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A RIGHT SELF-INJECTIVE RIGHT PERFECT RING IS QUASI-FROBENIUS

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Dedicated to Professor Nguyen Dinh Tri on his sixtieth birthday

Abstract. *The purpose of this note is to prove a theorem stated in the title, answering a known open question in the area affirmatively.*

All rings considered are associative with identity and all modules are right unitary modules. We are concerned with the categorical properties: A module M over a ring R (briefly denoted by M_R) is projective if and only if every R-epimorphism $N_R \to M_R$ splits. Dually, M_R is injective if and only if every R-monomorphism $M_R \to N_R$ splits.

A ring R is *quasi-Frobenius* (briefly, QF) if R has the descending chain condition on right ideals and R_R is injective. QF-rings form an interesting class of non-semisimple rings and they have been intensively studied (see, for example, Nagayama [27], [28], Morita and Tachikawa [26], Ikeda and Nakayama [19], Morita [25], Dieudonne [8], Faith [12], [13], Faith and Walker [15], Osofsky [27], Kato [21], Tachikawa [30], Lawrence [22], Gomez-Pardo [17], Menal [21], Armendariz [3], D.V.Huynh, N.V.Dung and R. Wisbauer [10], Armendariz and Park [4], Ara, Armendariz and Park [2],...). Many important properties equivalent to this definition (it is not the original one) have been obtained, however, several interesting questions remain open, for example questions recently mentioned again in Faith's survey article [14]:

(Q1) Is a right or left perfect right self-injective ring necessarily QF?

(Q2) Is a right and left perfect right self-injective ring necessarily QF?

(Q3) Is a semiprimary right self-injective ring necessarily QF?

We shall give a positive answer to a half of (Q1) therefore answering (Q2) and (Q3) affirmatively. Namely we prove the following theorem:

Theorem. *A right self-injective, right perfect ring is QF.*

Let M be a module and K be an index set with the cardinality $|K|$. Then $M^{(K)}$ denotes the direct sum of $|K|$ copies of M. The set of all positive integers is denoted by N, and $E(M)$ stands for the injective hull of the module M. For other notions used in this paper we refer to [1], [11], [30] and [31].

We start with a consideration on QF-3 rings, a generalization of QF-rings. A ring R is called right QF-3 provided there exists a (unique) minimal faithful right R-module U_R. That is, U_R is isomorphic to a direct summand of every faithful right R-module. It was proved that U_R is projective and isomorphic to the injective hull of a direct sum of finitely many simple right R-modules. One can identify U_R with a right ideal of R. Left QF-3 rings are defined similarly. Left and right QF-3 rings are simply called QF-3 rings. Basic properties of QF-3 rings can be found in [30].

We restate a result from our work with P.Dan [9, Theorem 3.2]:

Proposition. *For a ring R the following conditions are equivalent:*
(a) R is a semiprimary QF-3 ring.
(b) R is right perfect and $E(R_R^{(N)})$ is projective.
(c) R is right perfect satisfying the ascending chain condition on right annihilators and $E(R_R)$ is projective.

P r o o f. (a) $\Rightarrow$ (b) by [7, Theorem 1.3].

(b) $\Rightarrow$ (c). Assume (b). Since $E(R_R)$ is isomorphic to a direct summand of $E(R_R^{(\mathrm{N})})$, $E(R_R)$ is projective. Further, by (b) and [1, Theorem 27.11] we have

$$E(R_R^{(\mathrm{N})}) \cong e_1 R^{(I_1)} \oplus \ldots \oplus e_k R^{(I_k)}, \tag{1}$$

where $e_1, \ldots, e_k$ are primitive idempotents of R. By [11, Proposition 20.3A], to get (c) it is enough to show that $E(R_R)^{(\mathrm{N})}$ is injective. By (1) and since $E(R_R)$ is projective, there are subsets F_i of I_i ($i = 1, ..., k$) such that

$$E(R_R) \cong e_1 R^{(F_1)} \oplus \ldots \oplus e_k R^{(F_k)}. \tag{2}$$

Hence

$$E(R_R)^{(\mathrm{N})} \cong (e_1 R^{(F_1)} \oplus \ldots \oplus e_k R^{(F_k)})^{(\mathrm{N})}. \tag{3}$$

Since $E(R_R)^{(\mathrm{N})}$ can be embedded isomorphically in $E(R_R^{(\mathrm{N})})$, the injectivity of $E(R_R)^{(\mathrm{N})}$ follows from (3) and [11, Theorem 21.15].

(c) $\Rightarrow$ (a). Assume (c). In order to show (a), by [7, Theorem 1.3], it is enough to show that R contains a faithful $\sum$-injective right ideal eR. Note that $R = e_1 R \oplus \ldots \oplus e_n R$ where $\{e_i\}_{i=1}^n$ is a set of orthogonal primitive idempotents.

Since $E(R_R)$ is projective by (c), there is at least one e_i such that e_iR is injective. We may assume that $e_1R, \ldots, e_kR$ are injective and $e_{k+1}R, \ldots, e_nR$ are not. Put $e = e_1 + \ldots + e_k$. Then $eR = e_1R \oplus \ldots \oplus e_kR$ is a non-zero injective right ideal of R. Also, since $E(R_R)$ is projective, we may use (2) to see that $E(R_R)$ is embedded in $eR^{(F)}$ where $F = F_1 \cup \ldots \cup F_k$. Hence, if $P = ann_R(eR)$, then $E(R_R).P = 0$. It follows $P = 0$, proving that eR is faithful. By (c), R satisfies the ascending chain condition on annihilators of subsets of eR. Hence eR is $\sum$-injective by [11, Proposition 20.3A].

The proof of Proposition is complete.

Now we are at a position to prove our theorem.

P r o o f (of Theorem). Let R be a right self-injective, right perfect ring and let $\{e_i\}_{i=1}^n$ be a complete set of orthogonal primitive idempotents of R. We have

$$R = e_1R \oplus \ldots \oplus e_nR. \tag{4}$$

Let $\{e_{i_a}R\}_A$ be a family of submodules of R_R with $e_{i_a} \in \{e_i\}_{i=1}^n$. We consider an R-homomorphism $f : e_{i_a}R \to e_{i_b}R$. Since for each $r \in R$, $f(e_{i_a}r) = f(e_{i_a}).(e_{i_a}r)$, we may consider f as an element of R with the property $f.e_{i_a} = f$. Assume that f is non-isomorphic. Since every $e_{i_a}R$ is indecomposable and injective, we must have $ker(f) \neq 0$, i.e. f is non-monomorphic. It is then clear that $f(e_{i_a}R) \neq e_{i_b}R$. Since $e_{i_b}R$ is a local module, $f(e_{i_a}R)$ must be contained in the Jacobson radical J of R. But $f(e_{i_a}R) = f(e_{i_a})R$. It follows that $f(e_{i_a}) \in J$, implying $f \in J$, since we took f as an element of R with $f.e_{i_a} = f$.

On the other hand, by [6] (see also [1, Theorem 28.4] or [11, Theorem 22.29]) J is right T-nilpotent. That is, for every sequence $\{f_1, f_2, \ldots\}$ of elements of J, there exists a positive integer m such that $f_m f_{m-1} \ldots f_1 = 0$. From this and the previous observation we can show that the family $\{e_{i_a}R\}_A$ with infinite set A is locally-semi-transfinitely-nilpotent (briefly:lsTn, see the definition for example in [24, p.29]).

Thus we can apply [5, Theorem 4] or [20, Lemma 11, Proposition 6] to obtain that $M = \bigoplus_{a \in A} e_{i_a}R$ is an extending module in the sense that every submodule of M is contained as an essential submodule in a direct summand of M. We use this fact to show in the next step that $E(R_R^{(N)})$ is a projective R-module.

As is well-known, there exists an index set B such that we have an R-epimorphism

$$g : R_R^{(B)} \to E(R_R^{(\mathbb{N})}).$$

From (4) it follows

$$R_R^{(B)} = (e_1R \oplus \ldots \oplus e_nR)^{(B)}.$$

By the above consideration, $R_R^{(B)}$ is then an extending module. Therfore there is a direct decomposition $R_R^{(B)} = V \oplus W$ with $V \supseteq ker(g)$ and $ker(g)$ is essential in V. It follows the isomorphism

$$E(R_R^{(\mathbf{N})}) \cong V/ker(g) \oplus W. \tag{5}$$

We show that $V/ker(g) = 0$, implying the projectivity of $E(R_R^{(\mathbf{N})})$ by (5). Suppose on the contrary that $V/ker(g) \neq 0$. Then by using [24, Lemma 1.31] together with (4) and (5) we have the existence of an e_iR in (4) such that $V/ker(g)$ contains a non-zero submodule U isomorphic to a submodule of e_iR. Hence $V/ker(g)$, as an injective module by (5), contains $E(U)$ isomorphic to e_iR. This is a contradiction to the facts that $V/ker(g)$ is singular and e_iR is projective, cyclic. Thus $V/ker(g)$ has to be zero.

Now by Proposition we see that R has the ascending chain condition on right annihilators. Then by a well-known theorem of Faith [12] (see also [10, Theorem 24.20]), R is quasi-Frobenius.

The proof of Theorem is complete.

Let R be a right perfect ring and $\{e_iR\}$ be a family of submodules e_iR 's of R_R with primitive idempotents e_i 's. In case R being right self-injective we just gave a direct proof that this family is $lsTn$. In fact, this is also true for an arbitrary right perfect ring R. To see this we use [1, Theorem 28.14] and then [24, Theorem 2.25] (or [18, Theorem 1] and then [24, Theorem 2.25]). However to get the extending property for $\oplus e_iR$ we need the right injectivity of R. In this case, Theorem 4 of Baba-Harada in [5] is useful. The next step of the proof of Theorem could be done also by using Theorem III of [9] on co-H-rings. In fact, this was the way we used to obtain the result of this note in December 1988 after receiving a preprint of [5] from Professor Harada, and communicated with him at that time about this matter. I would like to seize this opportunity to thank him again for sending a preprint of [5] timely.

A ring R is called right $(F)PF$ (cf. [16]) if every (finitely generated) faithful right R-module generates all right R-modules. By [16, Proposition 2.2B], a right perfect right FPF ring is right self-injective and hence by our theorem it is then a QF-ring.

Inspired by a theorem of Faith [12], Armendariz and Park considered a restricted form of chain condition in a right self-injective ring and showed in [4, Theorem 3] that if R is a right self-injective ring such that $R/Soc(R_R)(Soc(R_R)$ denotes the right socle of $R)$ has the ascending chain condition on right annihilators, then R is semiprimary right PF. Using our theorem we see that in this case R is even a QF-ring.

Thus we obtain the following consequence of Theorem:

Corollary. *For a ring R the following conditions are equivalent:*

a) R is QF.

b) R is right perfect, right FPF.

c) R is right self-injective and $R/Soc(R_R)$ satisfies the ascending chain condition on right annihilators.

The equivalence a)⇔ b) gives a positive answer to question 10 in [16, Open-Questions].

REFERENCES

1. F. W. Anderson and K. R. Fuller: Rings and Categories of Modules. Springer-Verlag 1974.
2. P. Ara, E. P. Armendariz and J. K. Park: Continuous rings with restricted chain conditions. Proc. Inter. Symposium Semigroup Theory, Kyoto 1990.
3. E. P. Armendariz: Rings with DCC on essential left ideals. Comm. Algebra 8(3) (1980), 299-308.
4. E. P. Armendariz and J. K. Park: Self-injective rings with restricted chain conditions. To appear.
5. Y. Baba and M. Harada: On almost M-projectives and almost M-injectives. Tsukuba J. Math. 14 (1990), 53-69.
6. H. Bass: Finitistic dimension and a homological generalization of semiprimary rings. Trans. Amer. Math. Soc. 95(1960), 466-488.
7. R. R. Colby and Rutters: Generalization of QF-3 algebras. Trans. Amer. Math. Soc. 153 (1971), 371-386.
8. J. Dieudonne: Remarks on quasi-Frobenius rings, III. J. Math. 2(1958), 346-354.
9. Dinh Van Huynh and Phan Dan: Some characterizations of right co-H-rings. Inst. Math. Hanoi, Preprint Series No. 88.21 (1988).
10. Dinh Van Huynh, Nguyen V. Dung and R. Wisbauer: Quasi-injective modules with ACC or DCC on essential submodules. Arch. Math. 53 (1989), 252-255.
11. C. Faith: Algebra II: Ring Theory. Springer-Verlag 1976.
12. C. Faith: Rings with ascending chain condition on annihilators. Nagoya Math. J. 27(1966), 179-191.
13. C. Faith: Embedding modules in projectives. Advances in non-commutative Ring Theory Springer LNM 951 (1982), 21-42.
14. C. Faith: When self-injective rings are QF: A report on a problem. Lecture Notes of Centre Recerca Mathematica, U. Autonoma Barcelona, Spain (1990).

15. C. Faith and E. A. Walker: Direct sum representations of injective modules. J. Algebra 5(1967), 203-221.
16. C. Faith and S. Page: FPF Ring Theory. London Math. Soc. Lecture Note Series 88(1984).
17. J. L. Gomez-Pardo: Embedding cyclic and torsion free modules in free modules. Arch. Math. 44 (1985), 503-510.
18. M. Harada and T. Ishii: On perfect rings and the exchange property. Osaka J. Math. 12(1975), 483-491.
19. M. Ikeda and T. Nagayama: On some characteristic properties of quasi-Frobenius rings and regular rings. Proc. Amer. Math. Soc. 5 (1954), 15-19.
20. M. A. Kamal and B. J. Mueller: The structure of extending modules over noetherian rings. Osaka J. Math. 25(1988), 539-551.
21. T. Kato: Some characterizations of QF rings. Proc. Japan Acad. 44 (1968), 114-119.
22. J. Lawrence: A countable self-infective ring is quasi-Frobenius. Proc. Amer. Math. Soc. 65(1977), 217-220.
23. P. Menal: On the endomorphism ring of a free module. Publ. Math. A. U. B. 27(1982), 141-151.
24. S. H. Mohamed and B. J. Mueller: Continuous and discrete modules. London Math. Soc. Lecture Note Series 147 (1990).
25. K. Morita: Duality for modules and its applications to the theory of rings with minimum condition. Tokyo Kyoiku Daigaku. Sec. A 6 (1958), 83-142.
26. K. Morita and H. Tachikawa: Character modules, submodules of a free module, and quasi-Frobenius rings. Math. Z. 65 (1956), 414-428.
27. T. Nagayama: On Frobenius algebras, I. Ann. Math. 40 (1939), 611-633.
28. T. Nagayama: On Frobenius algebras, II. Ann. Math. 42 (1941), 1-21.
29. B. L. Osofsky: A generalization of quasi-Frobenius rings. J. Algebra 4(1966), 373-387.
30. H. Tachikawa: Quasi-Frobenius rings and generalizations. Springer LNM 351 (1973).
31. R. Wisbauer: Foundations of module and ring theory. Gordon and Breach, London-Tokyo e.a. (1991).

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