

ON THE RELATION BETWEEN CHIP FIRING GAMES AND PETRI NETS

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ABSTRACT. We represent a new approach to investigate the famous discrete dynamical system Chip Firing Game (CFG) by using Petri Net. First, we discuss about the definition and some main results of Chip Firing Game using different classical approaches. Then, we consider extensions of CFG, especially the class of coloured Chip Firing Games, which corresponds to the class of lower locally distributive lattice. Our main result is to construct injections mapping each Chip Firing Game or each coloured Chip Firing Game to a special Petri Net.

1. INTRODUCTION

The Chip Firing Game (CFG) was introduced by Bjorner, Lovasz and Shor in 1991 to illustrate the behaviors of distributed jobs in networks [2]. Then this model has become a very famous one which can be used to illustrate many systems in different science domains. For example, in complex systems research, CFG was considered as a paradigm for the so-called *self organized criticality* [17, 12, 4]; in economy or computer science, CFG was studied as a resource distribution systems [5, 10]. Because of this important role, many approaches to investigate the behavior of CFG were developed, from physics experimental techniques [17, 12] to other methods using algebraic structures [4], formal languages [2, 1] or enumerative combinatorics [11, 13].

Independently, there exists a very important model in Computer Sciences which was introduced in 1961 by C. A. Petri. This Petri Net (also known as a place/transition net or P/T net) is one of several mathematical modeling languages for the description of systems in computer science, system engineering and many other disciplines. Petri nets combine a well defined mathematical theory with a graphical representation of the dynamic behavior of systems. The theoretic aspect of Petri nets allow precise modeling and analysis of system behavior, while the graphical representation of Petri nets

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enable visualization of the model system state changes. This combination is the main reason for the great success of Petri nets.

In this paper, we discuss about the strong relation between these two significative models. More precisely, we give a constructive proof showing that there is an injection to map an arbitrary CFG to a special Petri Net. Moreover, variations of CFG or extensions of CFG can be also injected to Petri Net. One hand, this relation proves that the reachability graph of a special class of Petri Net have a strong structure - the structure of configuration space of CFG. On the other hand, this relation provides a new approach to study CFG using many research results and many simulation programs developed in Petri Net Theory. Especially, we prove that Petri Net can be used to harmonize many different kinds of variations of CFG. One of the most complicated and significative extensions of CFG is the coloured Chip Firing Game because of the lattice structure of their configuration space. In fact, in [13], the author proved that the class of configuration spaces of all CFG is a subset of the famous class of *LLD* lattices (lower locally distributive lattice), and this inclusion is strict. Then they introduced coloured Chip Firing Game, an extension of CFG for the object if that the set of configuration spaces of all coloured CFG are exactly the set of all *LLD* lattices. However, the definition of coloured CFG is quite complicated and the firing rule is not really similar to that of CFG. Our result at the end of this paper is to prove that all these models can be considered as subclass of Petri Net.

In Section 2, we first represent the definition of CFG and some important results of the order structure of this model, we also investigate coloured CFGs and their relation with *LLD* lattices. After that we talk about abelian version of CFG, that is widely studied in algebra. Then we introduced a new extension of CFG, the Conflicting Chip Firing Game which consists of firing each edge at a time and we show the order structure of this model using energy functions.

In Section 3, we recall the formal definition of Petri Net, and we prove the injection mapping a CFG to a special Petri Net. Similar results will be done for Abelian CFG, Conflicting CFG and coloured CFG.

2. CHIP FIRING GAMES AND RELATED MODELS

The goal of this section is to introduce some results of Chip firing game and its related model.

Chip firing game was introduced in 1991 in the Computer Science context [2]. A CFG is defined over a (directed) multigraph $G = (V, E)$, called the *support* or the *base* of the game. A weight $w(v)$ is associated with each vertex $v \in V$, which can be regarded as the number of *chips* stored at the *site* v . The CFG (denoted by $CFG(G)$) is then considered as a discrete dynamical system with the following rule, called the *firing rule*: a vertex containing at least as many chips as its outgoing degree (its number of going out edges)

transfers one chip along each of its outgoing edges. A *configuration* of CFG is a composition of n into V where n is the total number of chips which is constant over transfers process of CFG. The vertex v is said to be *firable* at the configuration a if the number of chips at v of a satisfying $a(v) \geq \text{deg}^+(v)$, where $\text{deg}^+(v)$ is outgoing degree of v . We call *configuration space*, and denote by $\text{CFG}(G, n, \mathcal{O})$, the set of all reachable configuration from the initial configuration \mathcal{O} . The configuration a is a *reachable* from another configuration b if a is obtained from b by a firing sequence, specially if b is the initial configuration, we say that a is *reachable*. If at a configuration μ there is no firing that is possible then μ is said to be a *fixed point*. We call the CFG be *strongly convergent* game if it has a unique fixed point. Notice that there exists CFGs with no fixed point. Let us present here one of most important result about the order structure of CFG.

Theorem 2.1. [11] *The configuration space of a CFG on a directed graph with no close component and with an arbitrary initial configuration \mathcal{O} ordered with the reachability relation is a lower locally distributive lattice.*

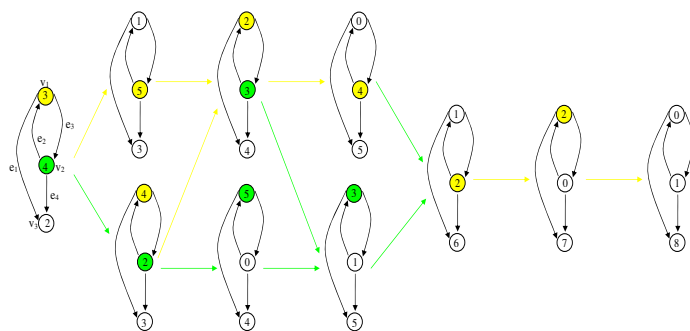


FIGURE 1. The configuration space of a CFG with 9 chips

Let us recall here some basic notion in the Order and Lattice Theory. A lattice L is an ordered set such that for any two elements a and b of L , there exists a unique smallest element which is greater than a and b (the supremum of a and b) and there exists a greatest element which is smaller than a and b (the infimum of a and b). A lattice L is distributive if it satisfies one of the two following laws of distributivity (which are equivalent):

$$\begin{aligned} \forall x, y, z \in L, x \wedge (y \vee z) &= (x \wedge y) \vee (x \wedge z) \\ \forall x, y, z \in L, x \vee (y \wedge z) &= (x \vee y) \wedge (x \vee z) \end{aligned}$$

A lattice is a *hypercube of dimension n* if it is isomorphic to the set of all subsets of a set of n elements, ordered by inclusion. It is also called a *boolean lattice*.

A lattice is *lower locally distributive* (denoted by *LLD* [15]) if the interval between an element and the infimum of all its lower covers is a hypercube.

This Theorem gives many consequences: first it proves the convergent property of CFG, then it shows that whatever the way of firings, after exactly the same time, the system reaches the same fixed point. Moreover this Theorem provides a characterization of order relation between two elements.

In [13], the authors proved that the set of configuration spaces of all CFG is strictly included in the set of *LLD* lattices and they introduced the coloured Chip Firing Game, that generates exactly the class of *LLD* lattices as follows.

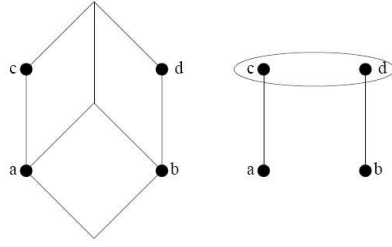


FIGURE 2. A *LLD* lattice and the order on its sup-irreducibles together. Successor relations are oriented upside down

Definition 2.2. For a graph $G = (V; E)$ and a set X of colours, we call a *coloured graph* the tuple $(V; E; X; col)$ where col is a mapping from E to X . The restriction of the graph to a colour $c \in X$ is the graph $(V; col^{-1}(c))$. A *coloured CFG* is defined over a directed coloured multi-graph $G = (V; E; X; col)$. A configuration is given by a function $\sigma : V \rightarrow \mathbb{N}^X$ which associates to a vertex a number of chips of each colour. Given a vertex v and a colour c , we will denote by $\sigma_c(v)$ the number of chips of colour c stored in v . To each vertex is also associated a state function: at any time, a vertex can be *open* or *closed*. The evolution rule for this model is to open a vertex. One can open vertex v if:

- v is closed
- there exists a colour $c \in X$ such that v can be fired (in the classical sense) in the restriction of the game to c (that is, there are at least as many chips of colour c in v as there are edges of colour c going out from v).

Opening a vertex consists in:

- marking it as open
- for each colour c in X , consider the restriction of the game to c and to the set of open vertices, and play the game until the final configuration is reached.

Notice that one must ensure that the movements of chips that occur when opening a vertex stops after some time. So we will consider only graphs in

which, for each colour c , the restriction of the game to c is a (classical) convergent CFG (this can be achieved by forbidding closed strongly connected component in the restriction of the graph to c [11]).

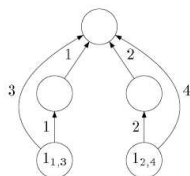


FIGURE 3. The coloured CFG \tilde{C} for the lattice of Figure 2.

At the beginning of an execution, all the vertices are closed. Since only closed vertices can be opened, coloured CFGs are convergent: after some time, no vertex can be opened and the final configuration is reached. They are also simple, therefore the configurations of the game are given by their shot-sets. The configuration space is ordered by the following relation: $\sigma \leq \sigma' \leftrightarrow s(\sigma) \subseteq s(\sigma')$.

The *restriction* of a coloured CFG to a colour $c \in X$ is the game defined over the restriction of the support graph to c such that, for each $v \in V$, the initial configuration is $\sigma_c(v)$. The restriction of the game to a set of vertices is the game played on the induced subgraph with the corresponding restriction of the initial configuration. In our figures, we will draw open vertices in gray. The colours will be represented by numbers, the colour of an edge being indicated by its label. In a vertex, a number N_{c_1, \dots, c_k} means that there are N chips of colour c_1 , N chips of colour c_k , and so on, in the vertex. For an example of execution of a coloured CFG see Figure 4.

The coloured Chip Firing Game is an extension of the classical Chip Firing Game model. Indeed we have the following result.

Theorem 2.3. [13] *All convergent classical CFGs are equivalent to coloured CFGs*

Theorem 2.4. [13] *The configuration space of a coloured CFG is a LLD lattice. Reciprocally, for any LLD lattice L , there exists a coloured CFG such that its configuration space is isomorphic to L .*

The class of CFGs and coloured CFGs are quite important because the class of LLD lattices is a classical object which is widely studied in algebra, in combinatorics and other mathematical domains.

Independently, the physicists studied the Abelian Sand Pile Model, (ASM) [6, 7] introduced in [17]: the model is defined over a finite two-dimensional grid, each cell containing a number of grains. The evolution rule then says that a cell which contains at least four grains can give one of them to each of its four neighbors. Therefore, its number of grains is decreased by four. If the cell is on the border of the grid, then some grains may fall to the

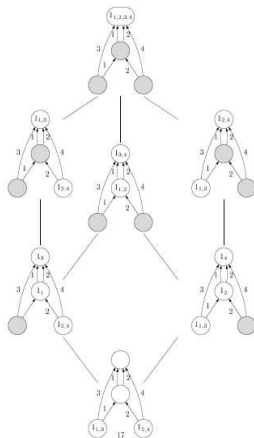


FIGURE 4. The configuration space of the coloured CFG obtained from the CFG \tilde{C} of Figure 3. Successor relations are oriented upside down

exterior, which simply stores the grains it receives. See Figure 5 for an example. This model has many important properties, and has mainly been studied from the algebraic point of view [7], [14]. It has been extended by Cori and Rossin in [3]: a number of gains is associated to each vertex of a given undirected connected graph with a special vertex called the sink. Any vertex except the sink can give a grain to each of its neighbors if it contains sufficiently many grains (i.e. at least as many grains as its degree). See Figure 5 (right) for an example. The algebraic properties of the original model are preserved, and this generalization received much attention since then. In [8], authors proved that one can consider Abelian Sand Pile Model as a variation of CFG (which will be called *Abelian CFG*) where the support directed graph is symmetric (except for the sink), or we can say that the support graph is an undirected graph.

Actually, there are many extensions of CFG obtained by relaxing the condition of firing, among them, a new one - the Conflicting Chip firing game - will be represented below. Let G be a directed graph, a vertex v of G is firable if it contains at least one chip and its firing is carried out by sending one chip along one edge from v to one of its neighbors. Each transition performs only one transfers chip along one edge. However, the firing of a chip along one edge may cause a conflict with the one along another edge. Hence we call this new model *Conflicting Chip Firing Game* (CCFG).

Keep in mind the notion of reachability relation and configuration space. We also denote by $CCFG(G, n)$ the set of all configurations of $CCFG(G, n)$ and call the *configuration space* of this game. This set is exactly the set of compositions of n into V .

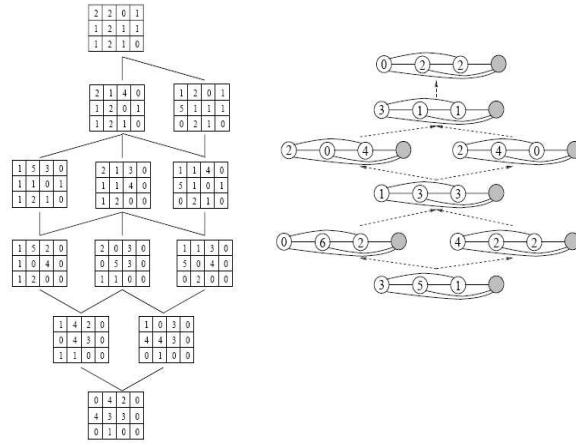


FIGURE 5. Left: an example of the original Abelian Sandpile Model on a 4×3 grid. Right: an example of the generalized Abelian Sandpile Model on a graph (the sink is the shaded vertex). Successor relations are oriented upside down.

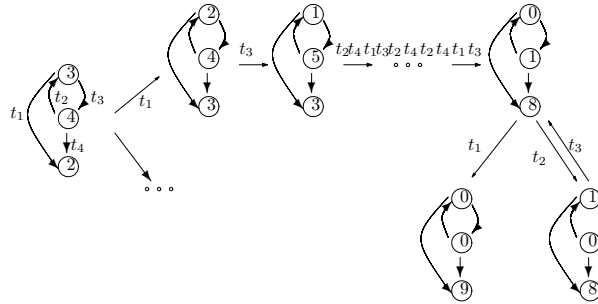


FIGURE 6. The configuration space of a CCFG with 9 chips

The following theorem proves the order structure of the configuration space of $CCFG$ on a special class of support graphs.

Theorem 2.5. [9] *The configuration space $CCFG(G, n)$ of conflicting Chip firing game on DAG G is ordered with the reflexive and transitive closure of the successor relation.*

Recall that a *directed acyclic graph* (DAG) is a directed graph without cycles.

Moreover, one can analyze this order relation as follows. Let $CCFG(G, n)$ on DAG $G = (V, E)$ and let $a = (a_1, a_2, \dots, a_{|V|})$ be a composition of n on V . It is easy to check that the vertices set V is ordered by the following relation \leq : for all $x, y \in V, y \leq x$ if and only if either $x = y$ or there is a path from x to y . We define $A \subseteq V$ is an *order filter* or, for short, *filter* (alternative terms are *increasing set* or *up-set*) if, whenever $x \in A, y \in V$

and $y \geq x$, we have $y \in A$. Denoted by $\mathcal{F}(V)$ the set of all filters of V and by $e(A, a) = \sum_{i \in A} a_i$ the *energy* of a on a subset $A \subseteq V$. A characterization of the order of $CCFG(G, n)$ is given in the following statement:

Theorem 2.6. [9] *Let a and b be two configurations of $CCFG(G, n)$. Then $a \geq b$ in $CCFG(G, n)$ if and only if $e(A, a) \geq e(A, b)$, for all filter $A \in \mathcal{F}(V)$.*

Now, for all model as CFG, Abelian CFG, CCFG or coloured CFG, we define the *reachability graph* the graph corresponds to the configuration space of the model, which means that vertices of this graph are configurations of the model, and edges of this graph are the successor relations of the model.

3. PETRI NETS AND THE RELATIONSHIP WITH CHIP FIRING GAME MODELS

3.1. Petri Net Definition. A Petri net is a particular kind of bipartite directed graphs populated by three types of objects. These objects are places, transitions, and directed arcs. Directed arcs connect places to transitions or transitions to places. In order to study the dynamic behavior of a Petri net model system in terms of its states and state changes, each place may potentially hold either none or a positive number of tokens.

A Petri net is formally defined as a 5-tuple $N = (P, T, I, O, M_0)$, where

- (1) $P = p_1, p_2, \dots, p_m$ is a finite set of places;
- (2) $T = t_1, t_2, \dots, t_n$ is a finite set of transitions, $P \cup T \neq \emptyset$, and $P \cap T = \emptyset$;
- (3) $I : P \times T \rightarrow \mathbb{N}$ is an input function that defines directed arcs from places to transitions, where \mathbb{N} is a set of nonnegative integers;
- (4) $O : T \times P \rightarrow \mathbb{N}$ is an output function that defines directed arcs from transitions to places; and
- (5) $M_0 : P \rightarrow \mathbb{N}$ is the initial marking.

A Petri net structure $N = (P, T, I, O)$ without any specific initial marking is denoted by N . A Petri net with the given initial marking is denoted by (N, M_0) .

A *marking* in a Petri net is an assignment of tokens to the places of a Petri net. The number and position of tokens may change during the execution of a Petri net. A marking is denoted by M , an m vector, where m is the total number of places. The p th component of M , denoted by $M(p)$, is the number of tokens in place p . A Petri net is called *preserve* if the total number of tokens is constant over execution process of net. A pair of a place p and a transition t is called a self-loop, if p is both an input place and an output place of t . A Petri net is said to be *pure* if it has no self-loops.

A Petri net graph is a Petri net structure as a bipartite directed multi-graph. Corresponding to the definition of Petri nets, a Petri net graph has two types of nodes. A *circle* represents a place and a *bar* or a *box* represents a transition. Directed arcs (arrows) connect places and transitions, with some arcs directed from places to transitions and other arcs directed from transitions to places. An arc directed from a place p_j to a transition t_i defines p_j to be an *input place* of t_i , denoted by $I(p_j, t_i) = 1$. An arc directed from a transition t_i to a place p_j defines p_j to be an *output place* of t_i , denoted by

$O(t_i, p_j) = 1$. If $I(p_j, t_i) = k$ (or $O(t_i, p_j) = k$), then there exist k directed (parallel) arcs connecting place p_j to transition t_i (or connecting transition t_i to place p_j). A circle contains k dot or a nonnegative integer number k represents a place contains k token.

Example 1: A simple Petri net.

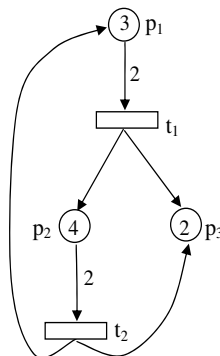


FIGURE 7. A simple Petri net

A transition t is said to be *enabled* or *firable* at marking M if each input place p of t contains at least the number of tokens equal to the weight of the directed arc connecting p to t , *i.e.*, $M(p) \geq I(p, t)$ for any p in P .

The firing of an enabled transition t removes from each input place p the number of tokens equal to the weight of the directed arc connecting p to t . It also deposits in each output place p the number of tokens equal to the weight of the directed arc connecting t to p . Mathematically, firing t at M yields a new marking

$$M'(p) = M(p) - I(p, t) + O(t, p) \text{ for any } p \text{ in } P.$$

Example 2: Consider the simple Petri net shown in Figure 7. The new token distribution of this Petri net is shown in Figure 8.

For a pure Petri net with m places and n transitions, the incidence matrix $A = (a_{p,t})_{m \times n}$ is an $n \times m$ matrix of integers and its typical entry is given by $a_{p,t} = a_{p,t}^+ - a_{p,t}^-$, where $a_{p,t}^- = I(p, t)$ is the weight of the arc from a place p to a transition t and $a_{p,t}^+ = O(p, t)$ is the weight of the arc from transition t to its output place p . One can use two matrices $A^+ = (a_{pt}^+)$ and $A^- = (a_{ij}^-)$ to analyze this Petri Net. Transition t is enabled at a marking M if and only if $a_{pt}^- \leq M(p)$ for all $p \in P$. In writing matrix equations, we write a marking M_k as an $m \times 1$ column vector. The j -th entry of M_k denotes the number of tokens in place j immediately after the k -th firing in some firing sequence. The k -th firing or *control vector* u_k is an $n \times 1$ column vector of $n - 1$ 0's and one nonzero entry, a 1 in the i -th position indicating that transition i fires at the k -th firing. We can write the following state equation for a Petri net (T. Murata [16]): $M_k = M_{k-1} + A^\top u_k, k = 1, 2, \dots$

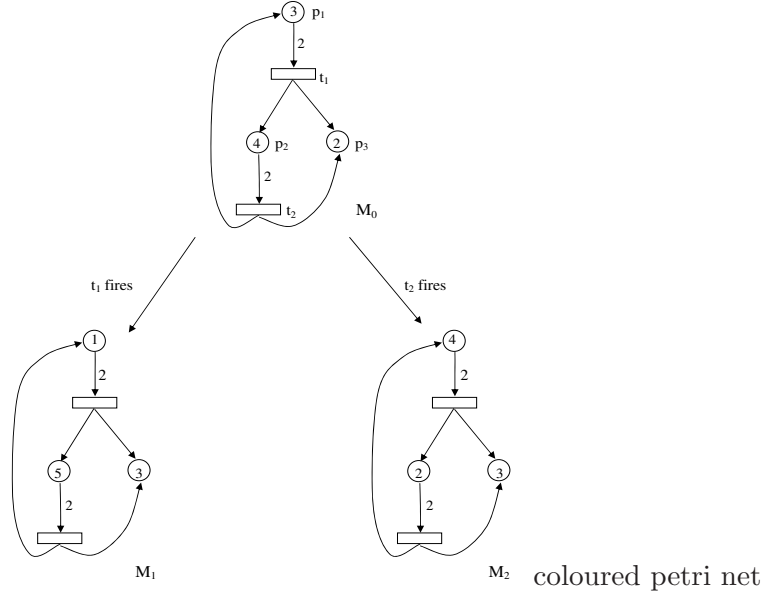


FIGURE 8. Firing of enabled transitions.

A marking M , is said to be *reachable* from a marking M_0 if there exists a sequence of firings that transforms M_0 to M . A *firing* or *occurrence sequence* is denoted by $\sigma = M_0 t_1 M_1 t_2 M_2 \dots t_n M_n$ or simply $\sigma = t_1 t_2 \dots t_n$. In this case, M_n is reachable from M_0 by σ and we write $M_0[\sigma > M_n$. A marking M_1 is said to be *immediately reachable* from M_0 if firing an enabled transition in M_0 results in M_1 . The set of all possible markings reachable from M_0 in a net (N, M_0) is denoted by $R(N, M_0)$ or simply $R(M_0)$. The set of all possible firing from M_0 in a net (N, M_0) is denoted by $L(N, M_0)$ (denoted by $CFG(G)$ or simply $L(M_0)$).

A transition t is said to be *live* in (N, M_0) if and only if for any $M \in R(N, M_0)$, there exists $M' \in R(N, M)$ such that t can be fired at M' . The Petri net (N, M_0) is said to be *live* if and only if every transition of N is live.

A place p is said to be *k-bounded* if the number of tokens in p is always less than or equal to k (k is a nonnegative integer number) for every marking $M \in R(M_0)$. It is *safe* if it is 1-bounded. A Petri net (N, M_0) is *k-bounded* (safe) if each place in P is *k-bounded* (safe).

Given a bounded Petri net (N, M_0) , from its initial marking M_0 , we can obtain as many new markings as the number of the enabled transitions. From each new marking, we can again reach more markings. Repeating the procedure over and over results in a graph representation of the markings. Nodes represent markings generated from M_0 , and each arc represents a transition firing, which transforms one marking to another. This graph is called *reachability graph* of the net (N, M_0) and denoted by (\mathcal{G}, M_0) .

3.2. The relationship between chip firing game models and petri nets. We first prove that Chip firing game can be considered as a subclasses of general Petri nets.

Given a CFG over a multigraph $G = (V, E)$. We will give a corresponding Petri nets structure N such that for every initial configuration O with n chips of $CFG(G)$, there exists an initial marking M_0 such that the reachability graph $R(N, M_0)$ is isomorphic to reachability graph of $CCFG(G, n, \mathcal{O})$.

Definition 3.1. The mapping ϕ from the set of Chip firing games to the set of Petri Net defined as follows. For a given (multi) directed graph G , $\phi(CFG(G))$ is the Petri Net $(N) = (P, T, I, O)$ such that:

- The set of places: $P = V$ and for each $p \in P$, $M(p)$ is exactly the number of *chips* stored at the corresponding vertex v in CGF.

- The set of transitions: for each $v \in V$ with $deg^+(v) > 0$, we define a transition $t(v)$ satisfying two following conditions:

- (i) there is an unique input place of $t(v)$ which is v and $I(v, t(v)) = deg^+(v)$,

- (ii) the output places u of $t(v)$ correspond to vertices $u \in V$ such that $(v, u) \in E$, and $O(t(v), u) = w(v, u)$, where $w(v, u)$ is the weight of arc (v, u) in G .

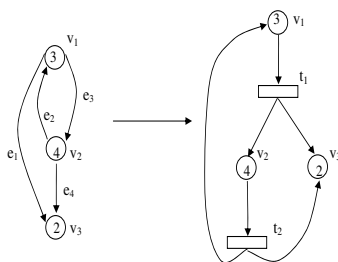


FIGURE 9. A CFG and its corresponding Petri Net

We consider now the reachability graphs of $CFG(G)$ and N for a given initial configuration.

Theorem 3.2. . Let G be a directed graph, and let O be an initial configuration. Let N be the Petri Net obtained from $CFG(G)$ by the map ϕ , and let M_0 be the marking of N which corresponds to O . Then the Petri Net (N, M_0) and the model $CCFG(G, n, \mathcal{O})$ have the same reachability graph.

Proof. We prove that the firing of a vertex v in G corresponds to the applying of the transition $t(v)$ in N . In fact, let a be a configuration of $CFG(G, n, \mathcal{O})$ and let $v \in V$ be any vertex. The vertex v is firable at the configuration a if and only if it contains at least as many chips as its outgoing degree. This equivalent to $I(v, t(v)) \leq M(v)$ and hence transition $t(v)$ is enabled.

When firing v the number of chips at v is reduced by $\deg^+(v)$ chips and the number of chips at neighbor vertices u of v is increased by $w(v, u)$ chips. Equivalently, when applying transition $t(v)$ in N , it removes from input place v $I(v, t(v))$ token, which is equal to $\deg^+(v)$. On the other hand, for an outgoing neighbor u of v , the output place of $t(u)$ is increased by $w(v, u)$ tokens. Moreover, the initial marking M_0 of this Petri Net is also the initial configuration space of $CFG(G, n, \mathcal{O})$. So, they have the same reachability graph. \square

Because the number of chips in a CFG is unchanged after firing sequences, so the following result is straightforward.

Corollary 3.3. *CFG is a preserve Petri net.*

Secondly, we consider Abelian CFG. Because this model is a kind of CFG, with the condition that the support graphs is an undirected graphs, so we analyze this condition for the corresponding Petri Net. Let $G = (V, E)$ be an undirected graph and let N be the Petri Net structure obtained from $CFG(G)$ by map ϕ . From the symmetrization of G , we have for all $u, v \in V$, $(u, v) \in E$ if and only if $(v, u) \in G$ and the weight of these two edges are the same. Or, from the definition of ϕ , we know that $w(v, u) = O(t(v), u)$ and $w(u, v) = O(t(u), v)$; moreover, the function O is represented by the matrix A^+ of N , that means $a_{u,v}^+ = O(t(v), u)$ and $a_{v,u}^+ = O(t(u), v)$. So the symmetrization of G corresponds to the symmetrization of the incidence matrix A^+ of N . Hence a subclass of Petri Net is in bijection with the class Abelian CFG.

Finally, we consider the Conflicting Chip Firing Games. Given a $CCFG(G)$ over a (multi) directed graph $G = (V, E)$. We define the map ψ also define the *reachability graph* of $CCFG(G, n, \mathcal{O})$ completely similar to the reachability graph of $CFG(G, n, \mathcal{O})$. This game is simulated by Petri nets as follows:

Definition 3.4. The mapping ψ from the set of Conflicting Chip firing games to the set of Petri Net defined as follows. For a given (multi) directed graph G , $\psi(CFG(G))$ is the Petri Net $(N) = (P, T, I, O)$ such that:

- The set of places $P = V$ and for each $p \in P$, $M(p)$ is exactly the number of *chips* stored at the corresponding v in CCFG.
- The set of transitions. For each arc $e = (u, v) \in E$, we defines a transition $t(e)$ satisfying two following conditions:
 - (i) u is unique input place of t and $I(u, t(e)) = 1$, $I(r(v), \theta(v)) = N + 1$
 - (ii) v is unique output place of t and $O(t(e), v) = 1$.

Note that in this Petri net, we have $|T| = |E|, |P| = |V|$.

Similar to theorem 3.2 we have the following result which show that Conflicting Chip Firing Games is a subclass of Petri Nets.

Theorem 3.5. *Let G be a directed graph, and let O be an initial configuration. Let N be the Petri Net obtained from $CCFG(G)$ by the map ψ , and*

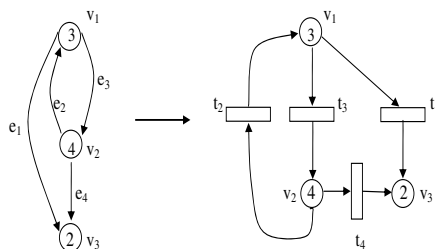


FIGURE 10. A Conflicting CFG and its corresponding Petri Net

let M_0 be the marking of N which corresponds to O . Then the Petri Net (N, M_0) and the model $CCFG(G, n, \mathcal{O})$ have the same reachability graph.

We now provide condition for *liveness* of this Petri net model.

Proposition 3.6. *The Petri Net (N, M_0) in definition 3.4 is live if and only if the graph G is strongly connected and M_0 has at least one token.*

The most complicated problem is to construct an injection from coloured CFGs to Petri Nets because different from other models in coloured CFGs, there are two kinds of transitions: the opening of a vertex and the normal transitions as in classical CFG. Other difficult is that in each vertex there are chips of different colours and the edges are also of different colours. In our first works, we tried to use coloured Petri Net, but the colour notion in Coloured CFGs and that of Coloured Petri Nets are quite different, so at the end, we propose to the following injection.

Let consider $ColCFG(G)$ a coloured CFG defined as in Definition 2.2 on a coloured graph $G = (V, E, X, col)$. Let m be the number of colors. For u, v in V and c in X , we write $d(v, c)$ the number of edges of colours c from v , and $d((v, u), c)$ the number of edges of colours c from v to u . Let N be a very great integer (which is greater than the total number of chips).

Definition 3.7. Let $ColCFG(G)$ be a coloured CFG on a coloured graph $G = (V, E, X, col)$. The mapping χ from the set of coloured Conflicting Chip firing games to the set of Petri Net defined as follows.

The image $\chi(ColCFG(G))$ is the Petri Net structure $(N(G)) = (P, T, I, O)$ such that:

- The set of places P : for each vertex v in V , there are $2m + 1$ places of P where m places denoted by $p(v, c)$, m places denoted by $q(v, c)$ and one place denoted by $r(v)$ for $c \in C$.
- The set of transitions T : for each vertex v in V , there are $2m + 1$ transitions of T where m transitions denoted by $t(v, c)$, m transitions denoted by $f(v, c)$ and one transition denoted by $\theta(v)$ for $c \in C$.

- The function I . For each $v \in V, c \in C$, $I(p(v, c), t(v, c)) = N + d(v, c)$, $I(r(v), \theta(v)) = N + 1$, and $I(q(v, c), f(v, c)) = N + d(v, c)$. All other values of function I equal to zero.
- The function O . For each $v \in V, c \in C$, $O(t(v, c), r(c)) = 1$, $O(\theta(v), q(v, c)) = N$, $O(f(v, c), q(v, c)) = N$, and for all neighbor u of v , $O(f(v, c), p(u, c)) = d((v, u), c)$ and $O(f(v, c), q(u, c)) = d((v, u), c)$. All other values of function O equal to zero.
- Condition added to N : operations of type θ can be realized if and only if all operations of type f are not possible.

We give here some explanations about the idea of this constructions. For obtaining a Petri Net without colours, we create m "copies" of vertices to separate chips and edges of different colours. The two kinds of operations in $ColCFG$ are coded as follows: the opening of a vertex v is coded by transition $\theta(v)$ and the firing of vertex v with colour c is coded by transition $f(v, c)$. And the first condition "there exists a colours c such that v can be fired" is coded by transition $t(v, c)$. The place $p(v, c)$ illustrates the chips of colours c in v before the opening of v , the place $q(v, c)$ illustrates the chips of colours c in v after the opening of v and the place $r(v)$ keep the condition to open or not v .

We give also the corresponding between configurations of $ColCFG(G)$ and $N(G)$

Definition 3.8. Let $ColCFG(G)$ be a Coloured CFG, let δ be an initial configuration (where $\delta(v, c)$ is the number of chips of colours c at vertex v), and let $N(G)$ be the corresponding Petri Net structure of $ColCFG(G)$. We define the corresponding initial marking M as follows.

- There are $N + \delta(v, c)$ tokens in $p(v, c)$.
- There are N tokens in $r(v)$.
- There are $\delta(v, c)$ tokens in $q(v, c)$.
- Each marking M is identified by its values set $\{q(v, c), c \in X\}$.

Let us now prove formally the correctness of this construction.

Theorem 3.9. Let $ColCFG(G)$ be a Coloured CFG, and let δ be an initial configuration. Let $N(G)$ be the Petri Net obtained from $ColCFG(G)$ by the map χ , and let M_0 be the marking of $N(G)$ which corresponds to δ . Then the Petri Net $(N(G), M_0)$ and the model $ColCFG(G, \delta)$ have the same reachability graph.

Proof. First, we prove that each operation in $ColCFG(G)$ corresponds to a transitions in $N(G)$. Our purpose is to prove that the opening of vertex v is coded by transition $\theta(v)$ and the firing of vertex v with colour c is coded by transition $f(v, c)$.

- At the beginning, all vertices of V are closed, there is no operations. In N , all operations "opening" $\theta(v)$ are not possible because $I(r(v), \theta(v)) = N + 1$ and there are N tokens in $r(v)$.

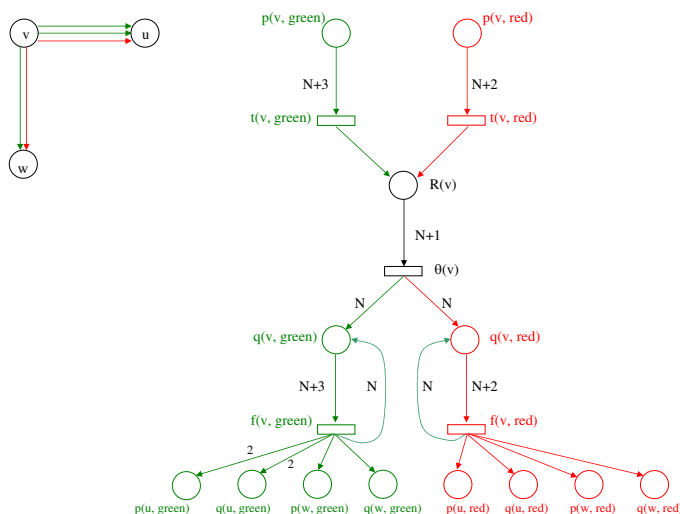


FIGURE 11. Left: A coloured CFG. Right: A part of the corresponding Petri Net

Moreover, before operation opening $\theta(v)$, all operations "firing" $f(v, c)$ are not possible because $I(q(v, c), f(v, c)) = N + d(v, c)$ and there are only $\delta(v, c)$ tokens in $q(v, c)$ which is smaller than the great value N .

- The condition for opening v is: "if there exists a colour c such that v can be fired". In N , the condition for the operation opening $\theta(v)$ is $I(r(v)) \geq I(r(v), \theta(v)) = N + 1$. Or at the beginning in $r(v)$ there are N tokens, then one can do $\theta(v)$ after $r(v)$ obtained at least one token. There m transition of output place $r(v)$ which are $t(v, c)$, so $r(v)$ obtained at least one token if at least one of there transition was done. The condition for transition $t(v, c)$ is the number of tokens at $p(v, c)$ is greater than $I(p(v, c), t(v, c))$, that means $N + \delta(v, c) \geq N + d(v, c)$, equivalently $\delta(v, c) \geq d(v, c)$, which is nothing but the condition to fire vertex v with colour c . So the condition to open a vertex v is corresponded to the condition to realize transition $\theta(v)$.
- The opening of vertex v consists of
 - marking it as open
 - for each colour c in X , consider the restriction of the game to c and to the set of open vertices, and play the game until the final configuration is reached.

Now in N , let us consider a v and c_0 such that $\delta(v, c_0) \geq d(v, c_0)$. Transition $t(v, c_0)$ is possible, and this transitions will be realized. For other colors c , if $\delta(v, c) \geq d(v, c)$ then $t(v, c)$ is possible, it will

be realized. Then transition $\theta(v)$ is possible and it will be realized. Now for each colour c , $q(v, c) = \delta(v, c) + N$, and $I(q(v, c), f(v, c)) = N + d(v, c)$. It is clear that $q(v, c_0) \geq I(q(v, c_0), f(v, c_0))$ so $f(v, c_0)$ is possible and it is realized. For other c if $\delta(v, c) \geq d(v, c)$ then $q(v, c) \geq I(q(v, c), f(v, c))$, and $f(v, c)$ is possible and it will be realized.

The tokens from $q(v, c)$ go to places $p(u, c)$ and $q(u, c)$ for neighbors u of v . If u is still closed, by the above statement $f(u, c)$ is not possible, that means u can not be fired, but this addition can change condition to opening u because $p(u, c)$ is increased as in a classical CFG. Otherwise, u is already opened, $q(u, c)$ can be fired as in a classical CFG.

- Property: in $ColCFG(G)$, each vertex v is opened at most once.
In N , if v is opened, that means $\theta(v)$ is already done, then $r(v)$ lost N tokens. So even if $r(v)$ can be obtained many tokens from transitions $t(v, c)$ (which gives only one token to $r(v)$ at each time), $r(v)$ can not reach N token. Or the condition of $\theta(t)$ is $r(v) \geq I(r(v), \theta(v)) = N + 1$, so $r(v)$ can not be done a new time.
- Condition in $ColCFG(G)$, each time opening a vertex, we play the classical CFG until the final configuration is reached. This condition corresponds to the last condition of N : the operation of type θ can be realized if and only if all operations of type f are not possible.

We now prove that each configuration in $ColCFG(G)$ is corresponded to a marking of $N(G)$. In fact, by Definition 3.8, each marking of $N(G)$ is identified by the set $\{q(v, c)\}$. At the beginning, there are $\delta(v, c)$ tokens in $q(v, c)$, so the set $\{q(v, c)\}$ illustrates exactly the numbers of chips of each colours in vertices of V .

After that, if a vertex v fires with colour c in G then v lost $d(v, c)$ chips. In $N(G)$, the corresponding operation $f(v, c)$ is realised, or $I(q(v, c), f(v, c)) = d(v, c)$, so $q(v, c)$ lost $d(v, c)$ tokens also. For a neighbor u of v in G , the firing of v adds $d((v, u), c)$ chips of colour c to u . In N , we have $O(f(v, c), q(u, c)) = d((v, u), c)$, that means after the firing $f(v, c)$, $q(u, c)$ is added $d((v, u), c)$ tokens of colours c .

So the modification of configuration after each firing in $ColCFG(G)$ corresponds to that of $N(G)$.

Because operations of $ColCFG(G)$ correspond to that of $N(G)$, and configurations of $ColCFG(G)$ corresponds to markings of $N(G)$, so the reachability graph of two models are the same. \square

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