectors $y_1, y_2 \in \mathbb{R}^n$,

$$D(t,x) = \big(d_{ij}(t,x)\big)_{n \times n} \quad \text{with} \quad d_{ij}(t,x) = \sum_{k=1}^{m} \Bigg(\sum_{r,l=1}^{n} f_{ik}^r(t) f_{jk}^l(t) x_r x_l \Bigg).$$

The central exponents of deterministic linear differential equations were initially introduced to give lower and upper estimates for Lyapunov exponents and are different from Lyapunov exponents in general, as shown by Example 13.5.1 in Bylov et al. [3, page 187]. Beside giving estimates for Lyapunov exponents the central exponents also serve as qualitative and quatitive characteristics of the equations under considerations. The auxiliary exponents γ_k are interesting from computational point of view: for their computation we do not have to follow trajectories of solutions on the whole time axis, but only compute the Cauchy matrix on each compact time interval.

In this paper, under the nondegeneracy condition specified above we will show that the central exponents Θ_k of the linear Ito stochastic differential equation (1) coincide with its auxiliary exponents γ_k, and, moreover, the first exponents coincide: $\Theta_1 = \lambda_1 = \Omega_1 = \gamma_1$. The observation on equality of exponents was made by one of the authors in 1993 (see [7]) for the case of equations (9). However, the proof given there is incomplete since the technique of changing from series of random variables forming a Markov chain to a series of independent random variables is not completely verified. Here in this paper we overcome this problem for the case of Θ_k and the first exponents by using another technique, namely we use the law of large numbers and inequalities provided by Rosenblatt-Roth [16] for a series of random variables depending on a Markov chain. Moreover, we are able to prove the results for a more general equations (1).

The paper is organized as follows. In Sec. 2 we prove some properties of central and auxiliary exponents of (1). Namely, using standard techniques in the theory of Lyapunov exponents, we show that central exponents are nonrandom and make upper and lower estimates for Lyapunov exponents; the auxiliary exponents are also nonrandom, and the biggest and least auxiliary exponents coincide with central exponents. Sec. 3 presents the main result of the paper: with assumption of the nondegeneracy condition, $\Theta_k = \gamma_k$ and $\Theta_1 = \lambda_1 = \Omega_1 = \gamma_1$.

2 Properties of central and auxiliary exponents

Theorem 2.1 *For any $k \in \{1, 2, ..., n\}$ we have $\gamma_k(\omega) = \limsup_{T\to+\infty} \gamma_k(T)$, hence the auxiliary exponents $\gamma_k(\omega)$ of (1) do not depend on $\omega \in \Omega$.*

Proof. It is known that for any $x \in \mathbb{R}^n$ the solution $\Phi_{s,t}(\omega)x$ of the Ito stochastic differential equation (1) satisfies the inequality

$$E\,\|\Phi_{s,t}(\omega)x - x\|^4 \leqslant K(t-s)^2(1 + \|x\|^4), \tag{11}$$

where K is a positive constant independent of t, s, x (see Khasminskii [11, page 80]). Let $\{e_1, e_2, ..., e_n\}$ denote the standard basis in $\mathbb{R}^n$. For any $x \in \mathbb{R}^n$, $\|x\| = 1$, we can represent $x = \sum\limits_{i=1}^{n} \beta_i e_i$, $\sum\limits_{i=1}^{n} \beta_i^2 = 1$. We have

$$\begin{aligned}
\left\|\Phi_{iT,(i+1)T}(\omega)\right\| &= \sup_{\|x\|=1} \left\|\Phi_{iT,(i+1)T}(\omega)x\right\| \\
&\leqslant \sum_{i=1}^{n} \left\|\Phi_{iT,(i+1)T}(\omega)e_i\right\| \\
&\leqslant \sum_{i=1}^{n} \left\|\Phi_{iT,(i+1)T}(\omega)e_i - e_i\right\| + n.
\end{aligned}$$

Using the inequality $(a+b)^4 \leqslant 8(a^4+b^4)$ we get

$$\left\|\Phi_{iT,(i+1)T}(\omega)\right\|^4 \leqslant 8^n \big(\sum_{i=1}^{n} \left\|\Phi_{iT,(i+1)T}(\omega)e_i - e_i\right\|^4 + n^4 \big).$$

Similarly, by considering the backward Ito differential equations we get

$$\left\|\Phi_{(i+1)T,iT}(\omega)\right\|^4 \leqslant 8^n \big(\sum_{i=1}^{n} \left\|\Phi_{(i+1)T,iT}(\omega)e_i - e_i\right\|^4 + n^4 \big).$$

Note that $\Phi_{(i+1)T,iT}(\omega) = \Phi^{-1}_{iT,(i+1)T}(\omega)$, hence

$$\frac{1}{\left\|\Phi_{iT,(i+1)T}(\omega)\right\|} \leqslant \left\|\Phi_{(i+1)T,iT}(\omega)\right\|.$$

By using the fact that $|\ln x| \leqslant x + \frac{1}{x}$ for all $x > 0$ we get

$$\begin{aligned}\Big(\frac{1}{T}\ln\big\|\Phi_{iT,(i+1)T}(\omega)\big\|\Big)^2 &\leqslant \frac{1}{T^2}\Big(\big\|\Phi_{iT,(i+1)T}(\omega)\big\| + \frac{1}{\big\|\Phi_{iT,(i+1)T}(\omega)\big\|}\Big)^2\\ &\leqslant \frac{2}{T^2}\Big(\big\|\Phi_{iT,(i+1)T}(\omega)\big\|^2 + \frac{1}{\big\|\Phi_{iT,(i+1)T}(\omega)\big\|^2}\Big)\\ &\leqslant \frac{2}{T^2}\Big(\big\|\Phi_{iT,(i+1)T}(\omega)\big\|^2 + \big\|\Phi_{(i+1)T,iT}(\omega)\big\|^2\Big)\\ &\leqslant \frac{2}{T^2}\Big(\sqrt{8^n\big(\sum_{i=1}^{n}\big\|\Phi_{iT,(i+1)T}(\omega)e_i - e_i\big\|^4 + n^4\big)}\\ &+ \sqrt{8^n\big(\sum_{i=1}^{n}\big\|\Phi_{(i+1)T,iT}(\omega)e_i - e_i\big\|^4 + n^4\big)}\Big)\end{aligned}$$

Using (11) and the assumption $T > 1$ we find a positive constant $M_1 > 0$ such that

$$E\big(\frac{1}{T}\ln\big\|\Phi_{iT,(i+1)T}(\omega)\big\|\big)^2 \leqslant M_1.$$

Similarly we can find a positive constant $M_2 > 0$ such that

$$E\big(\frac{1}{T}\ln\big\|\Phi_{(i+1)T,iT}(\omega)\big\|\big)^2 \leqslant M_2.$$

Fix an $k \in \{1, 2, ..., n\}$. Since

$$0 < d_n[\Phi_{iT,(i+1)T}(\omega)] \leqslant d_k[\Phi_{iT,(i+1)T}(\omega)] \leqslant d_1[\Phi_{iT,(i+1)T}(\omega)],$$

we have

$$\big(\frac{1}{T}\ln d_k[\Phi_{iT,(i+1)T}(\omega)]\big)^2 \leqslant \big(\frac{1}{T}\ln\big\|\Phi_{iT,(i+1)T}(\omega)\big\|\big)^2 + \big(\frac{1}{T}\ln\big\|\Phi_{(i+1)T,iT}(\omega)\big\|\big)^2.$$

Consequenly,

$$E\big(\frac{1}{T}\ln d_k[\Phi_{iT,(i+1)T}(\omega)]\big)^2 \leqslant M_1 + M_2.$$

Hence $\frac{1}{T}\ln d_k[\Phi_{iT,(i+1)T}(\omega)]$ $(i \in \{0, 1, 2, ...\})$ is a sequence of independent random variables having second moments bounded by $M_1 + M_2$. By virtue of the Kolmogorov strong law of large numbers (see Shiryaev [17, Theorem 2, page 389]), the following relation holds with probability 1

$$\lim_{m\to+\infty}\Big(\frac{1}{mT}\sum_{i=0}^{m-1}\ln d_k\big[\Phi_{iT,(i+1)T}(\omega)\big]-\frac{1}{mT}\sum_{i=0}^{m-1}\mathrm{E}\ \ln d_k\big[\Phi_{iT,(i+1)T}(\omega)\big]\Big)=0.$$

Consequently,

$$\limsup_{m\to+\infty}\frac{1}{mT}\sum_{i=0}^{m-1}\ln d_k\big[\Phi_{iT,(i+1)T}(\omega)\big]=\limsup_{m\to+\infty}\frac{1}{mT}\sum_{i=0}^{m-1}\mathrm{E}\ \ln d_k\big[\Phi_{iT,(i+1)T}(\omega)\big],$$

hence,

$$\gamma_k(\omega)=\limsup_{T\to+\infty}\gamma_k(T).$$

The theorem is proved. □

Theorem 2.2 *For any $k\in\{1,2,...,n\}$ the central exponents $\Theta_k(\omega)$ of (1) does not depend on $\omega\in\Omega$.*

Proof. Denote by $\{\mathcal{F}_s^t\}_{t\geq s\geq 0}$ the filtration of σ-algebras generated by the Wiener processes $(W_u^1,W_u^2,...,W_u^m)_{t\geq u\geq s}$ of the linear Ito stochastic differential equation (1) (see e.g. Arnold [1, pages 91-92]). Clearly the $\Phi_{0,t}(\omega)$ is adapted to the filtration $\{\mathcal{F}_0^t\}_{t\geq 0}$. From the fomula (5) it follows that the random variable $\Theta_k(\omega)$ is measurable with respect to the limit σ-algebra $\mathcal{F}_0^{+\infty}:=\lim\limits_{t\to+\infty}\mathcal{F}_0^t=\bigvee\limits_{t\geq 0}\mathcal{F}_0^t$. We note that for any fixed $k\in\{1,2,...,n\}$ and $N\in\mathbb{N}$,

$$\begin{aligned}\Theta_k(\omega)&=\sup_{V\in G_k}\sup_{T>1}\limsup_{m\to+\infty}\frac{1}{mT}\sum_{i=0}^{m-1}\ln\Big\|\Phi_{(i+1)T,iT}(\omega)|_{\Phi_{0,(i+1)T}(\omega)V}\Big\|^{-1}\\&=\sup_{V\in G_k}\sup_{T>1}\limsup_{m\to+\infty}\frac{1}{mT}\Big(\sum_{i=0}^{N}\ln\Big\|\Phi_{(i+1)T,iT}(\omega)|_{\Phi_{0,(i+1)T}(\omega)V}\Big\|^{-1}\\&\qquad+\sum_{i=N+1}^{m-1}\ln\Big\|\Phi_{(i+1)T,iT}(\omega)|_{\Phi_{0,(i+1)T}(\omega)V}\Big\|^{-1}\Big).\end{aligned}$$

Since N is a fixed number, $\sum_{i=0}^{N}\ln\Big\|\Phi_{(i+1)T,iT}(\omega)|_{\Phi_{0,(i+1)T}(\omega)V}\Big\|^{-1}$ is a random variable with finite second moment, hence the limit

$$\lim_{m\to+\infty}\frac{1}{mT}\sum_{i=0}^{N}\ln\Big\|\Phi_{(i+1)T,iT}(\omega)|_{\Phi_{0,(i+1)T}(\omega)V}\Big\|^{-1}=0$$

exists and the equality holds with probability 1. Therefore,

$$\begin{aligned}
\Theta_k(\omega) &= \sup_{V\in G_k}\sup_{T>1}\limsup_{m\to+\infty}\frac{1}{mT}\sum_{i=N+1}^{m-1}\ln\left\|\Phi_{(i+1)T,iT}(\omega)|_{\Phi_{0,(i+1)T}(\omega)V}\right\|^{-1}\\
&= \sup_{V\in G_k}\sup_{T>1}\limsup_{m\to+\infty}\frac{1}{mT}\sum_{i=N+1}^{m-1}\ln\left\|\Phi_{(i+1)T,iT}(\omega)|_{\Phi_{NT,(i+1)T}(\omega)\left(\Phi_{0,NT}(\omega)V\right)}\right\|^{-1}\\
&\leqslant \sup_{\widetilde{V}\in G_k}\sup_{T>1}\limsup_{m\to+\infty}\frac{1}{mT}\sum_{i=N+1}^{m-1}\ln\left\|\Phi_{(i+1)T,iT}(\omega)|_{\Phi_{NT,(i+1)T}(\omega)\widetilde{V}}\right\|^{-1} =: r(\omega).
\end{aligned}$$

On another hand, by the definition of $r(\omega)$ just given above, for any $\epsilon > 0$ and $\omega \in \Omega$ there exists $\widetilde{V}_1 \in G_k$ such that

$$r(\omega)-\epsilon < \sup_{T>1}\limsup_{m\to+\infty}\frac{1}{mT}\sum_{i=N+1}^{m-1}\ln\left\|\Phi_{(i+1)T,iT}(\omega)|_{\Phi_{NT,(i+1)T}(\omega)\widetilde{V}_1}\right\|^{-1}.$$

Let $\widetilde{V}_2 \in G_k$ denote the subspace $\Phi_{0,NT}^{-1}(\omega)\widetilde{V}_1$. We have $\Phi_{0,NT}(\omega)\widetilde{V}_2 = \widetilde{V}_1$, hence

$$\begin{aligned}
r(\omega)-\epsilon &< \sup_{T>1}\limsup_{m\to+\infty}\frac{1}{mT}\sum_{i=N+1}^{m-1}\ln\left\|\Phi_{(i+1)T,iT}(\omega)|_{\Phi_{NT,(i+1)T}(\omega)\Phi_{0,NT}\widetilde{V}_2}\right\|^{-1}\\
&= \sup_{T>1}\limsup_{m\to+\infty}\frac{1}{mT}\sum_{i=N+1}^{m-1}\ln\left\|\Phi_{(i+1)T,iT}(\omega)|_{\Phi_{0,(i+1)T}(\omega)\widetilde{V}_2}\right\|^{-1}\\
&= \sup_{T>1}\limsup_{m\to+\infty}\frac{1}{mT}\Bigg(\sum_{i=0}^{N}\ln\left\|\Phi_{(i+1)T,iT}(\omega)|_{\Phi_{0,(i+1)T}(\omega)\widetilde{V}_2}\right\|^{-1}\\
&\qquad\qquad + \sum_{i=N+1}^{m-1}\ln\left\|\Phi_{(i+1)T,iT}(\omega)|_{\Phi_{0,(i+1)T}(\omega)\widetilde{V}_2}\right\|^{-1}\Bigg)\\
&\leqslant \sup_{V\in G_k}\sup_{T>1}\limsup_{m\to+\infty}\frac{1}{mT}\sum_{i=0}^{m-1}\ln\left\|\Phi_{(i+1)T,iT}(\omega)|_{\Phi_{0,(i+1)T}(\omega)V}\right\|^{-1}\\
&= \Theta_k(\omega).
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary so we have $r(\omega) \leqslant \Theta_k(\omega)$. Thus for any fixed N we have $\Theta_k(\omega) = r(\omega) \in \mathcal{F}_{(N+1)T}^{+\infty}$ so $\Theta_k(\omega)$ is measuarable with respect to the $\mathcal{F}^{+\infty}_{(N+1)T} = \lim_{t\to+\infty} \mathcal{F}^{t}_{(N+1)T} = \bigvee_{t\geq (N+1)T} \mathcal{F}^{t}_{(N+1)T}$. Hence $\Theta_k(\omega)$ is measurable with respect to the tail σ-algebra $\bigcap\limits_{N=1}^{+\infty} \mathcal{F}^{+\infty}_{(N+1)T}$. By the zero-or-one law (see Shiryaev [17, page 381]) the random variable $\Theta_k(\omega)$ is degenerate, i.e. nonrandom. $\square$

Theorem 2.3 *For any $k \in \{1, 2, ..., n\}$ the central exponents $\Omega_k(\omega)$ of (1) does not depend on $\omega \in \Omega$.*

Proof. We note that for any fixed $k \in \{1, 2, ..., n\}$ and $N \in \mathbb{N}$,

$$\begin{aligned}\Omega_k(\omega) &= \inf_{U\in G_{n-k+1}} \inf_{T>1} \limsup_{m\to+\infty} \frac{1}{mT} \sum_{i=0}^{m-1} \ln \left\| \Phi_{iT,(i+1)T}(\omega)|_{\Phi_{0,iT}(\omega)U} \right\| \\ &= \inf_{U\in G_{n-k+1}} \inf_{T>1} \limsup_{m\to+\infty} \frac{1}{mT} \left(\sum_{i=0}^{N} \ln \left\| \Phi_{iT,(i+1)T}(\omega)|_{\Phi_{0,iT}(\omega)U} \right\| \right. \\ &\qquad\qquad \left. + \sum_{i=N+1}^{m-1} \ln \left\| \Phi_{iT,(i+1)T}(\omega)|_{\Phi_{0,iT}(\omega)U} \right\| \right).\end{aligned}$$

Using an argument similar to that of the proof of Theorem 2.2 we can show that $\Omega_k(\omega)$ is measurable with respect to the tail σ-algebra $\bigcap\limits_{N=1}^{+\infty} \mathcal{F}^{+\infty}_{(N+1)T}$, hence is degenerate, i.e. nonrandom. $\square$

Now, since the Lyapunov, central and auxiliary exponents are independent of ω, we will drop ω in their notations.

Theorem 2.4 *For any $k \in \{1, 2, ..., n\}$ the central exponents Ω_k of (1) is larger or equal to the Lyapunov exponents λ_k.*

Proof. Fixing an $\omega \in \Omega$, for any $\epsilon > 0$, by the definition of Ω_k, there exists an $U \in G_{n-k+1}$ such that

$$\inf_{T>1} \limsup_{m\to+\infty} \frac{1}{mT} \sum_{i=0}^{m-1} \ln \left\| \Phi_{iT,(i+1)T}(\omega)|_{\Phi_{0,iT}(\omega)U} \right\| < \Omega_k + \epsilon.$$

For any vector $x \in U$ and $T > 1$ we have

$$\begin{aligned}\|\Phi_{0,mT}(\omega)x\| &= \left\| \Phi_{(m-1)T,mT}(\omega) \circ \Phi_{0,(m-1)T}(\omega)x \right\| \\ &\leqslant \left\| \Phi_{(m-1)T,mT}(\omega)|_{\Phi_{0,iT}(\omega)U} \right\| \cdot \left\| \Phi_{0,(m-1)T}(\omega)x \right\| \leqslant ... \\ &\leqslant \left\| \Phi_{(m-1)T,mT}(\omega)|_{\Phi_{0,iT}(\omega)U} \right\| ... \|\Phi_{0,T}(\omega)|_U\| \, \|x\| \, .\end{aligned}$$

Therefore,

$$\limsup_{m\to+\infty} \frac{1}{mT} \ln \|\Phi_{0,mT}(\omega)x\| \leqslant \limsup_{m\to+\infty} \frac{1}{mT} \sum_{i=0}^{m-1} \ln \left\| \Phi_{iT,(i+1)T}(\omega)|_{\Phi_{0,iT}(\omega)U} \right\|.$$

By Theorem 3.4 of Nguyen Dinh Cong [8], for any $h \in \mathbb{R}^+$ we have

$$\lambda_k(\omega) = \min_{\tilde{U}\in G_{n-k+1}} \max_{x\in\tilde{U}} \limsup_{\substack{m\to+\infty \\ m\in\mathbb{N}}} \frac{1}{mh} \ln \|\Phi_{0,mh}(\omega)x\|.$$

Therefore,

$$\begin{aligned} \lambda_k &\leqslant \inf_{T>1} \max_{x\in U} \limsup_{m\to+\infty} \frac{1}{mT} \ln \|\Phi_{0,mT}(\omega)x\| \\ &\leqslant \inf_{T>1} \limsup_{m\to+\infty} \frac{1}{mT} \sum_{i=0}^{m-1} \ln \left\| \Phi_{iT,(i+1)T}(\omega)|_{\Phi_{0,iT}(\omega)U} \right\| < \Omega_k + \epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary we derive $\lambda_k \leqslant \Omega_k$. □

Theorem 2.5 *For any $k \in \{1, 2, ..., n\}$ the central exponents Θ_k of (1) is smaller or equal to the Lyapunov exponents λ_k.*

Proof. Fixing an $\omega \in \Omega$, for any $\epsilon > 0$, by the definition of Θ_k, there exists an $V \in G_k$ such that

$$\sup_{T>1} \limsup_{m\to+\infty} \frac{1}{mT} \sum_{i=0}^{m-1} \ln \left\| \Phi_{(i+1)T,iT}(\omega)|_{\Phi_{0,(i+1)T}(\omega)V} \right\|^{-1} \geq \Theta_k - \epsilon.$$

By the definition of λ_k and Theorem 3.4 of Nguyen Dinh Cong [8], there exists an $U \in G_{n-k+1}$ such that for any $h \in \mathbb{R}^+$

$$\lambda_k = \max_{x\in U} \limsup_{m\to+\infty} \frac{1}{mh} \ln \|\Phi_{0,mh}(\omega)x\|.$$

Since U, V are linear subspaces in $\mathbb{R}^n$, the dimension of U is $n - k + 1$ and the dimension of V is k, the dimension of the subspace $U \cap V$ is larger or equal to 1. Take an $x_0 \in U \cap V \setminus \{0\}$ we have

$$\begin{aligned} &\sup_{T>1} \limsup_{m\to+\infty} \frac{1}{mT} \sum_{i=0}^{m-1} \ln \left\| \Phi_{(i+1)T,iT}(\omega)|_{\Phi_{0,(i+1)T}(\omega)V} \right\|^{-1} \\ &\qquad \leqslant \sup_{T>1} \limsup_{m\to+\infty} \frac{1}{mT} \ln \frac{\|\Phi_{0,mT}(\omega)x_0\|}{\|x_0\|} \\ &\qquad \leqslant \sup_{T>1} \max_{x\in U_*} \limsup_{m\to+\infty} \frac{1}{mT} \ln \frac{\|\Phi_{0,mT}(\omega)x\|}{\|x\|} \\ &\qquad = \lambda_k. \end{aligned}$$

Therefore, $\Theta_k - \epsilon < \lambda_k$ for every $\epsilon > 0$, hence $\Theta_k \leqslant \lambda_k$. □

Theorem 2.6 *For any* $k \in \{1, 2, ..., n\}$ *the central exponents* Θ_k *of (1) is smaller or equal to the auxiliary exponents* γ_k.

Proof. First of all, as claimed in the Introduction, we will show that for definition of $\Theta_k(\omega)$ the formula (3) is equivalent to the formula (5). For $V_1 \in G_k, T \in \mathbb{R}^+, \omega \in \Omega, m \in \mathbb{N}$ we set

$$g(T, m, V_1) := \frac{1}{mT} \sum_{i=0}^{m-1} \ln \left\| \Phi_{(i+1)T,iT}(\omega)|_{\Phi_{0,(i+1)T}(\omega)V_1} \right\|^{-1}.$$

Then

$$\begin{aligned}
g(T,2m,V_1) &= \frac{1}{2mT}\sum_{i=0}^{2m-1} \ln \left\| \Phi_{(i+1)T,iT}(\omega)|_{\Phi_{0,(i+1)T}(\omega)V_1} \right\|^{-1} \\
&= \frac{1}{2mT}\sum_{i=0}^{2m-1} \ln \inf_{z\in V_1*} \frac{\left\|\Phi_{0,(i+1)T}(\omega)z\right\|}{\|\Phi_{0,iT}(\omega)z\|} \\
&= \frac{1}{2mT}\sum_{i=0}^{m-1} \left(\ln \inf_{z\in V_1*} \frac{\left\|\Phi_{0,(2i+1)T}(\omega)z\right\|}{\|\Phi_{0,2iT}(\omega)z\|} + \ln \inf_{z\in V_1*} \frac{\left\|\Phi_{0,(2i+2)T}(\omega)z\right\|}{\left\|\Phi_{0,(2i+1)T}(\omega)z\right\|} \right) \\
&\leqslant \frac{1}{2mT}\sum_{i=0}^{m-1} \ln \inf_{z\in V_1*} \frac{\left\|\Phi_{0,(2i+1)T}(\omega)z\right\|}{\|\Phi_{0,2iT}(\omega)z\|} \times \frac{\left\|\Phi_{0,(2i+2)T}(\omega)z\right\|}{\left\|\Phi_{0,(2i+1)T}(\omega)z\right\|} \\
&= \frac{1}{2mT}\sum_{i=0}^{m-1} \ln \inf_{z\in V_1*} \frac{\left\|\Phi_{0,(2i+2)T}(\omega)z\right\|}{\|\Phi_{0,2iT}(\omega)z\|} \\
&= \frac{1}{2mT}\sum_{i=0}^{m-1} \ln \left\| \Phi_{(i+1)2T,i2T}(\omega)|_{\Phi_{0,(i+1)2T}(\omega)V_1} \right\|^{-1} \\
&= g(2T,m,V_1).
\end{aligned}$$

Thus, for all $V_1 \in G_k, T \in \mathbb{R}^+, \omega \in \Omega, m \in \mathbb{N}$ we have

$$g(T,2m,V_1) \leqslant g(2T,m,V_1). \tag{12}$$

Now we will prove that for any fixed $V_1 \in G_k, T \in \mathbb{R}^+$ the following equality

$$\limsup_{m\to+\infty} g(T,2m,V_1) = \limsup_{m\to+\infty} g(T,m,V_1) \tag{13}$$

holds with probability 1. Note that

$$\frac{m+1}{m} g(T,m+1,V_1) - g(T,m,V_1) = \frac{1}{mT} \ln \left\| \Phi_{(m+1)T,mT}(\omega)|_{\Phi_{0,(m+1)T}(\omega)V_1} \right\|^{-1},$$

and

$$\begin{aligned}
-\frac{1}{mT}\ln \left\| \Phi^{-1}_{mT,(m+1)T}(\omega) \right\| \leqslant \frac{1}{mT} \ln \left\| \Phi_{(m+1)T,mT}(\omega)|_{\Phi_{0,(m+1)T}(\omega)V_1} \right\|^{-1} \\
\leqslant \frac{1}{mT}\ln \left\| \Phi_{mT,(m+1)T}(\omega) \right\|.
\end{aligned} \tag{14}$$

Put

$$\begin{aligned}
B_j &:= \left\{\omega \in \Omega | \left\|\Phi_{jT,(j+1)T}(\omega)\right\| \geq jT + n^2 e^{KT}\right\}, \quad j \in \{0,1,2...\}, \\
\tilde{B}_j &:= \left\{\omega \in \Omega | \left\|\Phi_{(j+1)T,jT}(\omega)\right\| \geq jT + n^2 e^{KT}\right\}, \quad j \in \{0,1,2...\}, \\
B &:= \bigcup_{i=1}^{+\infty} \bigcap_{j=i}^{+\infty} (\Omega \setminus (B_j \cup \tilde{B}_j)).
\end{aligned}$$

By Lemma 3.3 of Nguyen Dinh Cong [8] we have $\mathbb{P}(B) = 1$. Let $\omega \in B$ be arbitrary, then there exists $M(\omega) > 0$ such that for all $m > M(\omega)$ the following inequalities hold

$$\begin{aligned}
\frac{1}{mT} \ln \left\|\Phi_{mT,(m+1)T}(\omega)\right\| &\leqslant \frac{\ln(mT + n^2 e^{KT})}{mT}, && (15) \\
-\frac{1}{mT} \ln \left\|\Phi^{-1}_{mT,(m+1)T}(\omega)\right\| &\geq -\frac{\ln(mT + n^2 e^{KT})}{mT}. && (16)
\end{aligned}$$

From (14), (15) and (16) it follows that

$$\lim_{m \to +\infty} \Big(\frac{m+1}{m} g(T, m+1, V_1) - g(T, m, V_1) \Big) = 0$$

with probability 1, in particular,

$$\lim_{l \to +\infty} \Big(\frac{2l+1}{2l} g(T, 2l+1, V_1) - g(T, 2l, V_1) \Big) = 0.$$

Therefore, since $\lim\limits_{l \to +\infty} \frac{2l+1}{2l} = 1$, with probability 1 we have

$$\limsup_{l \to +\infty} g(T, 2l+1, V_1) = \limsup_{l \to +\infty} g(T, 2l, V_1),$$

from which (13) follows.

By (12) and (13), for all $V_1 \in G_k, T \in \mathbb{R}^+$ with probability 1 we have

$$\limsup_{m \to +\infty} g(T, m, V_1) \leqslant \limsup_{m \to +\infty} g(2T, m, V_1).$$

From this it implies that the formula (3) is equivalent to the formula (5). Moreover, this inequality also implies that

$$\sup_{T > 1} \limsup_{m \to +\infty} g(T, m, V_1) = \limsup_{T \to +\infty} \limsup_{m \to +\infty} g(T, m, V_1).$$

Therefore, taking into account Theorem 2.1 we have

$$\begin{aligned}
\Theta_k(\omega) &= \sup_{V\in G_k}\sup_{T>1}\limsup_{m\to+\infty}\frac{1}{mT}\sum_{i=0}^{m-1}\ln\left\|\Phi_{(i+1)T,iT}(\omega)|_{\Phi_{0,(i+1)T}(\omega)V}\right\|^{-1}\\
&= \sup_{V\in G_k}\sup_{T>1}\limsup_{m\to+\infty} g(T,m,V)\\
&= \sup_{V\in G_k}\limsup_{T\to+\infty}\limsup_{m\to+\infty} g(T,m,V)\\
&= \sup_{V\in G_k}\limsup_{T\to+\infty}\limsup_{m\to+\infty}\frac{1}{mT}\sum_{i=0}^{m-1}\ln\left\|\Phi_{(i+1)T,iT}(\omega)|_{\Phi_{0,(i+1)T}(\omega)V}\right\|^{-1}\\
&= \sup_{V\in G_k}\limsup_{T\to+\infty}\limsup_{m\to+\infty}\frac{1}{mT}\sum_{i=0}^{m-1}\ln\inf_{z\in\Phi_{0,iT}(\omega)V_*}\frac{\left\|\Phi_{iT,(i+1)T}(\omega)z\right\|}{\|z\|}\\
&\leqslant \limsup_{T\to+\infty}\limsup_{m\to+\infty}\frac{1}{mT}\sum_{i=0}^{m-1}\ln\, d_k\left[\Phi_{iT,(i+1)T}(\omega)\right]\\
&= \limsup_{T\to+\infty}\gamma_k(T)=\gamma_k.
\end{aligned}$$

The theorem is proved. ☐

Theorem 2.7 *For the equation (1) we always have*

$$\gamma_1=\Omega_1 \quad \textit{and} \quad \gamma_n=\Theta_n.$$

Proof. Note that

$$\begin{aligned}
d_1[\Phi_{iT,(i+1)T}(\omega)] &= \left\|\Phi_{iT,(i+1)T}(\omega)\right\|,\\
d_n[\Phi_{iT,(i+1)T}(\omega)] &= \left\|\Phi_{iT,(i+1)T}(\omega)^{-1}\right\|^{-1}.
\end{aligned}$$

Since the space G_1 consists of the single element $\mathbb{R}^n$, using (3) and the argument in the proof of Theorem 2.6 we get

$$\begin{aligned}
\Theta_n &= \sup_{T\in\mathbb{R}^+}\limsup_{m\to+\infty}\frac{1}{mT}\sum_{i=0}^{m-1}\ln d_n[\Phi_{iT,(i+1)T}(\omega)]\\
&= \limsup_{T\to+\infty}\limsup_{m\to+\infty}\frac{1}{mT}\sum_{i=0}^{m-1}\ln d_n[\Phi_{iT,(i+1)T}(\omega)]=\gamma_n.
\end{aligned}$$

Similarly, $\Omega_1=\gamma_1$. ☐

3 Lyapunov exponents of nondegenerate stochastic different equations coincide with central exponents

In the whole this section we will assume the *nondegeneracy condition (10)* for the equation (1).

Propotion 3.1 *For any $\epsilon > 0$ one can find $0 < \delta = \delta(\epsilon) < 1$ such that for any $V \in G_k$ and $U \in G_{n-k}$ $\quad (k \in \{1, 2, ..., n-1\})$ and any $\tau \in \mathbb{R}^+$ the set of $\omega \in \Omega$, for which*

$$\big[\Phi_{\tau,\tau+1}(\omega)V\big] \cap \hat{U}(\delta(\epsilon)) \not\equiv \{0\}$$

has $\mathbb{P}$-measure $\leqslant \epsilon$, where $\hat{U}(\varrho)$ denotes the cone consisting of vetors in $\mathbb{R}^n$ which make an angle $\leqslant \varrho$ with the subspace U.

Proof. The proof of this Proposition is completely similar to the proof of the Lemma 2 in Nguyen Dinh Cong [5] and Theorem in Nguyen Dinh Cong [7]. $\square$

Theorem 3.2 *There exists a positive constant c_1 such that for any $\epsilon \in (0, 1)$, $T > 1$ and $k \in \{1, 2, \cdots, n\}$ the following inequality holds*

$$\Theta_k \geq \gamma_k(T) + \frac{1}{T} \ln \frac{\delta(\epsilon)}{2} - 2c_1 \sqrt{\epsilon} \tag{17}$$

with probability 1, where $\delta = \delta(\epsilon)$ is determined according to Proposition 3.1.

Proof. Let $\epsilon \in (0, 1)$ and fix $k \in \{1, 2, ..., n\}$. Determine $\delta = \delta(\epsilon)$ from ϵ according to Proposition 3.1. Fix an arbitrary $T > 1$ and an arbitrary k-dimentional linear supspace V of $\mathbb{R}^n$. Let $i \in \{0, 1, 2, ...\}$, for brevity in expression, let Φ_i denotes the matrix $\Phi_{iT,(i+1)T}(\omega)$. Denote by $\{f_1, ..., f_k, f_{k+1}, ..., f_n\}$ such the eigenvectors, corresponding to the eigenvalues

$$d_1^2(\Phi_i) \geq \cdots \geq d_k^2(\Phi_i) \geq d_{k+1}^2(\Phi_i) \geq \cdots \geq d_n^2(\Phi_i)$$

of the matrix $\Phi_i^*\Phi_i$, that they form an orthonormal basis of $\mathbb{R}^n$. Furthermore, we denote by $U_{i,\omega}^{n-k}$ the linear subspace spanned by the last $n-k$ eigenvectors $\{f_{k+1}, ..., f_n\}$ of $\Phi_i^*\Phi_i$. We introduce some notations

$$\begin{aligned}
C_i &:= \big\{\omega \in \Omega : \big[\Phi_{0,iT}(\omega)V\big] \cap \hat{U}_{i,\omega}^{n-k}\big[\delta(\epsilon)\big] \not\equiv \{0\}\,\big\}, \\
\eta_i(\omega) &:= \frac{1}{T} \ln \,\big\|\Phi_{(i+1)T,iT}(\omega)\big\|\,, \\
\zeta_i(\omega) &:= \inf_{y \in V_*} \frac{1}{T} \ln \frac{\big\|\Phi_{0,(i+1)T}(\omega)y\big\|}{\|\Phi_{0,iT}(\omega)y\|}.
\end{aligned}$$

By the definition of the equation (1) the random variables $\eta_i(\omega)$ have second moments, bounded by a constant independent of $T > 1$ and $i \in \{0, 1, 2, ...\}$ (see H. Kunita [13, Lemma 4.5.3, page 156], notice the division by T in the formula defining $\eta_i(\omega)$). Let $\chi_i(\omega)$ denote the indicator funtion of the set C_i. By Proposition 3.1 and the Markov property of the solutions of the systems (1) we have

$$\mathbb{P}(C_i) \leqslant \epsilon, \qquad E(\chi_i(\omega)) \leqslant \epsilon.$$

By definition of C_i and χ_i, if $\chi_i(\omega) = 0$ then any vector of $\Phi_{0,iT}(\omega)V$ is separated from $U^{n-k}_{i,\omega}$ by an angle bigger than $\delta(\epsilon)$, hence

$$\begin{aligned}
\zeta_i(\omega) &= \inf_{z \in \Phi_{0,iT}(\omega)V_*} \frac{1}{T} \ln \frac{\|\Phi_i z\|}{\|z\|} \\
&\geq \inf_{z \in \Phi_{0,iT}(\omega)V_*} \frac{1}{T} \ln \Big(d_k(\Phi_i) \sin \angle(z, U^{n-k}_{i,\omega}) \Big) \\
&\geq \frac{1}{T} \ln \Big(d_k(\Phi_i) \sin \big[\delta(\epsilon)\big] \Big) \\
&\geq \frac{1}{T} \ln d_k \big[\Phi_{iT,(i+1)T}(\omega)\big] + \frac{1}{T} \ln \frac{\delta(\epsilon)}{2}.
\end{aligned}$$

If $\chi_i(\omega) = 1$ then

$$\zeta_i(\omega) \geq \inf_{z \in V_*} \frac{1}{T} \ln \frac{\big\|\Phi_{iT,(i+1)T}(\omega) z\big\|}{\|z\|} \geq -\frac{1}{T} \ln \Big\|\Phi^{-1}_{iT,(i+1)T}(\omega)\Big\| = -\eta_i(\omega).$$

Consequently,

$$\begin{aligned}
\zeta_i(\omega) &\geq \big[1 - \chi_i(\omega)\big] \Big(\frac{1}{T} \ln d_k \big[\Phi_{iT,(i+1)T}(\omega)\big] + \frac{1}{T} \ln \frac{\delta(\epsilon)}{2} \Big) - \chi_i(\omega) \eta_i(\omega) \\
&\geq \frac{1}{T} \ln d_k \big[\Phi_{iT,(i+1)T}(\omega)\big] + \frac{1}{T} \ln \frac{\delta(\epsilon)}{2} \\
&\qquad - \chi_i(\omega) \frac{1}{T} \ln d_k \big[\Phi_{iT,(i+1)T}(\omega)\big] - \chi_i(\omega) \eta_i(\omega).
\end{aligned}$$

It is easily seen that the random variables $\frac{1}{T} \ln d_k \big[\Phi_{iT,(i+1)T}(\omega)\big]$, $\zeta_i(\omega)$ have second moments bounded by a constant independent of $T > 1$, $i \in \{0, 1, 2, ...\}$ and $\epsilon \in (0, 1)$. Thererefore, there exists a positive constant $c_1 > 0$ which is independent of $T > 1,\ i \in \{0, 1, 2, ...\}$ and $\epsilon \in (0, 1)$ such that

$$\begin{aligned} E\left|\chi_i(\omega)\frac{1}{T}\ \ln\ d_k\big[\Phi_{iT,(i+1)T}(\omega)\big]\right| &\leqslant \frac{1}{T}\Big(E\chi_i^2(\omega)\Big)^{\frac{1}{2}}\Big(\mathrm{E}\ \ln^2 d_k\big[\Phi_{iT,(i+1)T}(\omega)\big]\Big)^{\frac{1}{2}} \\ &\leqslant c_1\Big(\int\limits_{\Omega}\chi_i^2(\omega)d\mathbb{P}\Big)^{\frac{1}{2}} = c_1\,\mathbb{P}\Big(\big\{\omega\,\big|\,\chi_i(\omega) = 1\big\}\Big)^{\frac{1}{2}} \\ &\leqslant c_1\sqrt{\epsilon}, \end{aligned}$$

and

$$E\left|\chi_i(\omega)\eta_i(\omega)\right| \leqslant \big[E\chi_i^2(\omega)\big]^{\frac{1}{2}}\ \big[E\eta_i^2(\omega)\big]^{\frac{1}{2}} \leqslant\ c_1\sqrt{\epsilon}.$$

Consequently,

$$E\zeta_i(\omega) \geq \frac{1}{T}E\ \ln\ d_k\big[\Phi_{iT,(i+1)T}(\omega)\big] + \frac{1}{T}\ \ln\ \frac{\delta(\epsilon)}{2} - 2c_1\sqrt{\epsilon}. \qquad (18)$$

Now we use results by Rosenblatt-Roth [16] to prove that the sequence of random variables $\zeta_i(\omega)$, $i = 1, 2, \ldots$, satisfies the strong law of large numbers. This is a crucial argument in the proof of this theorem. Note that the random variables $\zeta_i(\omega)$, $i = 1, 2, \ldots$, are not independent.

To this end we define a Markov chain in the state space $G_k \times Gl(n, \mathbb{R})$ with the Borel σ-algebra using the fundamental matrix of the equation (1) as follows:

Our Markov chain starts (at time $\tau = 0$) from the state $(V, I) \in G_k \times Gl(n, \mathbb{R})$. From the state $(V_1, Y_1) \in G_k \times Gl(n, \mathbb{R})$ at time $\tau = iT$ it goes to the state $(V_2, Y_2) \in G_k \times Gl(n, \mathbb{R})$ next time $\tau = (i + 1)T$ by the rule $V_2 = \Phi_{iT,(i+1)T}(\omega)V_1$, $Y_2 = \Phi_{(i+1)T,(i+2)T}(\omega)$.

Note that the second coordinate of our chain is a sequence of independent random variables, and the first coordinate is a Markov chain on the compact state space G_k generate by the solutions of the equation (1). The transition probability of our Markov chain is the product of transition probabilities on two coordinates because the second coordinate is independent on the present and past of the first coordinate. Denote by μ the Riemannian volume on the compact space G_k, and $P_i(V_1, B_1)$, where $V_1 \in G_k$ is a point and $B_1 \subset G_k$ is a measurable subset of G_k, the transition probability of the Markov chain of the first coordinate of our chain at the time moment $\tau = iT$. This Markov chain on G_k has density satisfying a parabolic partial equation which is determined by the equation (1) (see Khasminskii [11, page 96]). Since G_k is a compact manifold and our nondegeneracy condition (10) is uniform with respect to time we can find positive constants $K_3, K_4 > 0$ (see Aronson [2, page 891]) such that for any $i = 1, 2, \ldots$, any $V_1 \in G_k$ and any measurable subset $B \subset G_k$ we have

$$K_3 \mu(B) \leqslant P_i(V_1, B) \leqslant K_4 \mu(B). \tag{19}$$

where constants K_3, K_4 depend only n, T, μ_1, μ_2 and Lipschitz constant K of the equation (1). From this it follows that for any $i = 1, 2, \ldots$, any pair of points $V_1, V_2 \in G_k$ and any measurable subset $B \subset G_k$ we have

$$\sup |P_i(V_1, B) - P_i(V_2, B)| \leqslant \frac{K_4}{K_3 + K_4}. \tag{20}$$

Since the transition probability of our Markov chain on the product space $G_k \times Gl(n, \mathbb{R})$ is the product of two transition probabilities on its coordinates it is easily seen that the ergodic coefficient α_i of the transition function P_i of our Markov chain (see Dobrushin [9] and Rosenblatt-Roth [16] for definition of ergodic coefficient and its properties) satisfies for any $i = 1, 2, \ldots$ the inequality

$$\alpha_i = \alpha(P_i) \geq \frac{K_3}{K_3 + K_4}. \tag{21}$$

Thus for any $m \in \{1, 2, 3, ...\}$ we have

$$\alpha^{(m)} := \min_{0 \leqslant i \leqslant m-1} \alpha_i \geq \frac{K_3}{K_3 + K_4} > 0.$$

Let us come back to the random variables $\zeta_i(\omega)$ introduced above. We can consider them as random variables defined on our Markov chain as follows:

$$\zeta_i(\omega_i) = \zeta_i(V_i, \Phi_i) = \inf_{z \in V_i^*} \frac{1}{T} \ln \frac{\|\Phi_i(z)\|}{\|z\|}.$$

We known that ζ_i has second moments bounded by a constant c_2 independent of $T > 1$ and $i \in \{0, 1, 2, ...\}$, hence

$$0 \leqslant D\zeta_i \leqslant E|\zeta_i|^2 \leqslant c_2,$$

where $D\xi(\omega)$ denotes the variance of the random variable $\xi(\omega)$. This implies that

$$[m^2 \alpha^{(m)}]^{-1} \sum_{i=0}^{m-1} D\zeta_i \leqslant \frac{c_2}{m\alpha^{(m)}} < \frac{c_2(K_3 + K_4)}{mK_3} \to 0 \quad (m \to +\infty).$$

Therefore, according to Rosenblatt-Roth [16, Theorem 1, page 435] the sequence $\zeta_0, \zeta_1, \zeta_2, ...$ satisfies the strong law of large numbers, so we have with probability 1 the equalities

$$\begin{aligned}
\limsup_{m\to+\infty} \frac{1}{mT} \sum_{i=0}^{m-1} \ln \left\| \Phi_{(i+1)T,iT}(\omega)|_{\Phi_{0,(i+1)T}(\omega)V} \right\|^{-1} &= \limsup_{m\to+\infty} \frac{1}{m} \sum_{i=0}^{m-1} \zeta_i(\omega) \\
&= \limsup_{m\to+\infty} \frac{1}{m} \sum_{i=0}^{m-1} E\zeta_i(\omega) \\
&= \limsup_{m\to+\infty} \frac{1}{mT} \sum_{i=0}^{m-1} \mathrm{E} \ln \left\| \Phi_{(i+1)T,iT}(\omega)|_{\Phi_{0,(i+1)T}(\omega)V} \right\|^{-1}.
\end{aligned}$$

Using the definition of the central exponent Θ_k we get

$$\begin{aligned}
\Theta_k(\omega) &= \sup_{\widetilde{V}\in G_k} \sup_{T>1} \limsup_{m\to+\infty} \frac{1}{mT} \sum_{i=0}^{m-1} \ln \left\| \Phi_{(i+1)T,iT}(\omega)|_{\Phi_{0,(i+1)T}(\omega)\widetilde{V}} \right\|^{-1} \\
&\geq \limsup_{m\to+\infty} \frac{1}{mT} \sum_{i=0}^{m-1} \ln \left\| \Phi_{(i+1)T,iT}(\omega)|_{\Phi_{0,(i+1)T}(\omega)V} \right\|^{-1} \\
&= \limsup_{m\to+\infty} \frac{1}{m} \sum_{i=0}^{m-1} \zeta_i(\omega) = \limsup_{m\to+\infty} \frac{1}{m} \sum_{i=0}^{m-1} E(\zeta_i(\omega)) \\
&\geq \limsup_{m\to+\infty} \frac{1}{m} \sum_{i=0}^{m-1} \left(\frac{1}{T} E \ln d_k\left[\Phi_{iT,(i+1)T}(\omega)\right] + \frac{1}{T} \ln \frac{\delta(\epsilon)}{2} - 2c_1\sqrt{\epsilon} \right) \\
&= \limsup_{m\to+\infty} \frac{1}{mT} \sum_{i=0}^{m-1} E \ln d_k\left[\Phi_{iT,(i+1)T}(\omega)\right] + \frac{1}{T} \ln \frac{\delta(\epsilon)}{2} - 2c_1\sqrt{\epsilon}.
\end{aligned}$$

Consequently, with probability 1 we have

$$\Theta_k(\omega) \geq \gamma_k(T) + \frac{1}{T} \ln \frac{\delta(\epsilon)}{2} - 2c_1\sqrt{\epsilon}. \tag{22}$$

The theorem is proved. □

Theorem 3.3 *Assume the condition (10), then for any* $k \in \{1, 2, ..., n\}$ *we have*

$$\gamma_k = \Theta_k.$$

Proof. Fix $k \in \{1, 2, ..., n\}$. Taking into account Theorem 2.6 it suffices to prove that $\Theta_k \geq \gamma_k$. Due to Themrem 2.1 we only need to prove $\Theta_k \geq \limsup_{T\to+\infty} \gamma_k(T)$. To do this we will show that for any $\varrho > 0$, there exists $T_\varrho^{(1)} > 1$ such that for any $T > T_\varrho^{(1)}$ then

$$\gamma_k(T) < \Theta_k + \varrho.$$

By Theorem 3.2, for any $\epsilon \in (0, 1)$ and $T > 1$ we have

$$\gamma_k(T) + \frac{1}{T} \ln \frac{\delta(\epsilon)}{2} - 2c_1 \sqrt{\epsilon} \leqslant \Theta_k,$$

where $\delta = \delta(\epsilon)$ is as specified in Proposition 3.1. Fix an arbitrary $0 < \epsilon < \frac{\varrho^2}{16c_1^2}$. Since $\lim_{T\to+\infty} \frac{1}{T} \ln \frac{\delta(\epsilon)}{2} = 0$, for any $\varrho > 0$ there exists $T_\varrho^{(2)} > 1$ such that for any $T > T_\varrho^{(2)}$ we have

$$-\frac{\varrho}{2} < \frac{1}{T} \ln \frac{\delta(\epsilon)}{2} < \frac{\varrho}{2},$$

which implies for any $T > T_\varrho^{(2)}$ then

$$\gamma_k(T) - \frac{\varrho}{2} - 2c_1\sqrt{\epsilon} < \gamma_k(T) + \frac{1}{T} \ln \frac{\delta(\epsilon)}{2} - 2c_1 \sqrt{\epsilon} \leqslant \Theta_k.$$

Thus, taking into account the choice $0 < \epsilon < \min\left\{1, \frac{\varrho^2}{16c_1^2}\right\}$, we have that for any $\varrho > 0$, there exists $T_\varrho = \max\left\{T_\varrho^{(1)}, T_\varrho^{(2)}\right\}$ such that for any $T > T_\varrho$

$$\gamma_k(T) < \Theta_k + \frac{\varrho}{2} + 2c_1\frac{\varrho}{4c_1} < \Theta_k + \varrho.$$

Since $\varrho > 0$ is arbitrary, we obtain $\limsup_{T\to+\infty} \gamma_k(T) \leqslant \Theta_k$. □

Theorem 3.4 *Assume the condition (10) holds. Then the following equalitities hold*

$$\Theta_1 = \lambda_1 = \Omega_1 = \gamma_1.$$

Proof. By Theorem 2.7 we have

$$\gamma_1 = \Omega_1.$$

Due to assumption of the condition (10), Theorem 3.3 implies that

$$\gamma_1 = \Theta_1,$$

hence $\gamma_1 = \Omega_1 = \Theta_1$. By Theorems 2.4 and 2.5 we have $\Theta_1 \leqslant \lambda_1 \leqslant \Omega_1$. Consequently,

$$\Theta_1 = \lambda_1 = \Omega_1 = \gamma_1.$$

The theorem is proved. ☐

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