

# An improved stability criterion for a class of neutral differential equations

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## Abstract

This paper gives an improved criterion for asymptotical stability of a class of neutral differential equations. By introducing a new Lyapunov functional, we avoid the use of the stability assumption on the main operators and derive a novel stability criterion given in term of a LMI, which is less restricted than that given by J.H. Park [?] (2004) and Sun et al. [?] (2006).

*Key Words:* Asymptotical stability; Neutral differential equation; Lyapunov function; Linear matrix inequality.

## 1 Introduction

We are concerned with the asymptotic stability of the following neutral differential equation

$$\frac{d}{dt}[x(t) + px(t - \tau)] = -ax(t) + b \tanh x(t - \sigma), \quad t \geq 0, \quad (1.1)$$

where  $a, \tau$  and  $\sigma$  are positive constants,  $\sigma \geq \tau$ ,  $b, p$  are real numbers,  $|p| < 1$ . For each solution  $x(t)$  of equation (1.1), we assume the initial condition:

$$x(t) = \phi(t), \quad t \in [-\sigma, 0], \quad \phi \in C([-\sigma, 0], R).$$

Recently, the asymptotic stability of neutral differential equation (1.1) has been discussed in [?, ?, ?]. Using stability assumption on the operator

$$\mathcal{D}(x_t) = x(t) + px(t - \tau) + b \int_{t-\sigma}^t \tanh x(s) ds,$$

J.H. Park in [?] has proposed a delay-dependent criterion on  $\sigma$  for asymptotical stability of equation (??). Lately, constructing an improved Lyapunov function based on the operator

$$\mathcal{D}^*(x_t) = x(t) + px(t - \tau) + \alpha \int_{t-\tau}^t x(s)ds + b \int_{t-\sigma}^t \tanh x(s)ds,$$

Sun et al. in [?] have derived a less conservative stability criterion. However, this condition still depends on the stability of the operator  $\mathcal{D}^*$ . Our aim is to improve these criteria and to break away from some of the assumptions of previous papers. Compared with the result in [?, ?], our result has the following advantages:

- First, we introduce a new Lyapunov functional and avoid the use of the stability assumption on the operator  $\mathcal{D}$  in [?] or  $\mathcal{D}^*$  in [?]. Hence, our criterion will be less restricted than [?, ?](see Example 2).
- Second, using the operator

$$\mathcal{D}_1(x_t) = x(t) + px(t - \tau) + \alpha \int_{t-\tau}^t x(s)ds + b \int_{t-\sigma}^{t-\tau} \tanh x(s)ds,$$

the obtained condition depends not only on  $\tau$ , but also on  $\sigma - \tau$ . Thus, our criterion is more effective than [?] (see Example 1).

## 2 Main result

Throughout this section, the symbol  $*$  represents the elements below the main diagonal of a symmetric matrix. Also,  $X > Y$  means  $X - Y$  is positive definite.

Before present the main result, we need the following technical lemmas.

**Lemma 1.** (Completing the square) Assume that  $S \in \mathbb{R}^{n \times n}$  is a symmetric positive definite matrix. Then for every  $Q \in \mathbb{R}^{n \times n}$ ,

$$2\langle Qy, x \rangle \leq \langle QS^{-1}Q^T x, x \rangle + \langle Sy, y \rangle, \quad \forall x, y \in \mathbb{R}^n.$$

**Lemma 2.** [?] For any symmetric positive definite matrix  $M \in \mathbb{R}^{m \times n}$ , scalar  $\sigma \geq 0$  and vector function  $w : [0, \sigma] \rightarrow \mathbb{R}^n$  such that the integrations concerned are well defined, then

$$\left( \int_0^\sigma w(s)ds \right)^T M \left( \int_0^\sigma w(s)ds \right) \leq \sigma \int_0^\sigma w^T(s)Mw(s)ds.$$

For a real number  $\alpha$ , the equation (??) can be rewritten as follows:

$$\begin{aligned} \frac{d}{dt} [x(t) + px(t - \tau) + \alpha \int_{t-\tau}^t x(s)ds + b \int_{t-\sigma}^{t-\tau} \tanh x(s)ds] \\ = (\alpha - a)x(t) - \alpha x(t - \tau) + b \tanh x(t - \tau), \quad t \geq 0. \end{aligned} \tag{2.1}$$

Define the following operators

$$\mathcal{D}_1(x_t) = x(t) + px(t - \tau) + \alpha \int_{t-\tau}^t x(s)ds + b \int_{t-\sigma}^{t-\tau} \tanh x(s)ds, \quad \mathcal{D}_2(x_t) = x(t) + px(t - \tau),$$

Now, we have the following theorem.

**Theorem 1.** *The zero solution of equation (??) is uniformly asymptotically stable if there exist a constant  $\alpha$  with  $0 < |\alpha| < 1$  and the positive scalars  $\beta, \gamma, \eta, \theta, \mu$  such that the following linear matrix inequality holds*

$$\Omega = \begin{pmatrix} \Omega_{11} & p(\alpha - a) - \alpha & b & \alpha - a & b(\alpha - a) \\ \star & -\beta - 2p\alpha & pb & -\alpha & -b\alpha \\ \star & \star & \theta(\sigma - \tau)^2 - \eta & b & b^2 \\ \star & \star & \star & -\gamma & 0 \\ \star & \star & \star & \star & -\theta \end{pmatrix} < 0, \quad (2.2)$$

where  $\Omega_{11} = 2(\alpha - a) + \beta + \gamma\tau^2 + \eta$ .

*Proof.* Since  $\Omega < 0$ , there is a number  $\delta > 0$  such that

$$\Omega_1 = \begin{pmatrix} \Omega_{11} & p(\alpha - a) - \alpha & b & \alpha - a & b(\alpha - a) \\ \star & -\beta - 2p\alpha & pb & -\alpha & -b\alpha \\ \star & \star & \theta(\sigma - \tau)^2 - \eta + \delta & b & b^2 \\ \star & \star & \star & -\gamma & 0 \\ \star & \star & \star & \star & -\theta \end{pmatrix} < 0, \quad (2.3)$$

Consider the following Lyapunov functional

$$V = V_1 + V_2 + V_3 + V_4 + V_5 + V_6 + V_7$$

where  $V_1 = \mathcal{D}_1^T(x_t)\mathcal{D}_1(x_t)$ ,  $V_2 = \beta \int_{t-\tau}^t x^2(s)ds$ ,  $V_3 = \gamma\tau \int_{t-\tau}^t (\tau - t + s)(\alpha x(s))^2 ds$ ,  $V_4 = \theta(\sigma - \tau) \int_{t-\sigma}^{t-\tau} (s - t + \sigma) \tanh^2 x(s)ds$ ,  $V_5 = \eta \int_{t-\tau}^t \tanh^2 x(s)ds$ ,  $V_6 = \delta \int_{t-\sigma}^{t-\tau} \tanh^2 x(s)ds$ ,  $V_7 = \epsilon \mathcal{D}_2^2(x_t)$ ,  $\epsilon$  is a positive number that will be chosen later.

We first take the derivative of  $V^* = V_1 + V_2 + V_3 + V_4 + V_5 + V_6$  along the solution of equation (??) :

$$\begin{aligned} \frac{dV^*}{dt} = & 2 \left( x(t) + px(t - \tau) + \alpha \int_{t-\tau}^t x(s)ds + b \int_{t-\sigma}^{t-\tau} \tanh x(s)ds \right)^T \\ & \times \left( (\alpha - a)x(t) - \alpha x(t - \tau) + b \tanh x(t - \tau) \right) \\ & + \beta x^2(t) - \beta x^2(t - \tau) + \gamma\tau^2 \alpha^2 x^2(t) - \gamma\tau \int_{t-\tau}^t (\alpha x(s))^2 ds \\ & + \theta(\sigma - \tau)^2 \tanh^2 x(t - \tau) - \theta(\sigma - \tau) \int_{t-\sigma}^{t-\tau} \tanh^2 x(s)ds \\ & + \eta \tanh^2 x(t) - \eta \tanh^2 x(t - \tau) + \delta \tanh^2 x(t - \tau) - \delta \tanh^2 x(t - \sigma). \end{aligned}$$

Using Lemma ?? and  $\tanh^2 x(t) \leq x^2(t)$ , we have

$$\begin{aligned}
\frac{dV^*}{dt} &\leq 2\left(x(t) + px(t - \tau) + \alpha \int_{t-\tau}^t x(s)ds + b \int_{t-\sigma}^t \tanh x(s)ds\right)^T \\
&\quad \times \left((\alpha - a)x(t) - \alpha x(t - \tau) + b \tanh x(t - \tau)\right) \\
&\quad + \beta x^2(t) - \beta x^2(t - \tau) + \gamma \tau^2 \alpha^2 x^2(t) - \gamma \left(\int_{t-\tau}^t \alpha x(s)ds\right)^2 + \theta(\sigma - \tau)^2 \tanh^2 x(t - \tau) \\
&\quad - \theta \left(\int_{t-\sigma}^{t-\tau} \tanh x(s)ds\right)^2 + \eta x^2(t) - \eta \tanh^2 x(t - \tau) \\
&\quad + \delta \tanh^2 x(t - \tau) - \delta \tanh^2 x(t - \sigma).
\end{aligned}$$

Setting  $\alpha \int_{t-\tau}^t x(s)ds = u(t)$ ,  $\int_{t-\sigma}^{t-\tau} \tanh x(s)ds = v(t)$  and using  $0 < |\alpha| < 1$ , we have

$$\begin{aligned}
\frac{dV^*}{dt} &\leq [2(\alpha - a) + \beta + \gamma \tau^2 + \eta]x^2(t) + [2\alpha + 2p(\alpha - a)]x(t)x(t - \tau) \\
&\quad + 2bx(t) \tanh x(t - \tau) + 2(\alpha - a)x(t)u(t) + [2(\alpha - a)b]x(t)v(t) \\
&\quad - (2p\alpha + \beta^2)x^2(t - \tau) + 2pbx(t - \tau) \tanh x(t - \tau) - 2\alpha x(t - \tau)u(t) \\
&\quad - 2\alpha bx(t - \tau)v(t) + [\theta(\sigma - \tau)^2 - \eta + \delta] \tanh^2 x(t - \tau) - \delta \tanh^2 x(t - \sigma) \\
&\quad + 2b \tanh x(t - \tau)u(t) + 2b^2 \tanh x(t - \tau)v(t) - \gamma u^2(t) - \theta v^2(t) \\
&\leq \zeta^T(t) \Sigma \zeta(t),
\end{aligned}$$

where

$$\zeta(t) = \begin{pmatrix} x(t) \\ x(t - \tau) \\ \tanh x(t - \tau) \\ u(t) \\ v(t) \\ \tanh x(t - \sigma) \end{pmatrix},$$

$$\Sigma = \begin{pmatrix} \Omega_{11} & p(\alpha - a)ap - \alpha & b & \alpha - a & b(\alpha - a) & 0 \\ * & -\beta - 2p\alpha & pb & -\alpha & -b\alpha & 0 \\ * & * & \theta(\sigma - \tau)^2 - \eta + \delta & b & b^2 & 0 \\ * & * & * & -\gamma & 0 & 0 \\ * & * & * & * & -\theta & 0 \\ * & * & * & * & * & -\delta \end{pmatrix}.$$

Using the condition (??), we have  $\Sigma < 0$  and

$$\frac{dV^*}{dt} \leq \zeta^T(t) \Sigma \zeta(t) < 0.$$

Thus, there is a positive number  $\lambda$  such that

$$\frac{dV^*}{dt} \leq -\lambda \left( \|x(t)\|^2 + \|x(t - \tau)\|^2 + \|\tanh x(t - \tau)\|^2 + \|u(t)\|^2 + \|v(t)\|^2 + \|\tanh x(t - \sigma)\|^2 \right).$$

Now taking the derivative of  $V_7$  and using Lemma 1, we have

$$\begin{aligned}\frac{dV_7}{dt} &= 2\epsilon[x(t) + px(t - \tau)][-ax(t) + b \tanh x(t - \sigma)] \\ &= -2\epsilon ax^2(t) + 2\epsilon bx(t) \tanh x(t - \sigma) - 2\epsilon apx(t)x(t - \tau) + 2\epsilon bpx(t - \tau) \tanh x(t - \sigma) \\ &\leq \epsilon(-2a + |b|^2 + |ap|^2)x^2(t) + \epsilon(1 + |bp|^2)x^2(t - \tau) + 2\epsilon \tanh^2 x(t - \sigma)\end{aligned}$$

Choosing

$$\epsilon = \begin{cases} \frac{\lambda}{2} \min \left\{ \frac{1}{1+|bp|^2}, \frac{1}{2} \right\}, & \text{if } -2a + |b|^2 + |ap|^2 \leq 0 \\ \frac{\lambda}{2} \min \left\{ \frac{1}{-2a+|b|^2+|ap|^2}, \frac{1}{1+|bp|^2}, \frac{1}{2} \right\} & \text{if } -2a + |b|^2 + |ap|^2 > 0, \end{cases}$$

we obtain that  $\frac{dV}{dt} < -\frac{\lambda}{2}\|x(t)\|^2$ . On the other hand, we have  $V \geq \epsilon \mathcal{D}_2^2 x(t)$  and  $\mathcal{D}_2 x(t)$  is stable due to  $|p| < 1$ . Therefore, according to Theorem 8.1 in [?] (pp 292-293), the zero solution of equation (??) is uniformly asymptotically stable. The proof of the theorem is completed.  $\square$

**Remark 1.** Note that the stability criterion of (??) obtained in [?] is given as

$$\begin{pmatrix} -2a + 2\alpha + \beta + \gamma\tau^2 + \eta & p(\alpha - a) - \alpha & b & \alpha - a & b(\alpha - a) \\ \star & -\beta - 2p\alpha & pb & -\alpha & -b\alpha \\ \star & \star & \theta\sigma^2 - \eta & b & b^2 \\ \star & \star & \star & -\gamma & 0 \\ \star & \star & \star & \star & -\theta \end{pmatrix} < 0. \quad (2.4)$$

To compare with the stability criterion obtained in [4], we consider the case  $\alpha = 0$ . In this case, we have

$$\mathcal{D}_1(x_t) = x(t) + px(t - \tau) + b \int_{t-\sigma}^{t-\tau} \tanh x(s) ds, \quad V_3 = 0.$$

By the same arguments used in the proof of Theorem 1 with the Lyapunov functional chosen as

$$V = V_1 + V_2 + V_4 + V_5 + V_6 + V_7$$

we can derive the following stability criterion.

**Corollary 1.** *The zero solution of equation (??) is uniformly asymptotically stable if there exist the positive scalars  $\beta, \gamma, \theta, \eta$  such that the linear matrix inequality holds*

$$\bar{\Omega} = \begin{pmatrix} -2a + \beta + \eta & -ap & b & -ab \\ \star & -\beta & pb & 0 \\ \star & \star & \theta(\sigma - \tau)^2 - \eta & b^2 \\ \star & \star & \star & -\theta \end{pmatrix} < 0. \quad (2.5)$$

**Remark 2.** In [?], the delay-independent sufficient condition for the stability of the equation (??) is

$$1 + \frac{pa^2 + b^2}{2(1 - pb)} - a < 0 \quad \text{and} \quad pb < 1. \quad (2.6)$$

To show the advantages of our criterion, we consider the following examples

**Example 1.** Consider the following equation:

$$\frac{d}{dt}[x(t) + 0.6x(t - 0.15)] = -0.6x(t) + 0.493 \tanh x(t - 0.3). \quad (2.7)$$

By solving the LMI (??), we obtain its solution:

$$\beta = 1.3947, \quad \gamma = 9.0979, \quad \theta = 4.4533, \quad \eta = 0.7499, \quad \alpha = -0.8718,$$

which implies the asymptotical stability of equation (??). However, the LMI (??) is infeasible, that is the criterion of [?] can not be applied. Moreover, the criterion obtained in [?] cannot be applied since the inequality (??) does not hold. This shows that our criterion is more effective than that obtained in [?, ?].

**Example 2.** Consider the following equation:

$$\frac{d}{dt}[x(t) - 0.7x(t - 9.9)] = -0.6x(t) - 0.3 \tanh x(t - 10). \quad (2.8)$$

By solving the LMI (??), we obtain its solution:

$$\beta = 0.3923, \quad \gamma = 6.1108, \quad \theta = 4.9347, \quad \eta = 0.2840,$$

which implies the asymptotical stability of equation (??). However, the criterion obtained in [4] cannot be applied since the operator  $\mathcal{D}$  used in [?] is unstable. Indeed, we consider the equation  $\mathcal{D}x_t = 0$  with initial condition:

$$x(s) = c = \text{constant} > 0, \quad s \in [-10, 0].$$

If  $c \geq 1$  then  $\tanh(x(s)) > \frac{1}{2}$ ,  $\forall s \in [-10, 0]$  and hence  $x(t) \geq 0.7c + 1.5 > 1, \forall t \in (0, 10]$ . Similarly, we also have  $x(t) \geq 1, \forall t \in (10, 20]$ . Continuing the progress, we will have  $x(t) \geq 1, \forall t \in (0, +\infty)$ . If  $0 < c < 1$  then  $\tanh(x(s)) > \frac{x(s)}{2} = \frac{c}{2}$  and hence  $x(t) \geq 0.7c + 1.5c, \forall t \in (0, 10]$ . Similarly, we also have  $x(t) \geq 0.7c + 2 \times 1.5c, \forall t \in (10, 20]$ . Since  $c > 0$ , there is a natural number  $n_0 \in N$  such that  $0.7c + (n_0 - 1) \times 1.5c > 1$ . Then, we will have  $x(t) \geq 0.7c + n_0 \times 1.5c > 1, \forall t \in (10n_0, 10(n_0 + 1)]$ . Similar to above, we have  $x(t) \geq 1, \forall t \in [10n_0, +\infty)$ . Thus, the operator  $\mathcal{D}$  is unstable.

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