Journal of Mathematics *Tạp chí Toán học* **19**(2)(1991), 18—3?

AN EVOLUTION NONLINEAR MIXED PROBLEM*)

LE NGOC LANG and NGO VAN LUOC

Dedicated to the memory of Professor Le Van Thiem

Abstract. *The paper is devoted to an evolution nonlinear mixed problem. The existence and uniqueness theorems for the considered problem are proved by the Galerkin method. An approximate solution is obtained by an iterative method.*

1. INTRODUCTION

The paper deals with an initial boundary problem of the following evolution nonlinear equation

$$\frac{\partial p}{\partial t} - div[\lambda(p) grad p + \mu(p)\vec{\theta}] = f(p),$$

where λ, μ, f are given functions and θ is a given constant vector. This problem arises, for example, in filtration problem in cracked layers [1]. After the introduction, in the second section by the Galerkin method we shall show the existence and uniqueness of the solution for the considered problem. The third section deals with approximate solution of the problem by an iterrative method. The convergence of the proposed method is given.

*)This publication is completed with financial support from the National Basic Research Program in Natural Sciences.

2. EXISTENCE AND UNIQUENESS THEOREMS

Suppose that $G \subset R^n$ is a finite domain with the enough smooth boundary $\Gamma,\ \Gamma_1 \cup \Gamma_2,\ \Gamma_1 \cap \Gamma_2 = \emptyset,\ S = (0,T)$, where T is given positive number and $Q = G \times S$. Consider the following mixed problem

$$\frac{\partial p}{\partial t} - div\{\lambda(p)gradp + \mu(p)\vec{\theta}\} = f(p) \text{ in } Q, \tag{2.1}$$

$$p = 0 \text{ on } \Gamma_1,\ \lambda(p)(\frac{\partial p}{\partial n} + \mu(p)(\vec{\theta}, \vec{n})) = 0 \text{ on } \Gamma_2, \tag{2.2}$$

$$p\,|_{t=0} = p_a(x),\ x \in G, \tag{2.3}$$

where $\vec{n}$ is the outside normal vector of boundary Γ_2 and $p_a(x)$ is given function. The condition (2.2) shows that Γ_2 is an impervious boundary. Assume that

(I) The function $\lambda : R \to R$ is continuous and bounded from upper

$$\lambda(\xi) \leq M,\ \forall \xi \in R,\ M = \text{const}.$$

(II) The function λ is bounded from below $\lambda(\xi) \geq m,\ \forall \xi \in R,\ m = \text{const}$

(III) The functions $\mu : R \to R$ satisfies the condition $|\mu(\xi)| \leq c|\xi|,\ \forall \xi \in R.$

(IV) The function μ is Lipschitz continuous

$$|\mu(\xi_1) - \mu(\xi_2)| \leq c|\xi_1 - \xi_2|,\ \forall \xi_1, \xi_2 \in R.$$

(V) The function $f : R :\to R$ is continuous and $|f(\xi)| \leq c(|\xi| + 1)\quad \forall \xi \in R.$

(VI) The function f is Lipschitz continuous.

Here and in the sequence let C denote some constant. We introduce the Banach space

$$V = \{p \mid p \in H^1(G),\ p\,|_{\Gamma_1} = 0\},\ H = L^2(G)$$

with the norms

$$||p||_V^2 = ||p||^2 = \int_G (|p|^2 + |gradp|^2)dx,\ |p|_H^2 = \int_G |p|^2 dx.$$

It is easy to see that (see [4]) the space V is compactly imbedded into H and

$$|p|_H \leq C||p|| \qquad p \in Y.$$

Put

$$a(u,h)=\int_G \{\lambda(u) grad\ u\ grad\ h+\mu(u)\vec{\theta}\ grad\ h f(u)h\}da,\ \forall u,h\in Y.$$

In view of the Green formula we can deduce problem (2.1)–(2.3) to the following problem

$$\begin{cases} p'(h)+a(p,h)=0, \\ p(o)=p_a\in H,\ p\in W=\{p\mid p\in L^2(S,V),\ p'\in L^2(S,V^*)\}, \end{cases} \tag{2.4}$$

where V^* is the conjugate space of V. We shall call a solution of (2.4) to be a weak solution of problem (2.1)–(2.3).

Lemma 2.1. *Under assumptions (I), (III), (V) the following inequality holds*

$$|a(p,h)|\le C(||p||+1)||h|| \qquad \forall p,h\in V.$$

P r o o f. By virtue of Buniakovski inequality with all elements $p,h\in V$ we have

$$|a(p,h)|\le |\int_G\{\lambda(p) gradp\ gradh+\mu(p)\vec{\theta}\ gradh\}dh|+|\int_G f(p)hdx|$$

$$\le C||p||\,|\,||h||+C(\int_G(|p|+1)^2dx)^{1/2}||h||\le C(||p||+1)||h||$$

as was to be shown.

In view of Lemma 2.1 we can define the operator $A\in(V\to V^*)$ as follows

$$a(p,h)=(Ap,h)\qquad \forall p,h\in V,$$

from which it follows that problem (2.4) is equivalent to the Cauchy problem

$$p'+Ap=0,\ p(o)=p_a\in H,\ p\in W. \tag{2.5}$$

Lemma 2.2. *Suppose that conditions (I), (III)-(VI) are satisfied. Then operator A is continuous.*

P r o o f. With $p,\ p_n>h\in W$ we see that

$$||Ap_n - Ap||_* \le C\{(\int_G |\lambda(p_n) - \lambda(p)|^2 |gradp|^2 dx)^{1/2} + ||p_n - p||\}. \qquad (2.6)$$

Assume that the sequence $\{p_n\}$, $p_n \in V$ converges to the element p in the space V. Then we can choose a subsequence $\{p_{nk}\}$ such that

$$p_{nk} \longrightarrow p(x) \quad \text{a.e. in} \quad G.$$

Using the continuity of function λ we get

$$\lambda(p_{nk}(x)) \longrightarrow \lambda(p(x)) \quad \text{a.e. in} \quad G.$$

Then by virtue of Lebesgue theorem we see that

$$\lim_{k\to\infty} \int_G |\lambda(p_{nk}) - \lambda(p)|^2 |gradp|^2 dx \longrightarrow 0.$$

Since the last equality is valid for all subsequenceces which converge to $p(x)$ in V we have

$$\lim_{n\to\infty} \int_G |\lambda(p_n) - \lambda(p)|^2 |gradp|^2 dx = 0. \qquad (2.7)$$

From (2.6) and (2.7) we obtain Lemma 2.2. By the Galerkin method (see [5], for example) we shall show the existence of a weak solution for problem (2.5). In view of the separability of the space V there is a countable complete system of linearly independent elements $\{h_1, h_2, ...\}$ in the space V (and therefore in the space H). Suppose that $H_n(= V_n)$ is the linear envelope of the finite system $\{h_1, h_2, ..., h_n\}$. We shall identify H_n with H_n^*. Put $X_n = L^2(S, H_n)$ with the scalar product

$$< f, u > = \int_S (f(s), u(s))ds \quad \forall f, u \in X_n.$$

We introduce the operator $A_n : X_n \to X_n^*$.

$$< A_n u, v > = < Au, v > \quad \forall u, v \in X_n.$$

Suppose that $\{p_{an}\}$, $n = 1, 2, ...$ is a sequence of elements $p_{an} \in H_n$ such that p_{an} converges to p_a in H. We define $f_n \in X_n^*$ as follows

$$< f_n, v > = < f, v > \quad \forall v \in X_n.$$

Then the Galerkin equation corresponding to problem (2.5) has the form

$$p_{n'} + A_n p_n = 0,\ p_n(o) = p_{an},\ u_n \in V_n. \tag{2.8}$$

Note that we can take p_n as follows

$$p_n = \sum_{i=1}^{n} \alpha_{ni}(t) h_i,$$

where the coefficients α_{ni} depend only on t.

Lemma 2.3. *The operator $A_n : H_n \to H_n$ is continuous.*

P r o o f. From [5] with $u, v, h \in H_n$ we have

$$|(A_n u - A_n v, h)| \leq C(n)||Au - Av||_* |h|,$$

or

$$|A_n u - A_n v| \leq C(n)||Au - Av||_*. \tag{2.9}$$

Suppose that u_k converges to u in H_n. Then u_k converges to u also in V. Consequently, in view of the continuity of A and (2.9) we get

$$0 \leq \lim_{k\to\infty} |A_n u_k - A_n u| \leq C(n) \lim_{k\to\infty} ||Au_k - Au||_* = 0.$$

The lemma is proved.

Theorem 2.1. *Under conditions (II), (III), (V), (VI) there exists a solution of problem (2.8).*

P r o o f. By virtue of (2.8') we can deduce problem (2.8) to system of differential equation for $\alpha_{ni}(t)$, $i = 1, 2, ..., n$. Because of the Peano theorem and Lemma 2.3 it is enough to obtain a priori estimation for $p_n(t)$. For $t \in S$ we see that

$$0 = \int_0^t (p_n' + A_n p_n, p_n) ds$$

$$\geq \frac{1}{2}|p_n(t)|^2 - \frac{1}{2}|p_{an}|^2 + C_1 \int_0^t ||p_n(s)||^2 ds - C \int_0^t \int_G (|p_n| + t)|p_n| dx ds.$$

From this it follows that

$$|p_n(t)|^2 + C_1 \int_0^t ||p_n(s)||^2 ds \le C_2 + C_3 \int_0^t |p_n(s)|^2 ds, \tag{2.10}$$

where C_1, C_2 and C_3 are constant, independent of t and n.

Using Gronwall inequality we obtain

$$|p_n(t)|^2 \le CC^{C_3 T} = \text{const.} \tag{2.11}$$

So Theorem 2.1 is proved. It is easy to show the following

Lemma 2.4. *Under the asumptions of Lemma 2.3 we have the following estimations*

$$||p_n||_{L^2(S,V)} \le C, \tag{2.12}$$

$$||p_n'||_{L^2(S,V^*)} \le C. \tag{2.13}$$

Lemma 2.5. *Under the assumptions (I)–(VI) the sequence of solutions of Galerkin equations (2.8) is compact in* $L^2(S, H)$.

This lemma is an immediate consequence of Lemma 2.4 and the results in [6].

Theorem 2.2. *Suppose that the conditions (I)–(VI) are satisfied. then problem (2.5) has at least one solution.*

P r o o f. In view of Lemmas 2.4 and 2.5 there are a subsequence $\{p_i\}$ of the sequence $\{p_n\}$ and an element $p \in L^2(S, V)$ such that

$$p_i \rightharpoonup p \quad \text{in} \quad L^2(S, V), \tag{2.14}$$

$$p_i \rightharpoonup p \quad \text{in} \quad L^2(S, H). \tag{2.15}$$

From Lemma 2.1 and 2.5 we have

$$||Ap_n||_{L^2(S,V^*)} \le C(||p||_{L^2(S,V)} + 1) \le C.$$

Therefore we can choose a subsequence $\{p_i\}$ such that

$$Ap_i \rightharpoonup f \quad \text{in} \quad L^2(S, V^*).$$

By virtue of (2.8) with all elements $x \in \bigcup_n H_n$ and $\varphi \in \mathcal{D}(S)$ we get

$$0 = < p_i' + Ap_i, \varphi x > = (\int_S \varphi(t)(p_i'(t) + Ap_i(t))dt, x),$$

$$= (\int_S \varphi(t)p_i'(t)dt, x) + (\int_S \varphi(t)Ap_i(t)dt, x).$$

Because of (2.14) we see that p_i' converges to p' in $\mathcal{D}^*(S, V)$ (see [5], page 109). By letting $t \to \infty$ in the last equality we obtain

$$(p'(\varphi) + f(\varphi), x) = 0.$$

In view of the density of $\bigcup_n H_n$ in V we have

$$p' + f = 0 \quad \text{in} \quad L^2(S, V^*). \tag{2.16}$$

From (2.8) and (2.16) we see that

$$p(o) = p_a. \tag{2.17}$$

Now in view of the definition of operator A we have with all elements $h \in L^2(S, V)$:

$$< Ap_i, h > = \int_S \int_G \{[\lambda(p_i) grad p_i + \mu(p_i)\vec{\theta}] grad h - f(p_i)h\} dx ds. \tag{2.18}$$

By virtue of the continuity of the functional $b(v) = \int_S \int_G \lambda(p) grad v \; grad h \; dx ds$ we see that

$$\lim_{i \to \infty} \int_S \int_G [\lambda(p) grad p_i \; grad h] dx ds = \int_S \int_G \lambda(p)\lambda(p) grad p \; grad h \; dx ds. \tag{2.19}$$

Using (2.14) we can choose a subsequence $\{p_{ik}\}$ of the sequence $\{p_i\}$ such that

$$p_{ik} \xrightarrow{k} p(t, x) \quad \text{a.e. in} \quad S \times G.$$

From this and the continuity of the function λ it follows that

$$\lambda(p_{ik}(t, x)) \longrightarrow \lambda(p(t, x)) \quad \text{a.e. in} \quad S \times G.$$

In view of Lebesgue theorem we get

$$\lim_{k\to\infty}\Big(\int_S\int_G[(\lambda(p_{ik})-\lambda(p))gradh]^2dxds\Big)^{1/2}=0 \qquad (2.20)$$

Since the last equality holds for all subsequences converging to p in the space $L^2(S,H)$, from (2.20) we obtain

$$\lim_{i\to\infty}\Big(\int_S\int_G[(\lambda(p_i)-\lambda(p))gradh]^2dxds\Big)^{1/2}=0. \qquad (2.21)$$

Using (2.20) and (2.21) we have

$$\lim_{i\to\infty}\Big|\int_S\int_G[(\lambda(p_i)-\lambda(p))gradp\; gradh]^2dxds\Big|=0. \qquad (2.22)$$

Then by virtue of the condition (IV) we get

$$\lim_{i\to\infty}\int_S\int_G\mu(p_i)\vec{\theta}gradh\; dxds=\int_S\int_G\mu(p_i)\vec{\theta}gradh\; dxds. \qquad (2.23)$$

Analogously we can show that

$$\lim_{i\to\infty}\int_S\int_G f(p_i)h\; dxds=\int_S\int_G f(p)h\; dxds. \qquad (2.24)$$

From (2.18)–(2.24) we see that

$$\lim_{i\to\infty}<Ap_i,h>=\int_S\int_G[\lambda(p)gradp\,gradh+\mu(p)\vec{\theta}gradh-f(p)h]dxds$$
$$=<Ap,h>.$$

This completes the proof.

Theorem 2.3. *Assume the hypotheses of Theorem 2.2. In addition assume that there is a number $\beta_1>N$ such that for the solution p of the problem (2.5) the following inequality satisfies*

$$\int_S\Big[\int_G|gradp|^{\beta_1}dx\Big]^{\frac{2}{\beta_1-N}}ds<+\infty. \qquad (2.25)$$

Then the problem (2.5) has a unique solution.

P r o o f. Suppose that p_1 and p_2 are two solutions of the problems (2.5) and p_1 satisfies the condition (2.25). Then we have

$$0 = \frac{1}{2}|p_2(t) - p_1(t)|^2 + \int_0^t (Ap_2 - Ap_1, p_2 - p_1)ds, \tag{2.26}$$

where

$$\int_0^t (Ap_2 - Ap_1, p_2 - p_1)ds = \int_0^t \int_G \{[\lambda(p_2) grad p_2 - \lambda(p_1) grad p_1] grad(p_2 - p_1)$$

$$+ (\mu(p_2) - \mu(p_1))\vec{\theta} grad(p_2 - p_1) + (f(p_2) - f(p_1))(p_2 - p_1)\} dxds \tag{2.26'}$$

We shall estimate each member in the right part of (2.26'). With $\alpha_1 = \frac{2\beta_1}{\beta_1 - 2}$ and $\delta_1 > 0$ the following estimation holds

$$\int_0^t \int_G [\lambda(p_2) - \lambda(p_1)] grad(p_2 - p_1) dxds =$$

$$= \int_0^t \int_G |\lambda(p_2) grad^2(p_2 - p_1) + (\lambda(p_2) - \lambda(p_1)) grad p_1 grad(p_2 - p_1)| dxds$$

$$\geq C \int_0^t \int_G |grad(p_2 - p_1)|^2 dxds - C_1\delta_1 \int_0^t \int_G |grad(p_2 - p_1)|^2 dxds -$$

$$- \frac{C_1}{\delta_1} \int_0^t \{(\int_G |p_2 - p_1|^{\alpha_1} dx)^{2/\alpha_1} (\int_G |grad p_1|^{\beta_1} dx)^{2/\beta_1} ds. \tag{2.27}$$

Otherwise for $\delta_2 > 0$ we see that

$$\int_0^t \int_G (\mu(p_1) - \mu(p_2))\vec{\theta} grad(p_2 - p_1) dxds \geq -C \int_0^t |p_2 - p_1| \, |grad(p_2 - p_1)| dxds$$

$$\geq -C\delta_2 \int_0^t \int_G |grad(p_2 - p_1)|^2 dxds - \frac{C}{\delta_2} \int_0^t \int_G |p_2 - p_1|^2 dxds. \tag{2.28}$$

We have also

$$\int_0^t \int_G (f(p_2) - f(p_1))(p_2 - p_1)dxds \le C \int_0^t \int_G |p_2 - p_1|^2 dxds. \tag{2.29}$$

By virtue of (2.26)–(2.29) we obtain

$$\int_0^t (Ap_2 - Ap_1, p_2 - p_1)ds \ge (C - C_1\delta_1 - C\delta_2) \int_0^t ||p_2 - p_1||^2 ds -$$

$$-(C + \frac{C}{\delta_2} + C - C_1\delta_1 - C_2\delta_2) \int_0^t |p_2 - p_1|^2 ds$$

$$- \frac{C_1}{\delta} \int_0^t \{ (\int_G |p_2 - p_1|^{\alpha_1} dx)^{2/\alpha_1} (\int_G |gradp_1|^{\beta_1} dx)^{2/\beta_1} \} ds. \tag{2.30}$$

Using the imbedding theorem (see [1]) for $u \in H^1(G)$ we get

$$||u||_{L^p(G)} \le C||u||^{\alpha}_{H^1(G)} ||u||^{1-\alpha}_{L^2(G)}, \; 0 < \alpha < 1,$$

where $p \le \frac{2N}{N-\alpha}$ if $N > 2\alpha$ and p is arbitrary if $N \le 2\alpha$. Therefore, by choosing $\alpha = \frac{N}{\beta_1}$ we see that

$$||p||^2_{L^{\alpha_1}(G)} \le C||p||^{N/\beta_1}_{H^1(G)} ||p||^{2(1-\frac{N}{\beta_1})}_{L^2(G)}. \tag{2.31}$$

From (2.31) with $\alpha_2 = \frac{\beta_1}{N}$, $\beta_2 = \frac{\beta_1}{\beta_1 - N}$ and $\delta_2 > 0$ it follows that

$$\int_0^t \{ (\int_G |p_2 - p_1|^{\alpha_1} dx)^{2/\alpha_1} (\int_G |gradp_1|^{\beta_1} dx)^{2/\beta_1} \}$$

$$\le C_3\delta_3 \int_0^t ||p_2 - p_1|| ds + \frac{C_3}{\delta_3} \int_0^t |p_2 - p_1|^2 (\int_G |gradp_1|^{\beta_1} dx)^{\frac{2}{\beta_1 - N}} ds. \tag{2.32}$$

Put

$$k(s) = (\int_G |gradp_1(x,s)|^{\beta_1} dx)^{\frac{2}{\beta_1 - N}}.$$

In view of (2.25) we have $k(s) \in L^1(S)$. Using (2.30) and (2.32) we get

$$\int_0^t (Ap_2 - Ap_1, p_2 - p_1)ds \geq (C - C_1\delta_1 - C_2\delta_2 - \frac{C_1}{\delta_1}C_3\delta_3) \int_0^t |p_2 - p_1|^2 ds$$

$$- (2C + \frac{C}{\delta_2} - C_1\delta_1 - C_2\delta_2 + \frac{C_1C_3}{\delta_1\delta_3}) \int_0^t (1 + k(s))|p_2 - p_1|^2 ds. \tag{2.33}$$

By choosing δ_1, δ_2 and δ_3 such that $C - C_1\delta_1 - C_2\delta_2 - \frac{C_1}{\delta_1}C_3\delta_3 > 0$ from (2.26) and (2.33) we see that $|p_2(t) - p_1(t)|^2 \leq C \int_0^t (1 + k(s))|p_2(s) - p_1(s)|^2 ds$. By virtue of the last inequalilty and the Gronwal lemma we obtain $p_1(t) = p_2(t)$ as was to be shown.

Theorem 2.4. *Under the assumptions of Theorem 2.3 the sequence $\{p_n\}$ of solutions of Galerkin equation (2.8) strongly converges to the solution p of the problem (2.5) in the spaces $C(S, H)$ and $L^2(S, V)$.*

P r o o f. According to the results in [5] (see Lemma 1.5, page 209) there is a sequence $\{w_n\}$, $w_n \in C^1(S, V_n)$ such that $w_n \to p$ in w. From (2.5) and (2.8) we have

$$0 = \int_0^t (p_n'(s) - p'(s) + Ap_n(s) - Ap(s), p_n(s) - w_n(s))ds$$

$$\geq \frac{1}{2}|p_n(t) - p(t)|^2 - \frac{1}{2}|p_{an} - p_a|^2 + \int_0^t (Ap_n - Ap, p_n - p)ds$$

$$- 2||p_n - p||_{C(S,H)}||p - w_n||_{C(S,H)} - ||p_n - p||_{L^2(S,V)}||p' - w_n'||_{L^2(S,V^*)}$$
$$- ||Ap_n - Ap||_{L^2(S,V^*)}||p - w_n||_{L^2(S,V)}. \tag{2.34}$$

Since W is continuously imbedded into $C(S, H)$ and the sequences $\{p_n\}$ and $\{p_n\}$ are bounded in $C(S, H)$ and $L^2(S, V^*)$, respectively, from (2.34) we obtain

$$0 \geq \frac{1}{2}|p_n(t) - p(t)|^2 - \frac{1}{2}|p_{an} - p_a|^2 + \int_0^t (Ap_n - Ap, p_n - p)ds$$
$$- C||p - w_n||_W. \tag{2.35}$$

Using (2.33) and (2.35) and the Gronwall lemma we see that

$$\lim_{n\to\infty} ||p_n - p||_{C(S,H)} = 0, \qquad \lim_{n\to\infty} ||p_n - p||_{L^2(S,V)} = 0.$$

Theorem is proved.

3. ITERATIVE METHOD

This section is devoted to an iterative method for obtaining an approximate solution of the problem (2.5). Assume p_0 to be a given element of the space $L^2(S,V) \cap C(S,H)$. We shall construct the following iterative sequence $\{p_i\}$.

$$p_i' + B_i p_i = 0,\; p_i(o) = p_a \in H,\; p_i \in W,\; i = 1,2,3,..., \tag{3.1}$$

where the operator B_i is defined by the formula

$$(B_i p, h) = \int_G \{\lambda(p_{i-1} gradp\; gradh + m(p - p_{i-1})h +$$

$$+ \mu(p_{i-1})\vec{\theta} gradh - f(p_{i-1})h\}dx. \tag{3.2}$$

Theorem 3.1. *Suppose that all conditions in Theorem 2.4 are satisfied. Then the sequence $\{p_i\}$ defined by the iterative method (3.1), (3.2) strongly converges to the solution p of problem (2.5) in the spaces $C(S,H)$ and $L^2(S,V)$.*

P r o o f. It can easily be seen that with given element $p_{i-1} \in W$ the operator $B_i : L^2(S,V) \to L^2(S,V^*)$ is strongly monotone and Lipschitz continuous. Hence problem (3.1) has a unique solution $p_i \in W$. We consider the space $X = L^2(S,V) \cap C(S,V)$ with the norm

$$||x||^2_{X,k} = ||x||^2_{C,k} + a_0 ||x||^2_{L^2(S,V),k}, \quad x \in X,$$

where

$$||x||^2_{C,k} = \sup_{t\in S}(e^{-k(t)}|x(t)|^2),\; ||x||^2_{L^2(S,V),k} = \sup_{t\in S}(e^{-k(t)} \int_0^t ||z(s)||^2 ds),$$

$$k(t) = \eta \int_0^t \{\delta_2 + (\int_G |gradp|^{\beta_1} dx)^{\frac{2}{\beta_1 - N}}\}ds, \tag{3.3}$$

here η, δ_2 and a_0 are constants which will be choosen in the future. From (2.5) and (2.1) we see that

$$0 = \int_0^t (p_i'(s) - p'(s) + B_i(p_i(s) - Ap(s), p_i(s) - p(s))ds$$

$$= \frac{1}{2}|p_i(t) - p(t)|^2 + \int_0^2 \{\int_G [\lambda(p_{i-1}) gradp_i grad(p_i - p)$$

$$- \lambda(p) gradp\; grad(p_i - p) + (m(p_i - p) + m(p - p_{i-1}))(p_i - p) + (\mu(p_{i-1})$$

$$- \mu(p)\vec{\theta} grad(p_i - p) - (f(p_{i-1}) - f(p))(p_i - p)dx\}ds. \tag{3.4}$$

We shall estimate each member in the right part of (3.4). By an argument similar to that as in the proof of Theorem 2.3 we get

$$\int_0^t \int_G \{[\lambda(p_{i-1}) grad p_i - \lambda(p) grad p] grad(p_i - p) + m(p_i - p)^2\} dx ds \geq$$

$$\geq (m - C\delta_1) \int_0^t ||p_i - p||^2 ds - \frac{C\delta_2}{\delta_1} \int_0^t ||p_{i-1} - p||^2 ds -$$

$$- \frac{C}{\delta_1 \delta_2} \int_0^t |p_- p|^2 (\int_G |grad p|^{\beta_1} dx^{\frac{2}{\beta_1 - N}} ds. \tag{3.5}$$

Otherwise we have

$$\int_0^t \int_G m(p_i - p)(p - p_{i-1}) dx ds$$

$$\leq C\delta_1 \int_0^t ||p_i - p||^2 ds + \frac{C}{\delta_1} \int_0^t |p_{i-1} - p|^2 ds, \tag{3.6}$$

$$\int_0^t \int_G (\mu(p_{i-1}) - \mu(p) grad p)(p_i - p) dx ds$$

$$\leq C\delta_1 \int_0^t ||p_i(s) - p(s)||^2 + \frac{C}{\delta_1} \int_0^t |p_{i-1} - p|^2 ds. \tag{3.7}$$

$$\int_0^t \int_G (f(p_i - 1) - f(p))(p_i - p) dx ds$$

$$\leq C\delta_1 \int_0^t ||p_i(s) - p(s)||^2 + \frac{C}{\delta_1} \int_0^t |p_{i-1} - p|^2 ds. \tag{3.8}$$

From (3.4)–(3.8) it follows that

$$0 \geq \frac{1}{2}|p_i(t) - p(t)|^2 + (m - 4C\delta_1)\int_0^t ||p_i(s) - p(s)||^2 ds$$

$$- \frac{C\delta_2}{\delta_1}\int_0^t ||p_{i-1}(s) - p(s)||^2 ds - \frac{C}{\delta_1\delta_2}\int_0^t |p_{i-1}(s) - p(s)|^2$$

$$(\int_G |gradp|^{\beta_1} dx)^{\frac{2}{\beta_1 - N}} ds - \frac{3C}{\delta_1}\int_0^t |p_{i-1}(s) - p(s)|^2 ds.$$

Taking $C_1 = 4C$ in the best inequality we obtain

$$|p_i(t) - p(t)|^2 + 2(m - C_1\delta_1)\int_0^t ||p_i(s) - p(s)||^2 ds$$

$$\leq \frac{C_1\delta_2}{4\delta_1}\int_0^t ||p_{i-1}(s) - p(s)||^2 ds + \frac{3C_1}{4\delta_1\delta_2}\int_0^t k'(s)|p_{i-1}(s) - p(s)|^2 ds. \tag{3.9}$$

Taking $a_0 = 2(m - C_1\delta_1)$, $a_1 = \frac{C_1\delta_2}{4\delta_1}$, $a_2 = \frac{3C_1}{4\delta_1\delta_2\eta}$ and multiplying both parts of inequality (3.9) with $e^{-k(t)}$ we have

$$||p_i - p||^2_{X,k} \leq 2a_2(||p_i - p||^2_{C,k} + \frac{a_1}{a_2}||p_{i-1} - p||^2_{L^2(S,V),k}). \tag{3.10}$$

If we choose $\delta_1 < \frac{2m}{C}$, $\delta_2 < \frac{a_0\delta_1}{8C_1}$ and $\eta > \frac{8C_2}{\delta_1\delta_2}$ then we see that

$$a_0 > 0,\ C_2 < \frac{1}{8},\ \frac{a_1}{a_2} < a_0. \tag{3.11}$$

From (3.10) and ((3.11) it follows that

$$||p_i - p||^2_{X,k} \leq \frac{1}{2}||p_{i-1} - p||^2_{X,k} \leq \cdots \leq \frac{1}{2^i}||p_0 - p||^2_{X,k}.$$

Therefore we obtain

$$\lim_{i\to\infty} ||p_i - p||_{L^2(S,V)} = 0,\ \lim_{i\to\infty} ||p_i - p||_{C(S,H)} = 0.$$

The proof is complete.

Note that in the above iterative method instead of the nonlinear problem (2.5) we resolve the sequence of linear problem (3.1), (3.2).

REFERENCES

1. Pukhachev G.B., Isaev R.G.: Underground Hydraulics (In Russian), Nedra, Moscow, 1973.
2. P.Ia. Polubarinova Kochina: Developments of Researches on Filtration Theory. Nauka, Moscow, 1969.
3. Antosev S.N., Kajikhov A.V., Monakhov V.N.: Boundary Value Problems of Nonhomegeneous Fluid Mechanics (In Russian). Nauka, Novosibirsk, 1983.
4. Lions J.L.: Quelque méthodes de resolution des problemès aux limites non lineaires. Dunod, Paris, 1969.
5. Gajewski H., Gröger K., Zacharias K.: Nonlinear Operator Equations and Operator Differential Equations. Akademik Verlag, Berlin, 1974.
6. Gröger K.: Ein Kompaktheitssatz für Folgen von Näherungslösungen von Evolutionsgleichungen. Math. Nach., Band 72, 1976, 15–19.

Mine-Geology University of Hanoi
Tuliem, Hanoi, Vietnam.

Received March 10, 1991

Institute of Mathematics
P.O. Box 631 Boho
10000 Hanoi, Vietnam