New characterization of controllability via stabilizability and Riccati equation for LTV systems

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Abstract

This paper presents a new characterization of stabilizability via Riccati equation for linear time-varying (LTV) systems. An equivalence is given between the global null-controllability, complete stabilizability and the existence of the solution of some appropriate Riccati differential equation.

Keywords: Linear time-varying systems, controllability, stabilization, Riccati equation

1 Introduction

In the theory of control systems, the qualitative control problem has received considerable research interests in the past decades, see; e.g. [7, 9, 15] and the references therein. This problem, regarded as an extension of the classical Kalman result [5] on controllability and stability of linear systems, is to find an admissible control $u(t)$ such that the corresponding solution for the state vector $x(t)$ of the system has desired properties. Depending on the properties involved, one defines various qualitative problems. For example, the global null-controllability (GNC) problem of the linear system:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad t \geq 0,$$

concerns the question of finding an admissible control $u(t)$ which steers an arbitrary state x_0 of the system into the origin; the stabilizability problem is to find a control $u(t) = K(t)x(t)$

such that the zero solution of the closed-loop system

$$\dot{x}(t) = [A(t) + B(t)K(t)]x(t), t \geq 0$$

is asymptotically stable in the Lyapunov sense. In this case one says that the system is stabilizable with the stabilizing feedback control $u(t) = K(t)x(t)$. Various stability concepts can be adapted to investigate the stability property of the systems. One of the extended stability properties of control systems is the complete stabilizability, originally introduced by Wonham [13], which relates to a stronger stability of the system. Namely, the control system is completely stabilizable if for every number $\delta > 0$, there exists a feedback control $u(t) = K(t)x(t)$ such that the solution $x(t, x_0)$ of the closed-loop system satisfies the inequality

$$\exists N > 0 : \quad \|x(t, x_0)\| \leq Ne^{-\delta t}\|x_0\|, \forall t \geq 0.$$

This means that for every positive number $\delta > 0$, the system zero-input response of the closed-loop system decays faster than $e^{-\delta t}$. In other words, for any given positive number $\delta > 0$ for the decay rate, the system can be $\delta-$exponentially stabilizable. Such problem may arise in the speed control of real systems in manufacturing, communication networks, automotive and aircraft control. It is well known that if a linear time-invariant (LTI) control system is GNC then it is stabilizable, but the converse is not true. In [13], it is shown that if the LTI system is completely stabilizable, then it is GNC. This result has been extended to LTI systems in Hilbert spaces [15]. It is worth mentioning the result of [2], which provides an equivalence between the stabilizability and the existence of solutions of coupled Lyapunov-like equations for LTI control systems. For LTV systems, the first result on the relationship between GNC and Riccati differential equation (RDE) was given in [5], which proves that if LTV control system is GNC then the RDE

$$\dot{P}(t) + A^T(t)P(t) + P(t)A(t) - P(t)B(t)B^T(t)P(t) + Q(t) = 0,$$

where $Q(t) \geq 0$, has a positive semi-definite solution $P(t)$. However, the existence of the positive definite solution $P(t)$ of the above RDE is not sufficient for the GNC. This can be seen from the following example. Consider the following LTV control system

$$\dot{x} = \begin{pmatrix} 0.5(1-e^t) & 0 \\ 0 & 0.5(1-2e^t) \end{pmatrix} x + \begin{pmatrix} e^t \\ 0 \end{pmatrix} u.$$

The system is not GNC due to $\text{rank}[B(t), A(t)B(t)] < 2$ for all $t \geq 0$, however the corresponding RDE with $Q = 2I$ has a bounded positive definite solution $P(t) = e^{-t}I$. Some criteria for the stabilizability of LTV control systems were derived in [3, 10, 11] in terms of the uniformly positive definiteness of the solution. That is, $P(t)$ has to satisfy the inequality $\lambda_1 I \leq P(t) \leq \lambda_2 I, \quad \forall t \geq 0$. Note that the uniformly positive definiteness of the solution of this kind of RDE is still not sufficient for the GNC of LTV systems. For instant, consider a LTV control system with

$$A(t) = \begin{pmatrix} -0.5\cos t & 0 \\ 0 & 0.5\sin t - e^{-\cos t} \end{pmatrix}, \quad B(t) = \begin{pmatrix} e^{-\sin t} \\ 0 \end{pmatrix}.$$

It is easy to see that the system is not GNC, but the RDE, where $Q(t) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, has a uniformly positive definite solution $P(t) = \begin{pmatrix} e^{\sin t} & 0 \\ 0 & e^{\cos t} \end{pmatrix}$. In an attempt to seek a relationship between controllability and stabilizability, the authors of [4] introduced the concept of uniform global controllability, whereby it is shown that the LTV control system is completely stabilizable if and only if it is uniformly globally controllable. Notably, this result combining with the result of [5] shows that the complete stabilizability is a sufficient condition for the existence of the bounded positive semi-definite solution of RDE for LTV control systems. The relationship between controllability, stabilizability and the solution of periodic Riccati equations for periodic LTV control systems was given in [6]. It is interesting for the question as to whether there exists a relationship between the GNC, stabilizability and RDE. This has motivated our research. In this paper, we present a new characterization of controllability of LTV control systems through stabilizability and the existence of solutions of Riccati equations. It shows that the global null-controllability and complete stabilizability of LTV control systems are equivalent to the existence of bounded positive semi-definite solution of some appropriate RDE. The results obtained here can be considered as further extensions of [3, 4, 5, 11, 13] to LTV control systems.

The organization of the paper is as follows. Following the introduction and the problem motivation. Section 2 presents the notation, some definitions and propositions. The main result is given in Section 3 and followed by the cited references.

2 Preliminaries

Let us first introduce the notation used in this note. R^+ denotes the set of all real non-negative numbers; while R^n denotes the n-dimensional Euclidean space with the scalar product $\langle x, y\rangle$ of two vectors $x, y \in R^n$. $M^{n\times m}$ denote the space of all $(n \times m)-$ matrices. I, A^{-1} and A^T denote the identity matrix, the inverse and the transpose of the matrix A, respectively. A is symmetric if $A = A^T$. A matrix $Q \in M^{n\times n}$ is positive semi-definite ($Q \succeq 0$) if $\langle Qx, x\rangle \geq 0$, for all $x \in X$. If $\langle Qx, x\rangle > 0$, for $x \neq 0$ then Q is positive definite ($Q \succ 0$). $A \geq B$ means $A - B \succeq 0$; M^+ denotes the set of all symmetric positive definite constant matrices. $BM^{n\times m}(0,\infty)(BM^+(0,\infty)$, respectively) denote the set of all $n \times m-$matrix functions continuous and bounded on R^+ (symmetric positive semi-definite matrix functions continuous and bounded on $(0,\infty)$, respectively). $L_2([0,t], R^m)$ denotes the space of all square integrable on $[0,t]$ functions with valued in R^m.

Now, let us consider the following LTV control system, briefly denoted by $[A(t), B(t)]$:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad x(0) = x_0, \quad t \in R^+, \tag{2.1}$$

where $A \in BM^{n\times n}(0,\infty), B \in BM^{n\times m}(0,\infty)$ are given matrix functions. The control $u(t)$ is admissible if $u(s) \in L_2([0,t], R^m), \forall t \geq 0$. For every $x_0 \in R^n$, and admissible control $u(t)$, the solution of system (2.1) is given by

$$x(t) = U(t,0)x_0 + \int_0^t U(t,s)B(s)u(s)ds,$$

where $U(t,s)$ is the transition matrix of the unforced system $\dot{x}(t) = A(t)x(t)$. It is well known (see; e.g. [1]) that if $A \in BM(0,\infty)$ then the transition matrix $U(t,s)$ satisfies the condition

$$\exists M \geq 1, \alpha > 0 : \|U(t,s)\| \leq Me^{-\alpha|t-s|}, \quad \forall t,s \in R^+. \tag{2.2}$$

Definition 2.1. [5] Linear control system (2.1) is globally null-controllable (GNC) if for every $x_0 \in R^n$, there exist a number $T > 0$ and an admissible control $u(t)$ such that $x(T) = 0$.

We state the following well-known controllability criterion that will be used later.

Proposition 2.1. [1, 7] *Linear control system (2.1) is GNC if and only if one of the following conditions holds.*

(i) $\exists t > 0, c > 0 : \quad \int_0^t \|B^T(s)U^T(t,s)x\|^2 ds \geq c\|U^T(t,0)x\|^2, \quad \forall x \in R^n.$

(ii) $A(t), B(t)$ *are analytic on* R^+ *and rank* $M(t_0) = n$ *for some* $t_0 > 0$, *where* $M(t) = [M_0(t), M_1(t), ..., M_{n-1}(t)], M_0 := B(t), M_{i+1}(t) = -A(t)M_i(t) + \frac{d}{dt}M_i(t), \quad i = 0,1,...,n-2.$

Definition 2.2. [13] Linear control system (2.1) is completely stabilizable if for every number $\delta > 0$, there exists a feedback control $u(t) = K(t)x(t)$, where $K(t) \in BM^{m\times n}(0,\infty)$, such that the solution $x(t,x_0)$ of the closed-loop system satisfies the inequality

$$\exists N > 0 : \quad \|x(t,x_0)\| \leq Ne^{-\delta t}\|x_0\|, \quad \forall t \in R^+.$$

The solution to stabilization problem involves the following RDE

$$\dot{P}(t) + A^T(t)P(t) + P(t)A(t) - P(t)B(t)B^T(t)P(t) + Q(t) = 0. \tag{2.3}$$

Definition 2.3. Let $Q \in BM^+(0,\infty)$. Linear control system (2.1) is $Q-$stabilizable if for every initial state x_0, there is a control $u(t) \in L_2([0,\infty), R^m)$ such that the cost function

$$J(u) = \int_0^\infty [\|u(t)\|^2 + \langle Q(t)x(t), x(t)\rangle]dt,$$

where $x(t)$ is a solution of the system, exists and is finite.

Proposition 2.2. [1] *Let* $Q \in BM^+(0,\infty)$. *If linear control system (.1) is* $Q(t)-$*stabilizable, then the RDE (2.3) has a solution* $P \in BM^+(0,\infty)$.

Proposition 2.3. [14] *Consider nonlinear time-varying system* $\dot{x}(t) = f(t,x(t)), x(0) = x_0, f(t,0) = 0, t \in R^+$. *If there exists a Lyapunov function* $V(t,x) : R^+ \times R^n \to R$ *satisfying the following conditions:*

(i) $\exists \lambda_1 > 0, \lambda_2 > 0 : \quad \lambda_1\|x\|^2 \leq V(t,x) \leq \lambda_2\|x\|^2, \quad \forall t \in R^+,$

(ii) $\dot{V}_f(t,x) := \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}f(t,x(t)) \leq 0,$ *for all solutions* $x(t)$ *of the system,*

then the solution $x(t)$ *is bounded:* $\exists N > 0 : \|x(t,x_0)\| \leq N\|x_0\|, \forall t \in R^+.$

3 Main result

Given $\delta > 0$, we set $A_\delta(t) = A(t) + \delta I$. For $P, Q \in BM^+(0,\infty)$, consider the following RDE

$$(RDE_\delta) \qquad \dot{P}(t) + A_\delta^T(t)P(t) + P(t)A_\delta(t) - P(t)B(t)B^T(t)P(t) + Q(t) = 0.$$

Theorem 3.1. *The following statements are equivalent:*

(i) System $[A(t), B(t)]$ *is globally null-controllable.*

(ii) For every $\delta > 0, Q \in BM^+(0,\infty)$, RDE_δ *has a solution* $P \in BM^+(0,\infty)$.

(iii) System $[A(t), B(t)]$ *is completely stabilizable.*

Proof. (i)→(ii). Assume that system $[A(t), B(t)]$ is globally null-controllable. Then, by Definition 2.1, for every $x_0 \in X$ there are a time $h > 0$ and admissible control $u(s) \in L_2([0,h], R^m)$ such that

$$U(h,0)x_0 + \int_0^h U(t,s)B(s)u(s)ds = 0. \tag{3.1}$$

Let $\delta > 0$ be an arbitrary positive number. Multiplying both sides of (3.1) with $e^{\delta h}$ and observing $U_{A_\delta}(t,s) = e^{\delta(t-s)}U(t,s)$ we find

$$U_{A_\delta}(h,0)x_0 + \int_0^h U_{A_\delta}(h,s)B(s)\tilde{u}(s)ds = 0,$$

where $\tilde{u}(s) = e^{\delta s}u(s)$. This implies that the initial state x_0 can be steered to 0 by the admissible control $\tilde{u}(t)$ in the time h, i.e., the system $[A_\delta(t), B(t)]$:

$$\dot{y}(t) = A_\delta(t)y(t) + B(t)u(t), \quad t \in R^+, \tag{3.2}$$

is GNC. Therefore, for every initial state $x_0 \in X$ there is an admissible control $u_x(t) \in L_2([0,h], R^m)$ such that the solution $x(t)$ of the system (3.2) according to the control $u_x(t)$ satisfies $x(0) = x_0, x(h) = 0$. Define the control $\tilde{u}_x(t) \in L_2([0,\infty), R^m), t \geq 0$ as $\tilde{u}_x(t) = u_x(t)$, if $t \in [0,h]$, and $= 0$, if $t > h$. Then, taking any $Q \in BM^+(0,\infty)$ we have

$$J(\tilde{u}_x) = \int_0^\infty [\|\tilde{u}_x(t)\|^2 + \langle Q(t)x(t), x(t)\rangle]dt = \int_0^h [\|u_x(t)\|^2 + \langle Q(t)x(t), x(t)\rangle]dt < +\infty.$$

This means that system $[A_\delta(t), B(t)]$ is $Q-$stabilizable, and hence by Proposition 2.2, the RDE

$$\dot{P}(t) + A_\delta^T(t)P(t) + P(t)A_\delta(t) - P(t)B(t)B^T(t)P(t) + Q(t) = 0,$$

has a solution $P \in BM^+(0,\infty)$, which means (ii).

(ii)→(iii). Assume that the condition (ii) holds. For any $\delta > 0$, we define, the matrix function $Q \in BM^+(0,\infty)$ satisfying

$$Q(t) \geq A(t) + A^T(t) + 2\delta I + B(t)B^T(t), \quad t \in R^+. \tag{3.3}$$

Then, RDE_δ has a solution $P \in BM^+(0,\infty)$. We now rewrite this RDE_δ in the form

$$\dot{P}(t) + A_\delta^T(t)P(t) + P(t)A_\delta(t) - e^{-2\delta t}P(t)B_\delta(t)B_\delta^T(t)P(t) + Q(t) = 0, \tag{3.4}$$

where $B_\delta(t) = e^{\delta t}B(t)$. By using the transformation $y(t) = e^t x(t), t \in R^+$, system (2.1) is transformed to the system

$$\dot{y}(t) = A_\delta(t)y(t) + B_\delta(t)u(t), \quad y(0) = y_0, t \in R^+, \tag{3.5}$$

We first prove that the solution $y(t)$ of the system (3.5) is bounded on R^+. For this, we consider the following Lyapunov function

$$V(t,y) = \langle P(t)y, y\rangle + \|y\|^2, \quad t \in R^+,$$

where $P \in BM^+(0,\infty)$ is the solution of (3.4). It is easy to verify that the Lyapunov function $V(t,y)$ satisfies the inequality

$$\lambda_1\|y\|^2 \le V(t,y) \le \lambda_2\|y\|^2, \quad \forall t \in R^+,$$

for some $\lambda_1, \lambda_2 > 0$. Let us choose a feedback control of the form

$$u(t) = -\frac{e^{-2\delta t}}{2}B_\delta^T(t)[P(t) - I]y(t). \tag{3.6}$$

With the feedback control (3.6), taking the derivative of $V(.)$ in t along the solution of $y(t)$ of the closed-loop system of (3.5), we have

$$\begin{aligned}
\dot{V}(t,y(t)) &= \langle \dot{P}(t)y(t), y(t)\rangle + 2\langle P(t)\dot{y}(t), y(t)\rangle + 2\langle \dot{y}(t), y(t)\rangle \\
&= \langle \dot{P}(t)y(t), y(t)\rangle + \langle (A_\delta^T(t)P(t) + P(t)A_\delta(t))y(t), y(t)\rangle \\
&\quad +2\langle A_\delta(t)y(t), y(t)\rangle + 2\langle B_\delta(t)u(t), y(t)\rangle + 2\langle P(t)B_\delta(t)u(t), y(t)\rangle \\
&= \langle [\dot{P}(t) + A_\delta^T(t)P(t) + P(t)A_\delta(t) - e^{-2\delta t}P(t)B_\delta^T(t)B_\delta^T(t)P(t)]y(t), y(t)\rangle \\
&\quad +\langle [A_\delta(t) + A_\delta^T(t) + e^{-2\delta t}B_\delta(t)B_\delta^T(t)]y(t), y(t)\rangle \\
&= -\langle [Q(t) - (A(t) + A^T(t) + 2\delta I + B(t)B^T(t))]y(t), y(t)\rangle.
\end{aligned}$$

By choosing of $Q(t)$ from the condition (3.3), we get $\dot{V}(t,y(t)) \le 0, t \in R^+$, and then by Proposition 2.3, the solution $y(t)$ is bounded:

$$\exists N > 0 : \|y(t)\| \le N\|y_0\|, t \in R^+.$$

Returning to the solution $x(t)$ of system (2.1), by noting that $x(0) = y(0) = x_0$, we obtain

$$\|x(t)\| \le N\|x_0\|e^{-\delta t}, \quad \forall t \in R^+.$$

The last condition means that with the feedback control (3.6):

$$u(t) = -\frac{-e^{2\delta t}}{2}B_\delta^T(t)[P(t) - I]y(t)$$

$$= -\frac{1}{2}B^T(t)[P(t) - I]x(t) = K(t)x(t),$$

the zero solution of the closed-loop system:

$$\dot{x}(t) = [A(t) + B(t)K(t)]x(t), t \in R^+,$$

where

$$K(t) = -\frac{1}{2}B^T(t)[P(t) - I] \in BM^{m\times n}(0, \infty),$$

is exponentially stable with the decay rate $\delta > 0$.

(iii) →(i). Let the system $[A(t), B(t)]$ be completely stabilizable, but assume to the contrary that the system is not globally-null controllable. Taking $\delta > \alpha$, where $\alpha > 0$ is defined by the condition (2.2). From the complete stabilizability it follows that there is $K \in BM^{m\times n}(0, \infty)$ such that the solution $x(t, x_0)$ of the closed-loop system

$$\dot{x}(t) = [A(t) + B(t)K(t)]x(t)$$

satisfies the inequality:

$$\|x(t, x_0)\| = \|U_K(t, 0)x_0\| \le N\|x_0\|e^{-\delta t}, \quad \forall t \in R^+, \tag{3.7}$$

where $U_K(t, s)$ is the transition matrix of the closed-loop system. Substituting the feedback control $u(t) = K(t)x(t) = K(t)U_K(t, 0)x_0$ and the solution $x(t, x_0) = U_K(t, 0)x_0$ into the Cauchy solution of the nominal system:

$$x(t, x_0) = U(t, 0)x_0 + \int_0^t U(t, s)B(s)u(s)ds,$$

we obtain

$$U(t, 0)x_0 = U_K(t, 0)x_0 - \int_0^t U(t, s)B(s)K(s)U_K(s, 0)x_0 ds, \quad t \in R^+.$$

Since the above equation holds for all $x_0 \in R^n$, the following inequality holds for every $x \in R^n$:

$$\|U^T(t, 0)x\| \le \|U_K^T(t, 0)x\| + \int_0^t \|U_K^T(s, 0)K^T(s)B^T(s)U^T(t, s)x\|ds.$$

Taking condition (3.7) into account, we have

$$\begin{aligned} \|U^T(t, 0)x\| &\le Ne^{-\delta t}\|x\| + Nk\int_0^t e^{-\delta s}\|B^T(s)U^T(t, s)x\|ds \\ &\le Ne^{-\delta t}\|x\| + Nk(\int_0^t e^{-2\delta s}ds)^{1/2}(\int_0^t \|B^T(s)U^T(t, s)x\|^2ds)^{1/2}, \end{aligned} \tag{3.8}$$

where $k := \sup\{\|K(s)\| : s \in [0,\infty)\} < +\infty$. Setting $\beta(t) = (\int_0^t e^{-2\delta s} ds)^{1/2}$, we have

$$\beta(t) = (\frac{1}{2\delta} - \frac{1}{2\delta} e^{-2\delta t})^{1/2}. \tag{3.9}$$

By the contrary assumption, the system (2.1) is not globally null-controllable. Then, by Proposition 2.1 (i), for every $t > 0, c > 0$ and $\epsilon \in (0,1)$ satisfying

$$c < \Big[\frac{(1-\epsilon)\sqrt{2\delta}}{Nk}\Big]^2, \tag{3.10}$$

there exists $x_* \in R^n$ such that

$$\int_0^t \|B^T(s)U^T(t,s)x_*\| ds < c\|U^T(t,0)x_*\|^2. \tag{3.11}$$

It is obvious that $x_* \neq 0$, we can consider, without loss of generality, the inequality (14) holding for $\|x_*\| = 1$, otherwise we can take $x_1 = \frac{x_*}{\|x_*\|}$. Therefore, from (3.8) and (3.11), it follows that

$$\|U^T(t,0)x_*\| < Ne^{-\delta t} + \sqrt{c}Nk\beta(t)\|U^T(t,0)x_*\|. \tag{3.12}$$

On the other hand, we note that

$$1 = \|x_*\| = \|U^T(0,t)U^T(t,0)x_*\| \leq \|U^T(0,t)\|\|U^T(t,0)x_*\|,$$

which gives

$$\frac{1}{\|U^T(t,0)x_*\|} \leq \|U^T(0,t)\| \leq Me^{\alpha t}, \quad t \in R^+. \tag{3.13}$$

Therefore, combining (2.2), (3.12) and (3.13) gives

$$1 < \frac{Ne^{-\delta t}}{\|U^T(t,0)x_*\|} + \sqrt{c}Nk\beta(t) < NMe^{-(\delta-\alpha)t} + \sqrt{c}Nk\beta(t), \quad t \in R^+,$$

hence

$$1 - \sqrt{c}Nk\beta(t) < NMe^{-(\delta-\alpha)t}, \quad t \in R^+.$$

By letting t go to the infinity and noting (3.9) that $\beta(t) \to (1/\sqrt{2\delta})$, the right-hand side of the above inequality goes to 0 because $\delta > \alpha$, we have thus

$$1 - \sqrt{c}N\frac{1}{\sqrt{2\delta}}k \leq 0.$$

Then, from condition (3.10), we obtain the following inequality

$$\epsilon < 1 - \sqrt{c}N\frac{1}{\sqrt{2\delta}}k \leq 0,$$

which leads to a contradiction. This completes the proof of the theorem.

Remark 3.1. It is worth noting that the condition (ii) of Theorem 3.1 can be relaxed by the condition:

(ii): For every $\delta > 0$, there exists $Q \in M^+$ such that the RDE_δ, where $Q(t) = Q$ has a solution $P(t) \in BM^+(0, \infty)$.

Indeed, in this case, taking the Lyapunov function $V(t, y) = \langle P(t)y, y\rangle$, where $P(t) \in BM^+(0, \infty)$ is the solution of RDE (3.4) for some $Q(t) = Q \in M^+$. With the feedback control

$$u(t) = -\frac{e^{-2\delta t}}{2} B_\delta^T(t)P(t)y(t),$$

the derivative of $V(.)$ in t along the solution of $y(t)$ of the closed-loop system of (3.5) gives $\dot{V}(t, y(t)) \leq -\epsilon\|y(t)\|^2$, for some $\epsilon > 0$. Integrating both sides of the last inequality from 0 to t gives

$$V(t, y(t)) - V(0, y_0) \leq -\epsilon \int_0^t \|y(s)\|^2 ds.$$

Since $V(t, y) \geq 0$, we obtain that

$$\int_0^t \|y(t)\|^2 dt \leq \frac{\lambda_{\max}(P(0))}{\epsilon}\|y_0\|^2 < +\infty.$$

Let $U_\delta(t, s)$ be the transition matrix of the closed-loop system of (3.5). It is easy to veirfy that $U_\delta(t, s)$ satisfies the condition (2.2). For every $x \in R^n, t \in R^+$, we have

$$\begin{aligned} \frac{1 - e^{-2\alpha t}}{2\alpha}\|U_\delta(t, 0)x\|^2 &= \int_0^t e^{-2\alpha(t-s)}\|U_\delta(t, 0)x\|^2 ds \\ &\leq \int_0^t e^{-2\alpha(t-s)}\|U_\delta(t, s)\|^2\|U_\delta(s, 0)x\|^2 ds \\ &= M^2 \int_0^t \|U_\delta(s, 0)x\|^2 ds, \end{aligned}$$

and hence

$$\begin{aligned} \|y(t)\|^2 = \|U_\delta(t, 0)y_0\|^2 &\leq \frac{M^2 2\alpha}{1 - e^{-2\alpha t}} \int_0^t \|U_\delta(s, 0)y_0\|^2 ds \\ &= \frac{M^2 2\alpha}{1 - e^{-2\alpha t}} \int_0^t \|y(s)\|^2 ds \\ &\leq \frac{M^2 2\alpha}{1 - e^{-2\alpha t}} \frac{\lambda_{\max}(P(0))}{\epsilon}\|y_0\|^2. \end{aligned} \tag{3.14}$$

Letting $t \to \infty$, the right-hand side function is finite due to $(1 - e^{-2\alpha t}) \to 1$ and this implies that the solution $y(t)$, which is a continuous function, is bounded on R^+. The end of the proof is easily followed by the same argument used in the proof of Theorem 3.1.

Remark 3.2. The condition (ii) of Theorem 3.1 concerns with the solution of Riccati diferential equations. Note that the problem of solving Riccati differential equations is in general still complicated, however some various efficient approaches to solving this problem can be found, for instance, in [8, 12]. In the example below, we try to show the correctness of the obtained result, where the solution of $(RDE)_\delta$ is easily defined by solving some ordinary differential equations.

An example. Consider the LTV control system (2.1) in R^2, where

$$A(t) = \begin{pmatrix} -0.5\sin t & -1 \\ 1 & -0.5\sin t \end{pmatrix}, B(t) = \begin{pmatrix} \sqrt{e^{2\cos t} + 2e^{\cos t} - 1} & 1 \\ -1 & \sqrt{e^{2\cos t} + 2e^{\cos t} - 1} \end{pmatrix}.$$

To verify the global null-controllablity of the system we apply Proposition 2.1 (ii). We have

$$M(t) = \begin{pmatrix} b(t) & 1 & 0.5b(t)\sin t - 1 + \dot{b}(t) & 0.5\sin t + b(t) \\ -1 & b(t) & -0.5\sin t - b(t) & 0.5b(t)\sin t - 1 + \dot{b}(t) \end{pmatrix},$$

where $b(t) := \sqrt{e^{2\cos t} + 2e^{\cos t} - 1}$, then rank $M(t_0) = 2$ for, e.g. $t_0 = \frac{\pi}{2}$. For any $\delta > 0$, taking $Q = \delta^2 I \in M^+$, the RDE_δ has a solution $P(t) \in BM^+(0, \infty)$ defined as

$$P(t) = \begin{pmatrix} \delta e^{-\cos t} & 0 \\ 0 & \delta e^{-\cos t} \end{pmatrix}.$$

On the other hand, we can verify that the system is completely stabilizable with the feedback control

$$\begin{cases} u_1(t) = -0.5\delta e^{-\cos t}\sqrt{e^{2\cos t} + 2e^{\cos t} - 1}x_1(t) + 0.5\delta e^{-\cos t}x_2(t), \\ u_2(t) = -0.5\delta e^{-\cos t}x_1(t) - 0.5\delta e^{-\cos t}\sqrt{e^{2\cos t} + 2e^{\cos t} - 1}x_2(t). \end{cases}$$

From the estimation (3.14), we can define a number $N > 0$ such that the solution of the system satisfies the inequality

$$\|x(t, x_0)\| \leq Ne^{-\delta t}\|x_0\|, \quad \forall t \geq 0.$$

4 Conclusion

This papers has been concerned with the problem of controllability, stabilizability for linear time-varying systems. An equivalence between the global null-controllability, complete stabilizability and the existence of the solution of some appropriate Riccati differential equation is presented.

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