Coincidence of Lyapunov exponents and central exponents of linear Ito stochastic differential equations with nondegenerate stochastic term

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Abstract

In this paper we show that under a nondegeneracy condition Lyapunov exponents and central exponents of linear Ito stochastic differential equation coincide. Furthermore, as the stochastic term is small and tends to zero the highest Lyapunov exponent tends to the highest central exponent of the ordinary differential equation which is the deterministic part of the system.

Key words: Lyapunov exponents, central exponents, Lyapunov spectrum, nonautonomous stochastic differential equation, two-parameter stochastic flow.
MSC2000: Primary 60H10, 37H10, 34D08; secondary 60G17, 34F05, 93E15.

1 Introduction

In this paper we consider the linear Ito stochastic differential equation.

$$dX(t) = F_0(t)X(t)dt + \sigma \sum_{k=1}^{m} F_k(t)X(t)dW_t^k, \tag{1}$$

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where σ is a positive parameter and $F_k(t)\,(k \in \{0, 1, 2, ..., m\})$ are continuous matrix-valued functions bounded by a constant K, satisfies the following nondegeneracy condition of random perturbation:

There are positive numbers μ_1, μ_2 such that for any $x, y \in \mathbb{R}^n$ and $t \in \mathbb{R}^+$

$$\mu_1 \left\|x\right\|^2 \left\|y\right\|^2 \leqslant \sum_{k=1}^{m} \left\langle F_k(t)x, y\right\rangle \leqslant \mu_2 \left\|x\right\|^2 \left\|y\right\|^2 . \tag{2}$$

We study the Lyapunov and central exponents of equation (1). Using the ideas and methods by Millionshchikov [16, 17] and Nguyen Dinh Cong [4, 5, 6, 7, 8, 9] we shall prove that under the nondegeneracy condition (2) Lyapunov exponents and central exponents of the linear Ito stochastic differential equation (1) coincide. Furthermore, as the stochastic term is getting smaller and tends to zero (the intensity σ tends to zero) the highest Lyapunov exponent of (1) tends to the highest central exponent of the ordinary differential equation

$$dX(t) = F_0(t)X(t)dt, \tag{3}$$

which is the deterministic part of (1). Note that Nguyen Dinh Cong had in [4] similar results for the case the constant matrices $F_k(t)\,(k \in \{1, 2, ..., m\})$ ($F_k(t)$ are independent of t). However, the proof given there is incomplete since the technique of changing from series of random variables to independent variables is not completely verified. Here we use the strong law of large number for a series of random variables depending on a Markov chain provided by Rosenblatt-Roth [19] to solve this problem. Moreover we consider a more general case of variable matrices $F_k(t)\,(k \in \{1, 2, ..., m\})$.

The paper is organized as follows. In Sec. 2 we give the formulae defining exponents. In Sec. 3 we prove inequalities for estimation of distance between the central exponents and auxiliary exponents of (1); using those inequalities we prove our main results stating that the Lyapunov exponents of (1) coincide with central exponents of (1) and the highest Lypunov exponent of (1) tends to the highest central exponent of (3).

2 Definitions

It is known that with the above assumption the Cauchy problem of (1) has unique solution (see Khasminskii [13, page 79]). The linear Ito stochastic differential equations (1) generates a two-parameter stochastic flow $\Phi_{t_0,t}(\omega,\sigma)$ of linear operators of $\mathbb{R}^n$ (see Kunita [15, page 116]). The solution of (1) satisfying initial value condition $X(t_0) = x_0$, is a stochastic process given by the formula $X(t) = \Phi_{t_0,t}(\omega,\sigma)x_0$.

We denote by $\mathcal{G}_r$ the Grassmannian manifold of all r-dimentional subspace in $\mathbb{R}^n$. For a linear subspace U of $\mathbb{R}^n$ we denote by U_* the subset of all nonvanishing vectors of U. For any nondegenerate $n \times n$ matrix X let us denote by $d_1(X) \geq d_2(X) \geq ... \geq d_n(X)$ its singular numbers, i.e. they are the positive square roots of the eigenvalues of the matrix X^*X. Clearly, for any $k \in \{1, 2, ..., n\}$ we have

$$d_k(X) = \inf_{U\in\mathcal{G}_{n-k+1}} \sup_{x\in U_*} \frac{\|Xx\|}{\|x\|} = \sup_{V\in\mathcal{G}_k} \inf_{x\in V_*} \frac{\|Xx\|}{\|x\|}.$$

Definition 2.1 The random variables $\lambda_k(\omega,\sigma)$, $\Omega_k(\omega,\sigma)$, $\Theta_k(\omega,\sigma)$ ($k \in \{1, 2, ..., n\}$) defined by

$$\lambda_k(\omega,\sigma) := \min_{U\in\mathcal{G}_{n-k+1}} \max_{x\in U} \limsup_{t\to+\infty} \frac{1}{t} \ln \|\Phi_{0,t}(\omega,\sigma)x\|, \tag{4}$$

$$\Theta_k(\omega,\sigma) := \sup_{V\in\mathcal{G}_k} \sup_{T\in\mathbb{R}^+} \limsup_{m\to+\infty} \frac{1}{mT} \sum_{i=0}^{m-1} \ln \left\| \Phi_{(i+1)T,iT}(\omega,\sigma)\big|_{\Phi_{0,(i+1)T}(\omega,\sigma)V} \right\|^{-1}, \tag{5}$$

$$\Omega_k(\omega,\sigma) := \inf_{U\in\mathcal{G}_{n-k+1}} \inf_{T\in\mathbb{R}^+} \limsup_{m\to+\infty} \frac{1}{mT} \sum_{i=0}^{m-1} \ln \left\| \Phi_{iT,(i+1)T}(\omega,\sigma)\big|_{\Phi_{0,iT}(\omega,\sigma)U} \right\|, \tag{6}$$

where $\Phi|_S$ denotes the restriction of the operator Φ to S, are called *Lyapunov exponents* and *central exponents* of the equation (1), respectively.

It is not difficult to show (see Nguyen Dinh Cong and Nguyen Thi Thuy Quynh [10]) that formula (5) is equivalent to the following formula which can serve as a definition of $\Theta_k(\omega,\sigma)$ as well

$$\Theta_k(\omega,\sigma) := \sup_{V\in\mathcal{G}_k} \sup_{\substack{T\in\mathbb{R}\\ T>1}} \limsup_{m\to+\infty} \frac{1}{mT} \sum_{i=0}^{m-1} \ln \left\| \Phi_{(i+1)T,iT}(\omega,\sigma)\big|_{\Phi_{0,(i+1)T}(\omega,\sigma)V} \right\|^{-1}, \tag{7}$$

and the formula (6) defining $\Omega_k(\omega,\sigma)$ is equivalent to the following formula

$$\Omega_k(\omega,\sigma) := \inf_{U\in\mathcal{G}_{n-k+1}} \inf_{\substack{T\in\mathbb{R}\\ T>1}} \limsup_{m\to+\infty} \frac{1}{mT} \sum_{i=0}^{m-1} \ln \left\| \Phi_{iT,(i+1)T}(\omega,\sigma)\big|_{\Phi_{0,iT}(\omega,\sigma)U} \right\|. \tag{8}$$

Therefore, throughout this paper we assume $T \in \mathbb{R}$ and $T > 1$.

Definition 2.2 The random variables $\gamma_k(\omega,\sigma)$, $(k \in \{1,2,...,n\})$ defined by

$$\gamma_k(\omega,\sigma) := \limsup_{T\to+\infty} \limsup_{m\to+\infty} \frac{1}{mT} \sum_{i=0}^{m-1} \ln d_k \left[\Phi_{iT,(i+1)T}(\omega,\sigma)\right] \tag{9}$$

are called *auxiliary exponents* of the equation (1). The function $\gamma_k(T,\sigma)$ defined by

$$\gamma_k(T,\sigma) := \limsup_{m\to+\infty} \frac{1}{mT} \sum_{i=0}^{m-1} \mathbb{E} \ln d_k \left[\Phi_{iT,(i+1)T}(\omega,\sigma)\right] \tag{10}$$

where $\mathbb{E}\xi(\omega)$ denotes the expectation of the random variable $\xi(\omega)$, are called *auxiliary functions* of the equation (1).

We note that the Lyapunov exponents, central exponents and auxiliary exponents of (1) do not depend on ω (see Nguyen Dinh Cong and Nguyen Thi Thuy Quynh [10]). So we will drope ω and denote by $\lambda_k(\sigma)$, $\Theta_k(\sigma)$, $\Omega_k(\sigma)$ and $\gamma_k(\sigma)$ the Lyapunov exponents, central exponents and auxiliary exponents of (1), respectively.

3 Main result

For later use we recall some results on the strong law of large numbers for non-homogeneous Markov chains by Rosenblatt-Roth [19]. Let $(\mathcal{U}_i, \Sigma_i)$ be a sequence of measurable spaces, $i = 1,2,\ldots$; the stochastic transition functions $P_i(x_i, A_{i+1})$ with domains of definition $(\mathcal{U}_i, \Sigma_i, \mathcal{U}_{i+1}, \Sigma_{i+1})$, $(i = 1,2,\ldots)$, $x_i \in \mathcal{U}_i$, $A_{i+1} \in \Sigma_{i+1}$, define a Markov chain (on the above sequence of measurable spaces). Denote by $\alpha_i = \alpha(P_i)$ the ergodic coefficient of P_i (see Dobrushin [11] and Rosenblatt-Roth [18] for definition of ergodic coefficient and its properties). We consider the stochastic variables ξ_i, depending on $q \geq 1$ consecutive time-moments and suppose that all variances $D\xi_i$ are finite $(i = 1,2,\ldots)$. Put

$$\alpha^{(m)} = \min_{1\leqslant i\leqslant m} \alpha_i, \qquad D_m = \sum_{i=1}^{m} D\xi_i,$$

$$S_m = \sum_{i=1}^{m} \xi_i, \qquad U_m = \max_{1\leqslant s\leqslant m} |S_s - \mathbb{E}S_s|, \quad (m = 1,2,\ldots).$$

Propotion 3.1 *[19, Theorem 2, page 567] If $\alpha_i > \rho > 0$ for all $i = 1, 2, \ldots$ and $\sum_{m=1}^{+\infty} m^{-2} D\xi_m < +\infty$ then the sequence ξ_i $(i = 1, 2, \ldots)$ satisfies strong law of large numbers.*

Propotion 3.2 *[19, Lemma 1, page 568] If $\alpha_i > \rho > 0$ for all $i = 1, 2, \ldots$ then for any $m = 1, 2, \ldots$*

$$\mathbb{P}(U_m > \epsilon) < \frac{[20(1+\sqrt{6})]^2 \rho^2}{\epsilon^2} D_m.$$

Next we recall a technical results which first introduced by Millionshchikov and then developed by Nguyen Dinh Cong for various types of ordinary differential equations with random perturbations.

Propotion 3.3 *For any $\epsilon > 0$ one can find $0 < \delta = \delta(\epsilon) < 1$ such that for any $V \in \mathcal{G}_k$ and $U \in \mathcal{G}_{n-k}$ $\quad (k \in \{1, 2, ..., n-1\})$ and any $\tau \in \mathbb{R}^+$ the set of $\omega \in \Omega$, for which*

$$\left[\Phi_{\tau,\tau+1}(\omega,\sigma)V\right] \cap \hat{U}(\delta\sigma^{n^3}) \not\equiv \{0\}$$

has $\mathbb{P}$-measure $\leqslant \epsilon$, where $\hat{U}(\rho)$ denotes the cone consisting of vetors in $\mathbb{R}^n$ which make an angle $\leqslant \rho$ with the subspace U.

Proof. The proof of this Propotion is completely similar to the proof of Lemma 2 in Nguyen Dinh Cong [6] and Theorem in Nguyen Dinh Cong [8]. □

Now we are in a position to prove our main theorem which allows us to deduce the results claimed in Sec. 1.

Theorem 3.4 *There exists a positive constant c_1 such that for any $\epsilon \in (0, 1)$, $T \in \mathbb{R}, T > 1$ and $k \in \{1, 2, \cdots, n\}$ the following inequalities hold*

$$|\Omega_k(\sigma) - \gamma_k(T,\sigma)| \leqslant 2(c_1+1)\sqrt{\epsilon} - \frac{1}{T} \ln \frac{\delta\sigma^{n^3}}{2}, \tag{11}$$

$$|\Theta_k(\sigma) - \gamma_k(T,\sigma)| \leqslant 2(c_1+1)\sqrt{\epsilon} - \frac{1}{T} \ln \frac{\delta\sigma^{n^3}}{2}, \tag{12}$$

where $\delta = \delta(\epsilon)$ is determined according to Propotion 3.3.

Proof. For any $\epsilon \in (0,1)$, $T \in \mathbb{R}, T > 1$ and $k \in \{1, 2, \cdots, n\}$ in [10] we proved that there exist positive constants $c_1 > 0$ and $\delta(\epsilon) > 0$ such that

$$\gamma_k(T,\sigma) + \frac{1}{T}\ \ln \frac{\delta\sigma^{n^3}}{2} - 2c_1\sqrt{\epsilon} \leqslant \Theta_k(\sigma) \leqslant \Omega_k(\sigma). \tag{13}$$

Now we will estimate central exponents $\Omega_k(\sigma)$. To this end, for any $m \in \mathbb{N}$ we create a Markov chain of inverse time from mT to 0 on the measurable state space $\mathcal{G}_{n-k+1} \times Gl(n,\mathbb{R})$ with Borel σ-algebra. Our Markov chain starts (at $\tau = mT$) from the state $(U, I) \in \mathcal{G}_{n-k+1} \times Gl(n,\mathbb{R})$. From state $(U_1, Z_1) \in \mathcal{G}_{n-k+1} \times Gl(n,\mathbb{R})$ at time $\tau = (m-i)T$ it goes to the state $(U_2, Z_2) \in \mathcal{G}_{n-k+1} \times Gl(n,\mathbb{R})$ next time $\tau = (m-i-1)T$ by the rule $U_2 = \Phi_{(m-i)T,(m-i-1)T}(\omega,\sigma)U_1$, $Z_2 = \Phi_{(m-i)T,(m-i-1)T}(\omega,\sigma)$. Using the non-degeneracy condition (2) and estimation of the density of Markov transition probability (as the fundamental solution of a parabolic equation) we can prove that there exists a positive constant ν depend only n, T, μ_1, μ_2 and constants σ, K (for more details see the proof of Theorem 3.2 in [10]) such that for every $i = 0, 1, \ldots, m-1$ the ergodic coefficient β_i of stochastic transition function of our Markov chain satisfies following inequality

$$\beta_i > \nu > 0. \tag{14}$$

Fix a subspace $U \in \mathcal{G}_{n-k+1}$, for each $m \in \mathbb{N}$ we put

$$\zeta_i^m(\omega,\sigma) = \inf_{y \in U_*} \frac{1}{T} \ln \frac{\left\|\Phi_{mT,(m-i-1)T}(\omega,\sigma)y\right\|}{\left\|\Phi_{mT,(m-i)T}(\omega,\sigma)y\right\|} - \frac{1}{T} \ln d_{n-k+1}\left[\Phi_{(m-i)T,(m-i-1)T}(\omega,\sigma)\right].$$

Using the assumption on coefficients of (1), it is not difficult to show that the random variables $\zeta_i^m(\omega,\sigma)$, $i = 0, 1, \ldots, m-1$, have second moments bounded by a positive constant in dependent of T, i, m, ϵ. By changing the constant $c_1 > 0$ in (13) to a bigger constant if necessary we can assume that the variables $\zeta_i^m(\omega,\sigma)$, $i = 0, 1, \ldots, m-1$ have second moments bounded by the above constant $c_1 > 0$ (from formula (13)) and

$$\begin{aligned} \mathbb{E}\zeta_i^m(\omega,\sigma) &\geq \frac{1}{T}\ln\frac{\delta\sigma^{n^3}}{2} - 2c_1\sqrt{\epsilon}, \\ \frac{1}{m}\sum_{i=0}^{m-1}\mathbb{E}\zeta_i^m(\omega,\sigma) &\geq \frac{1}{T}\ln\frac{\delta\sigma^{n^3}}{2} - 2c_1\sqrt{\epsilon}. \end{aligned} \tag{15}$$

Take a large number $m \in \mathbb{N}$ and fix a number $l \in \mathbb{N}, l > 1$. For any $t \in \mathbb{N}$ satisfies $1 < l^t < l^{t+1} < m$ we consider a Markov chain of inverse time and with l^{t+1} steps

$$\zeta^m_{m-l^{t+1}}(\omega,\sigma), \zeta^m_{m-l^{t+1}+1}(\omega,\sigma), \ldots, \zeta^m_{m-1}(\omega,\sigma).$$

Put

$$\begin{aligned}
S^m_i(\omega) &:= \sum_{u=0}^{i-1} \zeta^m_{m-l^{t+1}+u}(\omega,\sigma) \quad (i = 1,2,\ldots,l^{t+1}),\\
U^m_{l^{t+1}}(\omega) &:= \max_{1\leqslant i\leqslant l^{t+1}} |S^m_i(\omega) - \mathbb{E}S^m_i(\omega)|,\\
\widetilde{S}^m_i(\omega) &:= \sum_{u=1}^{i} \zeta^m_{m-u}(\omega,\sigma) \quad (i = 1,2,\ldots,l^{t+1}),\\
\widetilde{U}^m_{l^{t+1}}(\omega) &:= \max_{1\leqslant i\leqslant l^{t+1}} |\widetilde{S}^m_i(\omega) - \mathbb{E}\widetilde{S}^m_i(\omega)|,
\end{aligned}$$

then

$$S^m_{l^{t+1}}(\omega) = \widetilde{S}^m_{l^{t+1}}(\omega),\ \widetilde{S}^m_i(\omega) = S^m_{l^{t+1}}(\omega) - S^m_{l^{t+1}-i}(\omega),\ S^m_i(\omega) = \widetilde{S}^m_{l^{t+1}}(\omega) - \widetilde{S}^m_{l^{t+1}-i}(\omega).$$

For any real number $a > 0$ we have

$$\begin{aligned}
\left\{\omega \,:\, \widetilde{U}^m_{l^{t+1}}(\omega) \geq 2a\right\} &= \left\{\omega \,:\, \max_{1\leqslant i\leqslant l^{t+1}} |\widetilde{S}^m_i(\omega) - \mathbb{E}\widetilde{S}^m_i(\omega)| \geq 2a\right\}\\
&= \left\{\omega \,:\, \max_{1\leqslant i\leqslant l^{t+1}} |[S^m_{l^{t+1}}(\omega) - S^m_{l^{t+1}-i}(\omega)] - [\mathbb{E}S^m_{l^{t+1}}(\omega) - \mathbb{E}S_{l^{t+1}-i}{}^m(\omega)]| \geq 2a\right\}\\
&= \left\{\omega \,:\, \max_{1\leqslant i\leqslant l^{t+1}} |[(S^m_{l^{t+1}}(\omega) - \mathbb{E}S^m_{l^{t+1}}(\omega)] - [S^m_{l^{t+1}-i}(\omega) - \mathbb{E}S^m_{l^{t+1}-i}(\omega)]| \geq 2a\right\}\\
&\subset \left\{\omega \,:\, \max_{1\leqslant i\leqslant l^{t+1}} |S^m_i(\omega) - \mathbb{E}S^m_i(\omega)| \geq a\right\} = \left\{\omega \,:\, U^m_{l^{t+1}}(\omega) \geq a\right\}.
\end{aligned}$$

Therefore,

$$\mathbb{P}(\left\{\omega \,:\, \widetilde{U}^m_{l^{t+1}}(\omega) \geq 2a\right\}) \leqslant \mathbb{P}(\{\omega \,:\, U^m_{l^{t+1}}(\omega) \geq a\}).$$

Arccording to Propotion 3.2, taking into account (14) we have

$$\begin{aligned}
\mathbb{P}(\left\{\omega \,:\, \widetilde{U}^m_{l^{t+1}}(\omega) \geq 2\sqrt{\epsilon}\right\}) &\leqslant \mathbb{P}(\left\{\omega \,:\, U^m_{l^{t+1}}(\omega) \geq \sqrt{\epsilon}\right\})\\
&\leqslant (20c_3)^2\epsilon^{-1}\nu^2 l^{t+1} c_1, \qquad (16)
\end{aligned}$$

where $c_3 = 1 + \sqrt{6}$. Put

$$\begin{aligned}
A_{m,t} &:= \left\{\omega\,:\,\max_{l^t\leqslant i<l^{t+1}} |S_i^m(\omega) - \mathbb{E}S_i^m(\omega)| \geq \sqrt{\epsilon} i\,\right\},\\
B_{m,t} &:= \left\{\omega\,:\,\max_{l^t\leqslant i<l^{t+1}} |S_i^m(\omega) - \mathbb{E}S_i^m(\omega)| \geq \sqrt{\epsilon} l^t\,\right\},\\
\widetilde{A}_{m,t} &:= \left\{\omega\,:\,\max_{l^t\leqslant i<l^{t+1}} |\widetilde{S}_i^m(\omega) - \mathbb{E}\widetilde{S}_i^m(\omega)| \geq 2\sqrt{\epsilon} i\right\},\\
\widetilde{B}_{m,t} &:= \left\{\omega\,:\,\max_{l^t\leqslant i<l^{t+1}} |\widetilde{S}_i^m(\omega) - \mathbb{E}\widetilde{S}_i^m(\omega)| \geq 2\sqrt{\epsilon} l^t\right\}
\end{aligned}$$

then $B_{m,t} \supset A_{m,t}$ and $\widetilde{B}_{m,t} \supset \widetilde{A}_{m,t}$. Therefore

$$\begin{aligned}
\mathbb{P}(\widetilde{A}_{m,t}) &\leqslant \mathbb{P}(\widetilde{B}_{m,t}) \leqslant \mathbb{P}\Big(\Big\{\omega\,:\,\widetilde{U}^m_{l^{t+1}}(\omega) \geq 2\sqrt{\epsilon} l^t\Big\}\Big)\\
&\leqslant \mathbb{P}\big(\big\{\omega\,:\,U^m_{l^{t+1}}(\omega) \geq \sqrt{\epsilon} l^t\big\}\big)\\
&\leqslant (20c_3)^2\epsilon^{-1}l^{-2t}\nu^2 l^{t+1}c_1\\
&\leqslant (20c_3)^2\epsilon^{-1}\nu^2 l^{-t+1}c_1.
\end{aligned}$$

Take $t^*, t_1 \in \mathbb{N}$, $t^* < t_1$ and $l^{t_1+1} < m$ we have

$$\begin{aligned}
\sum_{t=t^*}^{t_1}\mathbb{P}(\tilde{A}_{m,t}) &\leqslant \sum_{t=t^*}^{t_1}\mathbb{P}(\tilde{B}_{m,t})\\
&= \frac{(20c_3)^2\nu^2c_1}{\epsilon}\,[l^{-t^*+1} + l^{-t^*} + l^{-t^*-1}\ldots + l^{-t_1+1}]\\
&\leqslant \frac{(20c_3)^2\nu^2c_1}{\epsilon}\,l^{-t^*+1}[1 + l^{-1} + l^{-2} + \ldots + l^{-(t_1-t^*)} + \ldots]\\
&= \frac{(20c_3)^2\nu^2c_1}{\epsilon}\,l^{-t^*+1}\frac{1}{1-l^{-1}}. \qquad (17)
\end{aligned}$$

From (17) it follows that for any $\vartheta > 0$ there exists a number $t_0 \in \mathbb{N}$ large enough such that for any $k > t_0$, $m = l^k + 1$, we have

$$\mathbb{P}(F_m) \leqslant \sum_{t=t_0}^{k-1}\mathbb{P}(\widetilde{A}_{m,t}) < \vartheta,$$

where

$$F_m := \bigcup_{t=t_0}^{k-1}\widetilde{A}_{m,t}, \quad m = l^k + 1 \quad (k \geq t_0 + 1).$$

Put

$$H_m := \Omega \setminus F_m, \quad \hat{H}_{t_0} := \bigcap_{r=t_0+1}^{+\infty} \bigcup_{k=r}^{+\infty} H_{l^k+1},$$

then $\mathbb{P}(H_m) = 1 - \mathbb{P}(F_m) > 1 - \vartheta$, and since $H_{l^k+1} \supset H_{l^{k+1}+1} \supset H_{l^{k+2}+1} \supset \cdots$ we have

$$\mathbb{P}(\hat{H}_{t_0}) = \lim_{k \to +\infty} \mathbb{P}(H_{l^k+1}) \geq 1 - \vartheta.$$

We put further

$$\hat{H} := \bigcup_{\hat{t}=t_0}^{+\infty} \hat{H}_{\hat{t}}.$$

From (17) it follows that $\mathbb{P}(\hat{H}_{t_0})$ tends to 1 as $t_0 \to \infty$. Therefore

$$\mathbb{P}(\hat{H}) = \lim_{\hat{t} \to +\infty} \mathbb{P}(\hat{H}_{\hat{t}}) = 1.$$

For any $\omega_0 \in \hat{H}$ there exists a number $\hat{t} \in \mathbb{N}$ such that $\omega_0 \in \hat{H}_{\hat{t}}$. By definition of $\hat{H}_{\hat{t}}$ there exists an infinite sequence $k_1, k_2, \ldots \in \mathbb{N}$ satisfying $\hat{t} + 1 < k_1 < k_2 < \ldots$ such that $\omega_0 \in H_{l^{k_i}+1}$ for every $i = 1, 2, \ldots$. For any given number $j_0 \in \mathbb{N}$, we consider the case $m = l^{k_{j_0}} + 1$, since $\omega_0 \in H_{l^{k_{j_0}}+1}$ for any $t \in [\hat{t}, k_{j_0} - 1]$ we have $\omega_0 \notin \widetilde{A}_{m,t}$. Therefore, for every $u \in [l^{\hat{t}}, l^{k_{j_0}} - 1]$ we have

$$|\widetilde{S}^m_u(\omega_0) - \mathbb{E}\widetilde{S}^m_u(\omega)| < 2u\sqrt{\epsilon},$$

which implies

$$\frac{1}{u}\widetilde{S}^m_u(\omega_0) > \frac{1}{u}\mathbb{E}\widetilde{S}^m_u(\omega) - 2\sqrt{\epsilon}.$$

Consequently, using (15) we get

$$\begin{aligned}
\frac{1}{u}\widetilde{S}^m_u(\omega_0) &\geq \frac{1}{u}\sum_{i=m-u}^{m-1} \mathbb{E}\zeta^m_i(\omega, \sigma) - 2\sqrt{\epsilon} \\
&\geq \frac{1}{T}\ln\frac{\delta\sigma^{n^3}}{2} - 2c_1\sqrt{\epsilon} - 2\sqrt{\epsilon}.
\end{aligned}$$

On the other hand,

$$
\begin{aligned}
\frac{1}{u}\widetilde{S}_u^m(\omega_0) &= \frac{1}{u}\sum_{i=m-u}^{m-1}\zeta_i^m(\omega_0,\sigma) \\
&= -\sum_{j=0}^{u-1}\frac{1}{uT}\ln\left\|\Phi_{jT,(j+1)T}(\omega_0,\sigma)\big|_{\Phi_{0,jT}(\omega_0,\sigma)\circ\Phi_{mT,0}(\omega_0,\sigma)U}\right\| \\
&\qquad\qquad +\sum_{j=0}^{u-1}\frac{1}{uT}\ln d_k\big[\Phi_{jT,(j+1)T}(\omega_0,\sigma)\big].
\end{aligned}
$$

Therefore, for any $\omega_0 \in \hat{H}$ and $u \in [l^{\hat{t}}, l^{k_i}-1]$ $(i = 1, 2, \ldots)$ we have

$$
\begin{aligned}
&-\sum_{j=0}^{u-1}\frac{1}{uT}\ln\left\|\Phi_{jT,(j+1)T}(\omega_0,\sigma)\big|_{\Phi_{0,jT}(\omega_0,\sigma)\circ\Phi_{(l^{k_i}+1)T,0}(\omega_0,\sigma)U}\right\| \\
&\qquad +\sum_{j=0}^{u-1}\frac{1}{uT}\ln d_k\big[\Phi_{jT,(j+1)T}(\omega_0,\sigma)\big] \geq \frac{1}{T}\ln\frac{\delta\sigma^{n^3}}{2} - 2(c_1+1)\sqrt{\epsilon}. \qquad (18)
\end{aligned}
$$

Since a Grassmannian manifold $\mathcal{G}_{n-k+1}$ is compact, the sequence of its points $\big\{\Phi_{(l^{k_i}+1)T,0}(\omega_0,\sigma)U\big\}$ $(i = 1, 2, \ldots)$ contains a convergent subsequence. For simplicity of notation we assume that the sequence $\big\{\Phi_{(l^{k_i}+1)T,0}(\omega_0,\sigma)U\big\}$ $(i = 1, 2, \ldots)$ itself is convergent and the limit is a point $\widetilde{U} \in \mathcal{G}_{n-k+1}$. Using this convergence and noticing that the sequence $k_1 < k_2 < \ldots$ tends to $+\infty$, from (18) we get that for any fixed $u \geq l^{\hat{t}}$

$$
\begin{aligned}
&\sum_{j=0}^{u-1}\frac{1}{uT}\ln\left\|\Phi_{jT,(j+1)T}(\omega_0,\sigma)\big|_{\Phi_{0,jT}(\omega_0,\sigma)\widetilde{U}}\right\| \\
&\quad\leqslant \sum_{j=0}^{u-1}\frac{1}{uT}\ln d_k\big[\Phi_{jT,(j+1)T}(\omega_0,\sigma)\big] - \frac{1}{T}\ln\frac{\delta\sigma^{n^3}}{2} + 2(c_1+1)\sqrt{\epsilon}. \qquad (19)
\end{aligned}
$$

Since the random variables $d_k\big[\Phi_{jT,(j+i)T}(\omega,\sigma)\big]$ are independent and have variances bounded by a positive constant, the following equality holds with probability one

$$
\limsup_{m\to+\infty}\sum_{j=0}^{m-1}\frac{1}{mT}\ln d_k\big[\Phi_{jT,(j+i)T}(\omega,\sigma)\big] = \limsup_{m\to+\infty}\sum_{j=0}^{m-1}\frac{1}{mT}\ln \mathbb{E}d_k\big[\Phi_{jT,(j+i)T}(\omega,\sigma)\big].
$$

Therefore, since $\mathbb{P}(\hat{H}) = 1$ from (19) and definition of $\Omega_k(\sigma)$ we have for any $T > 1$ the following inequality

$$\begin{aligned}\Omega_k(\sigma) &\leqslant \limsup_{m\to+\infty} \sum_{j=0}^{m-1} \frac{1}{mT} \ln d_k\big[\Phi_{jT,(j+i)T}(\omega_0,\sigma)\big] + 2(c_1+1)\sqrt{\epsilon} - \frac{1}{T}\ln\frac{\delta\sigma^{n^3}}{2} \\ &\leqslant \limsup_{m\to+\infty} \sum_{j=0}^{m-1} \frac{1}{mT} \ln \mathbb{E} d_k\big[\Phi_{jT,(j+i)T}(\omega,\sigma)\big] + 2(c_1+1)\sqrt{\epsilon} - \frac{1}{T}\ln\frac{\delta\sigma^{n^3}}{2} \\ &\leqslant \gamma_k(T,\sigma) + 2(c_1+1)\sqrt{\epsilon} - \frac{1}{T}\ln\frac{\delta\sigma^{n^3}}{2}. \qquad (20)\end{aligned}$$

Combining inequality (20) with inequality (13), we get for any $T > 1$ inequalities

$$\begin{aligned}|\Omega_k(\sigma) - \gamma_k(T,\sigma)| &\leqslant 2(c_1+1)\sqrt{\epsilon} - \frac{1}{T}\ln\frac{\delta\sigma^{n^3}}{2}, \\ |\Theta_k(\sigma) - \gamma_k(T,\sigma)| &\leqslant 2(c_1+1)\sqrt{\epsilon} - \frac{1}{T}\ln\frac{\delta\sigma^{n^3}}{2}.\end{aligned}$$

The theorem is proved. □

Theorem 3.5 *Assume the condition (2) holds. Then for all* $k \in \{1,2,...,n\}$ *there exist limits*

$$\gamma_k(\sigma) := \lim_{T\to+\infty} \gamma_k(T,\sigma) = \gamma_k(\omega,\sigma), \qquad (21)$$

which coincide with auxiliary exponents of the equation (1), and the following equalities hold

$$\Omega_k(\sigma) = \lambda_k(\sigma) = \Theta_k(\sigma) = \gamma_k(\sigma).$$

Proof. Let $\epsilon \in (0,1)$ be arbitrary. Choose $T_\epsilon \in \mathbb{R}, T_\epsilon > 1$ to be so large that for any $T \geq T_\epsilon$ we have

$$|\frac{1}{T}\ \ln\frac{\delta\sigma^{n^3}}{2}| < \sqrt{\epsilon}.$$

Then (11) and (12) imply that

$$|\gamma_k(T_1,\sigma) - \gamma_k(T_2,\sigma)| \leqslant (4c_1+6)\sqrt{\epsilon}$$

for any $T_1, T_2 \in \mathbb{R}$, $T_1 \geq T_\epsilon$ and $T_2 \geq T_\epsilon$. Hence the limit in (21) exists. By choosing big $T > 1$ we can make the term $\frac{1}{T}\ln\frac{\delta\sigma^{n^3}}{2}$ arbitrarily small and $\gamma_k(T,\sigma)$ arbitrarily close to $\gamma_k(\sigma)$, hence we also get the following inequalities

$$\begin{aligned} |\Omega_k(\sigma) - \gamma_k(\sigma)| &\leqslant 3(c_1+1)\sqrt{\epsilon}, \\ |\Theta_k(\sigma) - \gamma_k(\sigma)| &\leqslant 3(c_1+1)\sqrt{\epsilon}. \end{aligned}$$

Since $\epsilon \in (0,1)$ is arbitrary, this implies that

$$\Omega_k(\sigma) = \gamma_k(\sigma) = \Theta_k(\sigma). \tag{22}$$

Now, since $\Theta_k(\sigma) \leqslant \lambda(\sigma) \leqslant \Omega_k(\sigma)$ (see [10]) we conclude that

$$\Omega_k(\sigma) = \lambda_k(\sigma) = \Theta_k(\sigma) = \gamma_k(\sigma). \tag{23}$$

The theorem is proved. □

Theorem 3.6 *Assume the condition (2) holds. Then the highest Lyapunov exponent $\lambda_1(\sigma)$ of the Ito differential equation (1) tends to the highest central exponents Ω_1 of the deterministic part of (1), i.e. of the ordinary differential equation (3), as σ tends to zero.*

Proof. The proof of this theorem is based on Theorem 3.4, definitions of λ_1, Ω_1, and is completely similar to the proof of Theorem 1 in Nguyen Dinh Cong [4] and Theorem in Nguyen Dinh Cong [8]. □

Acknowledgment

This work was supported by the Vietnam National Foundation for Science and Technology Development.

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