Improved exponential stability for time-varying systems with nonlinear delayed perturbations

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Abstract

In this paper, a new sufficient delay dependent exponential stability condition for a class of linear time-varying systems with nonlinear delayed perturbations is derived by using an improved Lyapunov-Krasovskii functional. The proposed exponential stability conditions are formulated in terms of the solution of Lyapunov differential equations. The approach allows for computation of the bounds that characterize the exponential stability rate of the solution. Compared with existing results, our conditions are shown to be less conservative. Numerical examples are given to illustrate the effectiveness of the obtained conditions.

Key words. Linear time-varying system, exponential stability, nonlinear perturbation, time delay, Lyapunov equation.

1 Introduction

Stability analysis of nonlinear delay systems is fundamental to many applied problems. In the literature, various theoretical stability results mostly are based on the Lyapunov function method [1, 4, 10]. The stability property of a working point is really robust if it still holds when perturbations make the system vary. Numerous papers deal with the stability of linear time-invariant (LTI) delay perturbed systems [2, 3, 6, 7, 8, 11], while only few papers [4, 9, 10, 12] give stability conditions for linear time-varying (LTV) delay systems among those [12] dealing with the exponential stability of perturbed systems. The aim of this paper is to establish sufficient delay dependent exponential stability condition for a LTV system with nonlinear delayed perturbations by employing an improved Lyapunov-Krasovskii functional.

Consider a LTV system with nonlinear delayed perturbations of the form

$$\begin{aligned} &\dot{x}(t) = A(t)x(t) + f(t, x(t - h(t))), \quad t \geq 0, \\ &x(t) = \phi(t), \quad t \in [-h, 0], \end{aligned} \tag{1}$$

where $x(t) \in R^n$, where $A(t) \in R^n$ is matrix function continuous on $R^+, \phi(t) \in C([-h, 0], R^n)$ is the initial function with the norm $\|\phi\| = \sup_{s\in[-h,0]} \|\phi(s)\|; h(t)$ is time-varying delay function satisfies

$$0 \leq h(t) \leq h, \quad \dot{h}(t) \leq \delta < 1, \quad \forall t \geq 0,$$

and the nonlinear perturbation $f(.)$, throughout this paper , satisfies

$$\exists \gamma > 0 : \quad \|f(t, y\| \leq \gamma \|y\|, \quad \forall t \geq 0, y \in R^n.$$

Definition 2.1. The system (1) is said to be exponentially stable, if there exist positive numbers α, N such that every solution $x(t, \phi)$ of the system satisfies

$$\|x(t, \phi)\| \leq N e^{-\alpha(t - t_0)} ||\phi||, \quad \forall t \geq t_0 \geq 0.$$

the following technical proposition is needed for the proof of the main reesult.

Proposition 2.1. *Let Q, S are symmetric matrices of appropriate dimensions and $S > 0$. Then*

$$2\langle Qy, x\rangle - \langle Sw, w\rangle \leq \langle QS^{-1}Q^T x, x\rangle, \quad \forall (x, y, w).$$

2 Main result

In this section, $BM^+(0, \infty)$ denotes all symmetric positive semidefinite matrix functions bounded on R^+ and $x_t := x(t + s), s \in [-h, 0]$. Let us set

$$P_\beta(t) = P(t) + \beta I, \quad \eta(A) = \frac{1}{2}\lambda_{\max}(A + A^T),$$

$$\bar{\eta}(A) = \sup_{t \in R^+} \eta(A(t)), \quad p = \sup_{t \in R^+} \|P(t)\|,$$

$$N = p + \beta + h\epsilon_2 + 2h^2\epsilon_3, \quad \eta = \epsilon_3 h e^{2\alpha h} + \epsilon_1 + \epsilon_2 + 2\alpha\beta.$$

Theorem 3.1. Assume that the condition (2) holds and there exist positive numbers $\alpha, \beta, \epsilon_1, \epsilon_2, \epsilon_3$ and $P(t) \in BM^+(0,\infty), \epsilon_1 - 2\beta\bar{\eta}(A) > 0$, satisfying the following Lyapunov differential equation (LDE)

$$\dot{P}(t) + A(t)^T P(t) + P(t)A(t) + 2\alpha P(t) + \eta I = 0. \tag{2}$$

Then, the system (1) is exponentially stable if

$$\gamma \leq \frac{\sqrt{\epsilon_2(1-\delta)(\epsilon_1 - 2\beta\bar{\eta}(A))}}{(p+\beta)e^{\alpha h}}. \tag{3}$$

Moreover, the solution $x(t,\phi)$ satisfies the condition

$$\|x(t,\phi)\| \leq N\|\phi\|e^{-\alpha(t-t_0)}, \quad t \geq t_0 \geq 0.$$

Proof. Consider the following Lyapunov-Krasovskii functional

$$V(t,x_t) = V_1(.) + V_2(.) + V_3(.),$$

where

$$V_1(x) = \langle P_\beta(t)x, x\rangle,$$
$$V_2(t,x_t) = \epsilon_2 \int_{t-h(t)}^{t} e^{2\alpha(s-t)}\|x(s)\|^2 ds,$$
$$V_3(t,x_t) = \epsilon_3 \int_{-h}^{0}\int_{t+\tau-h(t+\tau)}^{t} e^{2\alpha(s+h-t)}\|x(s)\|^2 ds d\tau.$$

It is easy to see that

$$\lambda_1\|x(t)\|^2 \leq V(t,x_t) \leq \lambda_2\|x_t\|^2, \quad t \geq 0,$$

for some $\lambda_1 > 0, \lambda_2 > 0$. Without loss of generality, we consider $t_0 = 0$. By using the following differentiation rule

$$\frac{d}{dt}\Big(\int_{-h}^{0}\int_{u(t+s\tau)}^{t} f(s)dsd\tau\Big) = hf(t) - \int_{-h}^{0}\dot{u}(t+s)f(u(t+sds,$$

the time derivative of $V(t,x_t)$ along the solution of the system gives

$$\begin{aligned}
\dot{V}_1(t,x(t)) =& \langle(\dot{P} + A^T P_\beta + P_\beta A)x(t), x(t)\rangle + 2\langle P_\beta f(t, x(t-h(t)), x(t)\rangle \\
\leq& \langle(\dot{P} + A^T P_\beta + P_\beta A)x(t), x(t)\rangle + 2(p+\beta)\gamma\|x(t-h(t))\|\|x(t)\| \qquad (4)\\
\dot{V}_2(t,x_t) =& -2\alpha V_2(t,x_t) + \epsilon_2\|x(t)\|^2 - \epsilon_2 e^{-2\alpha h(t)}(1-\dot{h}(t))\|x(t-h(t))\|^2 \\
\leq& -2\alpha V_2(t,x_t) + \epsilon_2\|x(t)\|^2 - \epsilon_2 e^{-2h\alpha}(1-\delta)\|x(t-h(t))\|^2. \qquad (5)\\
\dot{V}_3(t,x_t) \leq& -2\alpha V_3(t,x_t) + \epsilon_3 h e^{2h\alpha}\|x(t)\|^2 \\
&- \epsilon_3 e^{-2h\alpha}(1-\delta)\int_{-h}^{0}\|x(t+s-h(t+s))\|^2 ds \\
\leq& -2\alpha V_3(t,x_t) + \epsilon_3 h e^{2h\alpha}\|x(t)\|^2, \qquad (6)
\end{aligned}$$

because of the last integral item is non-negative. Therefore, from (4)-(6) it follows that

$$\begin{aligned}\dot{V}(t,x_t)+2\alpha V(t,x_t) &\le \langle[\dot{P}+A^TP_\beta+P_\beta A+2\alpha P_\beta+(\epsilon_2+\epsilon_3he^{2h\alpha})I]x(t),x(t)\rangle\\ &\quad +2(p+\beta)\gamma\|x(t-h(t))\|\|x(t)\|-\epsilon_2e^{-2h\alpha}(1-\delta)\|x(t-h(t))\|^2.\end{aligned}$$

Using Proposition 2.1, we have

$$2(p+\beta)\gamma\|x(t-h(t))\|\|x(t)\|-\epsilon_2e^{-2h\alpha}(1-\delta)\|x(t-h(t))\|^2\le\frac{(p+\beta)^2\gamma^2e^{2h\alpha}}{\epsilon_2(1-\delta)}\|x(t)\|^2,$$

and hence

$$\begin{aligned}\dot{V}(t,x_t)+2\alpha V(t,x_t) &\le \langle[\dot{P}+A^TP+PA+2\alpha P+(2\alpha\beta+\epsilon_2+\epsilon_3he^{2h\alpha})I]x(t),x(t)\rangle\\ &\quad +\beta\langle(A^T+A)x(t),x(t)\rangle+\frac{(p+\beta)^2\gamma^2e^{2h\alpha}}{\epsilon_2(1-\delta)}\|x(t)\|^2.\end{aligned}$$

Since $P(t)$ is a solution of LDE (2) and

$$\beta\langle(A^T(t)+A(t))x,x\rangle\le2\beta\bar{\eta}(A)\|x\|^2,$$

we have

$$\dot{V}(t,x_t)+2\alpha V(t,x_t)\le-[\epsilon_1-2\beta\bar{\eta}(A)-\frac{(p+\beta)^2\gamma^2e^{2h\alpha}}{\epsilon_2(1-\delta)}]\|x(t)\|^2.$$

Taking the condition (3) into account, we have

$$\dot{V}(t,x_t)\le-2\alpha V(t,x_t),\quad t\ge0,$$

which gives

$$V(t,x_t)\le V(0,x_0)e^{-2\alpha t},\quad t\ge0.$$

One the other hand, from the expression of $V(.)$ we have

$$\beta\|x(t)\|^2\le V(t,x_t),\quad t\ge0,$$

and then

$$\|x(t,\phi)\|\le\sqrt{\frac{V(0,x_0)}{\beta}}e^{-\alpha t},\quad t\ge0.$$

Estimating $V(0,x_0)$ gives

$$\begin{aligned}V(0,x_0) &\le (p+\beta+h\epsilon_2)\|\phi\|^2+\epsilon_3\int_{-h}^0\int_{\tau-h(\tau)}^0e^{2\alpha(s+h)}\|x(s)\|^2dsd\tau\\ &\le (p+\beta+h\epsilon_2)\|\phi\|^2+2h^2\epsilon_3\|\phi\|^2\\ &\le N\|\phi\|^2.\end{aligned}$$

the proof is complete.

As an consequence of Theorem 3.1 we obtain the following simple stability condition, which extends the condition obtained in [2, 8, 11] for LTI systems.

Corollary 3.1. With the same assumption stated in Theorem 3.1, the system (1) is exponentially stable if

$$\gamma \leq \frac{\sqrt{\epsilon_2(1-\delta)}}{p+\beta}. \tag{8}$$

Moreover, the solution $x(t, \phi)$ satisfies the condition

$$\|x(t,\phi)\| \leq N\|\phi\|e^{-\alpha t}, \quad t \geq 0.$$

3 Examples and simulation

The first example illustrates the effectiveness of Theorem 3.1.

Example 3.1. Consider the non-autonomous systems with nonlinear time-delay perturbations (1) with time-varying delay $h(t) = h\cos^2(0.45)t$,

$$f(t, x(t-h(t))) = \begin{bmatrix} -\gamma\sin(t)[x_2(t-h(t))] \\ \gamma\cos(t)[x_1(t-h(t))] \end{bmatrix},$$

where h, $\gamma > 0$ will be chosen later, and

$$A(t) = \begin{bmatrix} a(t) & 1 \\ -1 & a(t) \end{bmatrix},$$

where

$$a\,(t) = -0.5\cos t - 10e^{(-\sin t)+4} - 5.1e^{-\sin t} - 1.$$

By using MATHEMATICA, we obtain

$$\bar{\eta}(A) = sup_{t\in R^+}(\frac{1}{2}\lambda_{max}(A+A^T)) = sup_{t\in R^+}(-0.5\cos t - 10e^{-\sin t+10} - 5.1e^{-\sin t} - 1) \approx -81033.7$$

. Let $\beta = 0.01, \epsilon_1 = e^{10}, \epsilon_2 = e^{10}, \alpha = 2$ and $\epsilon_3 = \frac{1}{he^{2h}}$. With the same notations as in Theorem 3.1, it's straightforward to show that we may take $h = 2$, $\delta = 0.9$ and $\gamma = 3465$. Then $\eta = 1.02 + 2e^{10}$ and $\epsilon_1 = e^{10} \geq 2\beta\bar{\eta}(A) \approx 2(0.01)(-81033.7) = -1620.67$. We can verify that a solution P of LDE (2) is given by

$$P(t) = \begin{bmatrix} \frac{e^{\sin t}}{10} & 0 \\ 0 & \frac{e^{\sin t}}{10} \end{bmatrix}.$$

We have $p = sup_{t\in R^+}\|P(t)\| = \frac{e}{10}$ and

$$\frac{\sqrt{\epsilon_2(1-\delta)(\epsilon_1 - 2\beta\bar{\eta}(A))}}{(p+\beta)e^{\alpha h}} \geq \frac{\sqrt{e^{10}(1-0.9)(e^4 - 2(0.01)(-81033))}}{(\frac{e}{10}+0.01)e^h} \approx \frac{25608.1}{e^h}.$$

We may easily show that all assumptions of Theorem 3.1 are satisfied with $\gamma = \frac{25608.1}{e^h}$. Therefore, the system is robustly stable and the solution satisfies

$$\|x(t,\phi)\| \leq N\|\phi\|e^{-t}, \quad t \in R^+,$$

where $N = p+\beta+h\epsilon_2+2h^2\epsilon_3$. We see that the perturbation bound γ in this example is much greater than the one considered in [7, 8, 9], as shown in the Table 1 below. The simulation of this example is shown in Fig 1.

Delay bound (h)	2	4	6	8	10
Perturbation bound (γ)	3465.68	469.03	63.47	8.59	1.16

Table 1

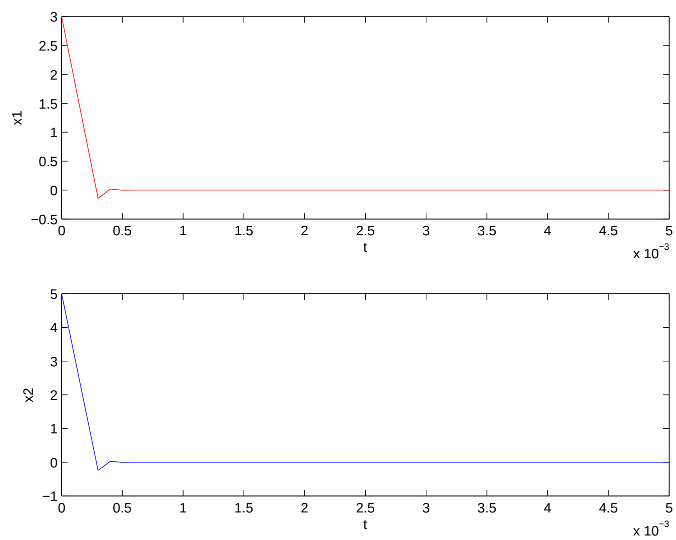

Figure 1: The trajectories of x_1 and x_2 in Example 3.1

The following example illustrates the effectiveness of Corollary 3.1.

Example 3.2. Let us consider the following system

$$\dot{x} = Ax(t) + Bx(t-h(t)) + Gx(t-h(t)),$$

where $A = \begin{bmatrix} -1.2 & 0.1 \\ -0.1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} -0.3 & 0.35 \\ -0.5 & -0.4 \end{bmatrix},$

$Gx(t-h(t)) = \begin{bmatrix} \beta(x(t-h(t)),t)\,x_1(t-h(t)) \\ 0 \end{bmatrix}$, with $|\beta(x(t-h(t)),t)| \leq 0.1$ whatever the values of the parameters, $h(t)$ is differentiable and β is piecewise continuous, such that $h(t) \leq h$, $\dot{h}(t) \leq \delta < 1, \forall t \in R^+$. Let $f(t, x(t-h(t))) = Bx(t-h(t)) + Gx(t-h(t))$. One may see that

$$\begin{aligned} \|f(t, x(t-h(t)))\| &\leq (\|B\| + |\beta(x(t-h(t)),t)|) \\ &\approx 0.741 \end{aligned}$$

With the same notations as in Theorem 3.1, we take $h = 1.25$, $\alpha = 0.1$, $\delta = 0.1$, $\beta = 0.000625$, $\gamma = 0.747$, $\epsilon_1 = 0.5$, $\epsilon_2 = 0.5$ and $\epsilon_3 = 0.000625$. A solution of RDE (2) is given by

$$P = \begin{bmatrix} 1.03391 & -0.011361 \\ -0.011361 & 1.26114 \end{bmatrix}$$

and $p = \|P\| = 1.2617$. From this, we may verify that all assumptions of Theorem 3.1 hold and hence the system is exponentially stable. The following Table 2 compares the upper bound of delay of our main result, with the same perturbation bound $\gamma = 0.747$, with some known results in previous literatures [1, 6] in case the delay is constant.

Criteria	Upper bound of delay (h)
1.Niculescu et al.	
[6, Theorem 2 with the estimation $\left\|e^{(A+B)t}\right\| \leq e^{\eta(A+B)t} = e^{-1.36t}$]	1.07588
2. Bartholomeus et al. [1, Theorem 1]	
2.1 Without decomposition of B	1.07588
2.2 With decomposition $B = \begin{bmatrix} -0.3 & 0 \\ 0 & -0.4 \end{bmatrix} + \begin{bmatrix} 0 & 0.35 \\ -0.5 & 0 \end{bmatrix}$	1.14382
3.Our result (Theorem 3.1)	≥ 1.25

Table 2

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