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ON THE QUALITATIVE PROPERTIES OF THE SOLUTION SET TO FUNCTIONAL DIFFERENTIAL INCLUSIONS IN BANACH SPACES

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Dedicated to the memory of Professor Le Van Thiem

Abstract. *For a class of upper semicontinuous functional differential inclusions in Banach spaces with weakly compact convex right-hand side containing parameters, the set of solutions is proved to be a nonempty weakly compact set which depends upper semicontinuously on the initial function and parameters. As an application of the obtained results, an optimal control problem for a system of functional differential equations is studied. The existence of optimal solutions and the continuity of the Bellman marginal function are provided.*

1. INTRODUCTION

In this paper we are concerned with some qualitative properties of the set of Carathéodory solutions for a functional differential inclusions (FDI) depending on parameters of the form

$$\dot{x}(t) \in G(t, x_t, \xi), \quad t \in [0, T],$$
$$x(t) = \varphi(t), \quad t \in [-h, 0].$$

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We shall show that under suitable continuity and convexity assumptions on the multifunction G, the set of all solutions of the above functional differential inclusion is a upper semicontinuous multifunction of the initial value φ and the parameter ξ, with nonempty compact values in $C_{E_\sigma}(-h,T)$. For ordinary differential inclusions, topological properties of the solution set has been extensively investigated in recent years, see e.g. [2,3,6,10]. However, still little is known in the case of functional differential inclusions, see e.g.[5], and the results have been obtained so far only for FDIs in finite dimensional spaces. In the present paper, we consider the case of general functional differental inclusions in Banach spaces, following the lines of research developed in [3]. The paper can also be considered as a continuation of [7]. The results mentioned above will be established in Sections 2 and 3. As an application of these results, we consider in Section 4 an optimal control problem for a dynamical system described by a functional differential equation, the existence of optimal solutions and the conuinuity of the corresponding Bellman function will be proved.

Throughout this paper, E denotes a separable Banach space with the norm $\|\cdot\|$ and with the strong dual E'. E_σ and E'_s stand for the spaces E and E' equipped with the weak topologies $\sigma(E,E')$ and $\sigma(E',E)$, respectively. The Banach space of all linear bounded operators from E into E is denoted by $L(E)$. For $a,b \in R$, $C_E(a,b)$ and $C_{E_\sigma}(a,b)$ denote the spaces of continuous functions from $[a,b]$ to E and E_σ, respectively, endowed with the topology of uniform convergence. It is clear that $C_E(a,b) \subset C_{E_\sigma}(a,b)$. The space $C_E(a,b)$ endowed with the induced topology of uniform convergence of $C_{E_\sigma}(a,b)$ will be denoted by $C_E^\sigma(a,b)$. The space of all Bochner integrable functions from $[a,b]$ to E is denoted by $L_E^1(a,b)$. It is well-known that $(L_E^1(a,b))' = L_{E'_s}^\infty(a,b)$, the last denoting the space of essentially bounded measurable functions from $[a,b]$ to E'_s. For the sake of brevity, we shall omit sometimes the interval of definition (a,b) in the above notations of functions spaces when no confusion can arise. Finally, let $\Gamma(t)$ be a measurable multifunction from $[0,T]$ to E, then we shall denote by S_Γ the set of all measurable selections of Γ. For the basic theory of measurable multifunctions, we refer to [3].

2. DEPENDENCE ON THE INITIAL VALUES

Let E be a separable Banach space. For given $T > 0$ and $h \geq 0$, let $F : [0,T] \times C_E(-h,0) \to E$ be a multifunction, satisfying the following assumptions:

(2.i) For each $(t,\varphi) \in [0,T] \times C_E(-h,0)$, $F(t,\varphi)$ is a nonempty convex $\sigma(E,E')$ compact subset of E,

(2.ii) For every $\varphi \in C_E(-h,0)$, $F(.,\varphi)$ is a measurable multifunction on $[0,T]$;

(2.iii) For every $t \in [0,T]$, $F(t,.)$ is a upper semicontinuous (u.s.c.) function from $C_E^\sigma(-h,0)$ into E_σ, (2.iv) Linear growth condition : There exists a balanced convex $\sigma(E, E')$ compact set K and a positive integrable function $\alpha(.) : [0,T] \to R$ such that, for every $\varphi \in C_E(-h,0)$, $F(t,\varphi) \subset \alpha(t)(1 + \|\varphi\|)K$.

Let $A : [0,T] \to L(E)$ be a Bochner integrable function, i.e.

$$\int_0^T \|A(t)\| dt < \infty.$$

We consider the functional differential inclusion of the form

$$\dot{x}(t) \in A(t)x(t) + F(t, x_t), \; t \in [0,T], \tag{2.1}$$

with the initial condition

$$x(t) = \varphi(t), \quad t \in [-h,0], \tag{2.2}$$

where φ is a given element in $C_E(-h,0)$ and, by definition, $x_t(\theta) = x(t+\theta)$ for $\theta \in [-h,0]$.

A function $x(.) \in C_E(-h,T)$ satisfying (2.2) is said to be a solution of FDI (2.1)-(2.2) if $x(.)$ is absolutely continuous on $[0,T]$ and verifies (2.1) for almost $t \in [0,T]$.

Let $M \subset C_E(-h,0)$ be a given compact subset such that M is compact in $C_{E_\sigma}(-h,0)$.

Theorem 2.1. *Under the above hypotheses, for every $\varphi \in M$, the set $N(\varphi)$ of all solutions of (2.1)-(2.2) is a nonempty subset of $C_E(-h,T)$. Moreover, the multifunction $N : M \to C_{E_\sigma}(-h,T)$ is u.s.c.*

P r o o f. The first part of the proof is similar to the one of Theorem 3.1 in [7], but slightly more complicated because of the presence of the linear term $A(t)x$ in the right-hand side of (2.1). We provide it here in details, for completeness. Let us denote $a = \sup\{\|\varphi(t)\|, t \in [-h,0], \varphi \in M\}$, $c_1 = \sup\{\|x\|, x \in K\}$ and $c = \max\{c_1, 1\}$. Since K is weakly compact, we have $c < \infty$. Suppose that $x(.)$ is a sulution of the FDI (2.1)-(2.2), then it is obvious that, for $t \in [0,T]$,

$$x(t) \in \varphi(0) + \int_0^t [A(s)x(s) + F(s, x_s)] ds.$$

Therefore, by (2.iv),

$$\|x(t)\| \leq a + \int_0^t [\|A(s)\| \|x(s)\| + c\alpha(s)(1 + \|x_s\|)] ds.$$

It follows that

$$\|x_t\| \leq a + c\int_0^t (\|A(s)\| + \alpha(s))(1 + \|x_s\|)ds,$$

which, by Gronwall's lemma, implies that

$$\|x_t\| \leq z(t) := (a+1)\exp\left(c\int_0^t [\|A(s)\| + \alpha(s)]ds\right) - 1, \tag{2.3}$$

for $t \in [0, T]$.

Let us introduce the following multifunction from $[0, T]$ into E

$$\Gamma(t) = \alpha(t)(1 + z(t))K.$$

Then, it is well-known that the set S_Γ of all measurable selections of Γ is compact in the weak topology $\sigma(L^1_E, L^\infty_{E'_s})$. For each $f \in S_\Gamma$ and $\varphi \in C_E(-h, 0)$ let us define the function

$$x^{f,\varphi}(t) = \begin{cases} \varphi(t), \text{ for } t \in [-h, 0], \\ \Phi(t)\varphi(0) + \Phi(t)\int_0^t \Phi^{-1}(s)f(s)ds, \text{ for } t \in [0, T], \end{cases}$$

where the operator function $\Phi : [0, T] \to L(E)$ is the fundamental solution operator of the linear equation $\dot{x}(t) = A(t)x(t)$, i.e. $\Phi(t)$ satisfies the equation $\dot{\Phi}(t) = A(t)\Phi(t)$ and $\Phi(0) = I$ (the identity operator in E). In what follows, we omit the superscript φ in the notation $x^{f,\varphi}(t)$, for the sake of simplicity. It is obvious that, for $t \in [0, T]$,

$$x^f(0) = \varphi(0) \text{ and } \dot{x}^f(t) = A(t)x^f(t) + f(t). \tag{2.4}$$

We shall show that

$$\|x^f(t)\| \leq z(t), \quad t \in [0, T]. \tag{2.5}$$

Indeed, $\|x^f(0)\| = \|\varphi(0)\| \leq a = z(0)$. Assume the contrary, that there exists t_1, t_2 such that $0 \leq t_1 < t_2 \leq T$ and

$$\|x^f(t)\| \leq z(t) \text{ for } t \in [0, t_1] \quad \text{and} \quad \|x^f(t)\| > z(t) \text{ for } t \in (t_1, t_2]. \tag{2.6}$$

Since $z(.)$ is a nondecreasing function, this implies that $\|x^f_t\| = \sup\{\|x^f(t+\theta)\|, \theta \in [-h, 0]\} \leq z(t)$ for $t \in [0, t_1]$ and

$$\|x_t^f\| \geq \|x^f(t)\| > z(t), \tag{2.7}$$

for $t \in (t_1, t_2]$. Consequently, by (2.6) and (2.7), the following inequality holds for $t \in (t_1, t_2]$

$$\|x^f(t)\| \leq a + c\int_0^{t_1} (\|A(s)\| + \alpha(s))(1 + z(s))ds +$$

$$+ c\int_{t_1}^{t} (\|A(s)\| + \alpha(s))(1 + \|x_s\|)ds =$$

$$= z(t_1) + c\int_{t_1}^{t} (\|A(s)\| + \alpha(s))(1 + \|x_s\|)ds.$$

By the same reasoning as above, one can replace $\|x^f(t)\|$ by $\|x_t^f\|$ in the left-hand side of this inequality. Then, by the Gronwall's lemma, it follows that

$$\|x_t^f\| \leq (z(t_1) + 1)\exp(c\int_{t_1}^{t} (\|A(s)\| + \alpha(s))ds - 1 =$$

$$= (a + 1)\exp(c\int_0^{t_1} (\|A(s)\| + \alpha(s))ds +$$

$$+ c\int_{t_1}^{t} (\|A(s)\| + \alpha(s))ds) - 1 =$$

$$= (a + 1)\exp\,(c\int_0^{t} (\|A(s)\| + \alpha(s))ds - 1 = z(t),$$

for $t \in (t_1, t_2]$, conflicting with (2.7). Thus (2.5) is proved.

Now, let us introduce a multifunction Ψ by setting, for each $f \in S_\Gamma$,

$\Psi(f) = \{g : [0, T] \to E \ : \ g$ is measurable and $g(t) \in F(t, x_t^f)$ a.e $\}$.

It is obvious that $\Psi(f)$ is a nonempty and convex set. Moreover, for every $g \in \Psi(f)$ and a.e. $t \in [0, T]$, by (2.iv) and (2.5),

$$g(t) \in F(t, x_t^f) \subset \alpha(t)(1 + \|x_t^f\|)K \subset \alpha(t)(1 + z(t))K = \Gamma(t).$$

Thus, $\Psi(f)$ is a multifunction with nonempty convex values from the $\sigma(L^1_E, L^\infty_{E'_\sigma})$ compact set S_Γ into itself. Preceeding in exactly the same way as in the proof of Theorem 3.1 [7], we obtain that Ψ admits a fixed point $f_0 \in S_\Gamma$, i.e. $f_0(t) \in F(t, x^{f_0}_t)$ a.e. $t \in [0,T]$, and hence, in view of (2.4), $x^{f_0}(.)$ is a solution of FDI (2.1)-(2.2). Thus, we have shown that, for each $\varphi \in M$ the set $N(\varphi)$ of solutions of (2.1)-(2.2) is nonempty subset of the set X defined by

$$X = \{x^{f,\varphi}(.) : f \in S_\Gamma, \varphi \in M\}.$$

We claim that X is compact when regarded as a subset of $C_{E_\sigma}(-h,T)$. To this end, by virtue of the Ascoli's theorem, we have to prove that the set $X(t) := \{x(t) : x \in X\}$ is compact in E_σ for $t \in [-h,T]$ and X is closed equicontinuous subset of $C_{E_\sigma}(-h,T)$. The last is obvious because X is clearly equicontinuous in $C_E(-h,T)$. Further, since M is compact in $C_{E_\sigma}(-h,0)$, it follows that for every $t \in [-h,0]$, $M(t) := \{\varphi(t) : \varphi \in M\}$ is compact in E_σ. On the other hand, since S_Γ is weakly compact in L^1_E and the linear maps $x \to \Phi(t)x$ and $f \to \int_0^t \Phi^{-1}(s)f(s)ds$ are weakly continuous for each fixed $t \in [0,T]$, it follows that the set

$$X(t) = \Phi(t)M + \Phi(t)\{\int_0^t \Phi^{-1}(s)f(s)ds : f \in S_\Gamma\}$$

is weakly compact for each $t \in [0,T]$. Finally, the closeness of X can be proved in the same way as in Theorem 3.1 of [7]. Therefore, according to the Berge's theorem, to complete the proof of the theorem, it suffices to show that the graph of the multifunction $\varphi \to N(\varphi)$ is closed in $M \times X$. For this, let (φ^n, x^n) be a sequence in the graph of N which converges to $(\varphi, x) \in M \times X$. Then, clearly, for each $t \in [0,T], x^n_t$ converges to x_t in $C_{E_\sigma}(-h,0)$, and for each $t \in [-h,0], \varphi^n(t)$ converges to $\varphi(t)$ in E_σ. On the other hand, by definition, we have

$$x^n(t) = \Phi(t)\varphi^n(0) + \Phi(t)\int_0^t \Phi^{-1}(s)f_n(s)ds, \qquad t \in [0,T]$$

with $f_n(s) \in F(s, x^n_s)$ for almost $s \in [0,T]$, and

$$x(t) = \Phi(t)\varphi(0) + \Phi(t)\int_0^t \Phi^{-1}(s)f(s)ds, \qquad t \in [0,T] \tag{2.8}$$

with $f \in S_\Gamma$. Since S_Γ is weakly compact in L^1_E, a subsequence of $\{f_n\}$ (kept with the same indices) converges to some $g \in S_\Gamma$ in the weak topology $\sigma(L^1_E, L^\infty_{E'_s})$, that means

$$\lim_{n\to\infty} \int_0^t < h(s), f_n(s) > ds = \int_0^t < h(s), g(s) > ds \tag{2.9}$$

for each $h(.) \in L^\infty_{E'_s}$ and each fixed $t \in [0,T]$. In particular, by taking $h(.) = \gamma(.)e'$ with $e' \in E'$ and $\gamma \in L^\infty_R(0,T)$, (2.9) yields

$$< e', f_n > \to < e', g >, \quad \text{as} \quad n \to \infty$$

for the weak topology $\sigma(L^1_R, L^\infty_R)$. Therefore, by the Convergence Theorem (see Theorem VI.6 in [3]), it follows that

$$g(s) \in F(s, x_s), \quad \text{for almost} \quad s \in [0,T]. \tag{2.10}$$

Furthermore, since $\varphi^n(0) \to \varphi(0)$, in E_σ it follows that, for each $e' \in E'$ and $t \in [0,T]$,

$$\lim_{n\to\infty} < e', \Phi(t)\varphi^n(0) > = < e', \Phi(t)\varphi(0) > .$$

Consequently, the above representation of x^n and x implies that

$$\lim_{n\to\infty} \int_0^t < \Phi^{*-1}(s)\Phi^*(t)e', f_n(s) > ds = \int_0^t < \Phi^{*-1}(s)\Phi^*(t)e', f(s) > ds.$$

In combining with (2.9), this gives

$$\Phi(t)\int_0^t \Phi^{-1}(s)f(s)ds = \Phi(t)\int_0^t \Phi^{-1}(s)g(s)ds,$$

which implies, in view of (2.8) and (2.10), that $x \in N(\varphi)$. Thus the graph of N is closed, completing the proof of the theorem.

Corollary 2.2. *For every $\varphi \in M$, the set $N(\varphi)$ of solutions of the FDI (2.1)-(2.2) is compact in $C_{E_\sigma}(-h,T)$. Moreover, the set $N = \bigcup_{\varphi\in M} N(\varphi)$ is compact in $C_{E_\sigma}(-h,T)$.*

3. DEPENDENCE ON PARAMETERS

Let Q be a compact metric space endowed with the metric ρ and E be a separable Banach space. Let $T > 0, h \geq 0$ and $\varphi \in C_E(-h, 0)$ be given. In this section we shall be concerned with the FDI of the form

$$\dot{x}(t) \in A(t, \xi) + F(t, x_t, \xi), \quad t \in [0, T], \tag{3.1}$$

with the initial condition

$$x(t) = \varphi(t), \quad t \in [-h, 0], \tag{3.2}$$

where the right-hand side of (3.1) satisfies the following assumptions:

(3.i) For every $(t, \varphi, \xi) \in [0, T] \times C_E(-h, 0) \times Q$, $F(t, \varphi, \xi)$ is a nonempty convex $\sigma(E, E')$ compact subset of E,

(3.ii) For every $(\varphi, \xi) \in C_E(-h, 0) \times Q$, $F(., \varphi, \xi)$ is a measurable multifunction on $[0, T]$;

(3.iii) For every $t \in [0, T]$, $F(t, ., .)$ is a u.s.c. multifunction from $C_E^{\sigma}(-h, 0) \times Q$ into E_{σ},

(3.iv) Linear growth condition : For all $(t, \varphi, \xi) \in [0, T] \times C_E(-h, 0) \times Q$. $F(t, \varphi, \xi) \subset \alpha(t)(1 + \|\varphi\|)K$, where $\alpha(.)$ and K are as in the condition (2.iv) in the Section 2;

(3.v) For every $(t, \xi) \in [0, T] \times Q$, $A(t, \xi) \in L(E)$ and moreover there exists a positive integrable function γ such that $\|A(t, \xi_1) - A(t, \xi_2)\| \leq \gamma(t)\rho(\xi_1, \xi_2)$, for $\xi_1, \xi_2 \in Q$ and $t \in [0, T]$.

Theorem 3.1. *Under the above hypotheses, for every $\xi \in Q$, the set $N(\xi)$ of all solutions of (3.1)-(3.2) is a nonempty subset of $C_E(-h, T)$. Moreover, the multifunction $N : Q \to C_{E_\sigma}(-h, T)$ is u.s.c.*

P r o o f. Let, for every $\xi \in Q, \Phi_\xi(t)$ be the fundamental solution operator of the linear equation $\dot{x}(t) = A(t, \xi)x(t)$. It is well-known that $\Phi_\xi(t)$ can be defined as

$$\Phi_\xi(t) = I + \int_0^t A(s, \xi)ds + \int_0^t A(s, \xi) \int_0^s A(s_1, \xi)dsds_1 + \cdots \tag{3.3}$$

Since, by (3.v) and the compactness of Q, $\|A(t, \xi)\| \leq q\gamma(t)$ for some $q < \infty$, an easy computation shows that

$$\|\Phi_\xi(t)\| \leq \exp(pt), \quad t \in [0, T] \tag{3.4}$$

where $p = q\int_0^T \gamma(s)ds$. Moreover, by (3.v) and (3.3), it is easily seen that $\Phi_\xi(t)$ is strongly uniformly continuous in ξ. Now, as in the proof of Theorem 3.1, let us denote

$$a = \sup\{\|\varphi(t)\|, \quad t \in [-h,0]\},$$
$$c_1 = \sup\{\|x\|, \quad x \in K\}, \; c = \max\{c_1, 1\},$$
$$z(t) = (a+1)\exp(c\int_0^t [q\gamma(s) + \alpha(s)]ds) - 1, \quad t \in [0,T],$$

and

$$\Gamma(t) = \alpha(t)(1 + z(t))K, \quad t \in [0,T].$$

For each $f \in S_\Gamma$ and $\xi \in Q$, let us define

$$x^{f,\xi}(t) = \begin{cases} \varphi(t), & t \in [-h,0], \\ \Phi_\xi(t)\varphi(0) + \Phi_\xi(t)\int_0^t \Phi_\xi^{-1}(s)f(s)ds, & t \in [0,T], \end{cases} \tag{3.5}$$

and we put

$$H = \{x^{f,\xi}(.) : f \in S_\Gamma, \xi \in Q\}.$$

Then, by Theorem 2.1, for each fixed $\xi \in Q$, the set $N(\xi)$ of all solutions of (3.1)-(3.2) is a nonempty subset of H. Further, arguing as in the proof of the Theorem 2.1, we can show that H is compact when regarded as a subset of $C_{E_\sigma}(-h,T)$. Thus, again by virtue of Berge's theorem, the upper semicontinuity of the multivalued function $\xi \to N(\xi)$ will be proved by showing that the graph of N is closed in the compact space $Q \times H$. To this end, let (ξ_n, x^n) be a sequence in the graph of N converging to $(\xi, x) \in Q \times H$. It follows that, for every $t \in [0,T]$, $x_t^n \to x_t$ in the topology of $C_{E_\sigma}(-h,T)$ as $n \to \infty$. On the other hand, let f_n and f be the corresponding measurable selections of Γ for which the integral representations (3.5) of x^n and x hold, repectively. Then, it is obvious that for each fixed $t \in [0,T]$ $\Phi_{\xi_n}(t)\varphi(0) \to \Phi_\xi(t)\varphi(0)$ in E_σ as $n \to \infty$. This implies

$$\lim_{n\to\infty} \Phi_{\xi_n}(t)\int_0^t \Phi_{\xi_n}^{-1} f_n(s)ds = \Phi_\xi(t)\int_0^t \Phi_\xi^{-1}(s)f(s)ds \tag{3.6}$$

in E_σ. Without loss of generality, we can assume that f_n converges to $g \in S_\Gamma$ in the weak topology $\sigma(L^1_E, L^\infty_{E'_s})$. Since $x^n \in N(\xi_n)$, by definition we have $f_n(s) \in F(s, x^n_s, \xi_n)$. Therefore, we can apply the Covergence Theorem to assert that $g(s) \in F(s, x_s, \xi)$ a.e.on $[0,T]$. Now, since f_n tends to g in the weak topology of L^1_E, it follows that, for each $e' \in E'$ and $t \in [0,T]$

$$< e', \Phi_\xi(t)\Phi_\xi^{-1}(.)f_n(.) > \to < e', \Phi_\xi(t)\Phi_\xi^{-1}(.)g(.) > \tag{3.7}$$

as $n \to \infty$ in the weak topology $\sigma(L^1_R, L^\infty_R)$. We have for all t and n

$$|\int_0^t < e', \Phi_\xi(t)\Phi_\xi^{-1}(s)(f(s) - g(s)) > ds| \le$$

$$\le |\int_0^t < e', \Phi_\xi(t)\Phi_\xi^{-1}(s)f(s) - \Phi_{\xi_n}(t)\Phi_{\xi_n}^{-1}(s)f_n(s) > ds| +$$

$$+ |\int_0^t < e', (\Phi_{\xi_n}(t)\Phi_{\xi_n}^{-1}(s) - \Phi_\xi(t)\Phi_\xi^{-1}(s))f_n(s) > ds| +$$

$$+ |\int_0^t < e', \Phi_\xi(t)\Phi_\xi^{-1}(s)(f_n(s) - g(s)) > ds|.$$

We observe that, in view of (3.6) and (3.7), the first term and the third term in the right-hand side of the above inequality tend to 0 as $n \to \infty$. Regarding the second term we can write

$$|\int_0^t < e', (\Phi_{\xi_n}(t)\Phi_{\xi_n}^{-1}(s) - \Phi_\xi(t)\Phi_\xi^{-1}(s))f_n(s) > ds| \le$$

$$\le \int_0^t \|e'\| \|| \Phi_{\xi_n}(t)\Phi_{\xi_n}^{-1}(s) - \Phi_\xi(t)\Phi_\xi^{-1}(s)\|\| \|(f_n(s)\| ds.$$

Therefore, since $\Phi_\xi(.)$ and $\Phi_\xi^{-1}(.)$ are uniformly continuous in ξ and f_n is uniformly integrable bounded, it follows that the second term also tends to 0 as $n \to \infty$. Hence, we obtain

$$\int_0^t \Phi_\xi(t)\Phi_\xi^{-1}(s)f(s)ds = \int_0^t \Phi_\xi(t)\Phi_\xi^{-1}(s)g(s)ds.$$

Thus, $x = x^{f,\xi} = x^{g,\xi}$ with $g(s) \in F(s, x_s, \xi)$ which means that $x \in N(\xi)$ and completes the proof of the theorem.

Corollary 3.2. *The set $N(\xi)$ of all solutions of the FDI (3.1)-(3.2) is compact in $C_{E_\sigma}(-h,T)$. Moreover, the set $N = \bigcup_{\xi \in Q} N(\xi)$ is compact in $C_{E_\sigma}(-h,T)$.*

4. APPLICATION TO AN OPTIMAL CONTROL PROBLEM

In this section we apply some of results obtained in the previous sections to an optimization problem involving the system of functional diffential equations.

Let E and U be separable Banach spaces, $T > 0$ and $h \geq 0$ be given and M be a convex subset of $C_E(-h,0)$ such that M is compact in the topology of uniform convergence of $C_{E_\sigma}(-h,0)$. Let $\Omega : [0,T] \to U$ be a measurable multifuntion with nonempty weakly compact values in U. Let $f : [0,T] \times C_E(-h,0) \times U \to E$ be a given function satisfying the following assumptions

(4.i) $f(.,\varphi,u)$ is a measurable function, for each $(\varphi,u) \in C_E(-h,0) \times U$;

(4.ii) $f(t,.,.) : C_E^\sigma(-h,0) \times U \to E_\sigma$ is a continuous function, for each t,

(4.iii) $f(t,\varphi,\Omega(t))$is a nonempty convex set, for each $(t,\varphi) \in [0,T] \times C_E$;

(4.iv) There exists a positive integrable function $\alpha(.)$ and a balanced convex weakly compact set B such that

$$f(t,\varphi,\Omega(t)) \subset (1 + \|\varphi\|)\alpha(t)B.$$

Let $\psi : C_E(-h,0) \times C_E(-h,T) \to R$ be a continuous convex function. We consider the following optimal control problem (P):

$$J(\varphi^0) := \min\{\psi(\varphi^0, x(.)) : x(.) \in N_f(\varphi^0)\},$$

where $N_f(\varphi^0)$ denotes the set of all admissible trajectories of the following control system (CS):

$$\dot{x}(t) = f(t, x_t, u(t)), \quad \text{a.e } t \in [0,T], \tag{4.1}$$

$$x(t) = \varphi^0(t), \quad t \in [-h,0], \tag{4.2}$$

$$u(t) \in \Omega(t), \quad \text{a.e } t \in [0,T]. \tag{4.3}$$

We recall that a continuous function $x(.) : [-h,T] \to E$ is said to be an admissible trajectory of the control system (CS) if there exists a measurable function $u(.) : [0,T] \to U$ satisfying (4.3) such that (4.1) and (4.2) hold.

Let us put $G(t,\varphi) = f(t,\varphi,\Omega(t))$ and denote by $N_G(\varphi^0)$ the set of all solutions of the following functional differential inclution

$$\dot{x}(t) \in G(t, x_t), \quad \text{a.e } t \in [0, T], \tag{4.4}$$
$$x(t) = \varphi^0(t), \quad t \in [-h, 0], \tag{4.5}$$

Then it is readily verified that the above FDI satisfies all the hypotheses of Theorem 2.1. Therefore, for each $\varphi^0 \in M$, $N_G(\varphi^0)$ is a nonempty subset of $C_E(-h,T)$ which is compact in the topology of $C_{E_\sigma}(-h,T)$, and moreover $N_G(.)$ is u.s.c. multifunction from M into $C_{E_\sigma}(-h,T)$

Lemma 4.1. $N_f(\varphi^0) = N_G(\varphi^0)$, $\forall \varphi^0 \in C_E(-h,0)$.

P r o o f. The inclusion $N_f(\varphi^0) \subset N_G(\varphi^0)$ being obvious, let us assume that $x(.)$ satisfies (4.4)-(4.5). Then, by putting $g := f$, $\Theta(t) := \dot{x}(t)$, $\Sigma(t) := \{x_t\} \times \Omega(t)$, we get $\theta(t) \in g(t, \Sigma(t))$ and we readily verify that all the hypotheses of the Fillipov-Castaing implicit functions theorem (see Theorem III. 38 in [3]) are satisfied. This implies that there exists a measurable selection $\sigma(t) \in \Sigma(t)$ such that $\Theta(t) = g(t, \sigma(t))$, or equivalently, $\dot{x}(t) = f(t, x_t, v(t))$ with some measurable function $v(t) \in \Omega(t)$. Thus $x(.) \in N_f(\varphi^0)$ and the lemma is proved.

Lemma 4.2. *Let $Z \subset C_E(a,b)$ be a convex compact subset. Then Z is compact when regarded as a subset of $C_{E_\sigma}(a,b)$.*

P r o o f. Since Z is compact in $C_E(a,b)$ it follows that $Z(t) := \{x(t) : x \in Z\}$ is a compact subset of E and Z is equicontinuous in C_E. Therefore, $Z(t)$ is weakly compact and Z is equicontinuous in C_{E_σ} and thus, by Ascoli's Theorem, Z is relatively compact in C_{E_σ}. It remains to show that Z is closed in C_{E_σ}. In fact, by Proposition 0.4.9 in [4], Z is closed in the pointwise convergence topology on $E^{[a,b]}$ and hence, closed in the product topology of $Y := \Pi\{E^t,\ t \in [a,b]\}$. Since Z is convex, this implies that Z is closed in the weak topology $\sigma(Y, Y')$. On the other hand, by Theorem 4.3 in [8, Chapter 4] $\sigma(Y,Y') = \Pi\{\sigma(E^t, E'^t),\ t \in [a,b]\}$ it follows that Z is closed in the product topology of pointwise convergence on $E_\sigma^{[a,b]}$. The equicontinuity of Z in C_{E_σ} implies, again in view of Proposition 0.4.9 in [4], that Z is closed in the topology of uniform convergence of C_{E_σ} as was to be shown.

From the proof of the above lemma, we obtain

Corollary 4.3. *If Z is a closed convex equicontinuous subset of C_E then Z is a closed subset of C_{E_σ}*

Now we state the main result of the Section.

Theorem 4.4. *Under the mentioned assumptions, the optimization problem (P) admits an optimal solution and, moreover, the Bellman function $J(\varphi^0)$ is lower semicontinuous function from M into R.*

P r o o f. Since, by Lemma 4.1, $N_f(\varphi^0) = N_G(\varphi^0)$ for each $\varphi^0 \in M$, it follows, by Theorem 2.1, that $N_f(\varphi^0)$ is a nonempty compact subset in $C_{E_\sigma}(-h,T)$.

We notice that, as was shown in the proof of Theorem 2.1, $N_f(\varphi^0)$ is a subset of the convex set

$$X := \{x^{g,\varphi} : \ g \in S_\Gamma, \varphi \in M\},$$

where, by definition

$$\Gamma(t) = (1 + z(t))\alpha(t)B,$$

$$z(t) = (a+1)\exp(c\int_0^t \alpha(s)ds) - 1,$$

$$a = \sup\{\|\varphi(\theta)\|, \varphi \in M, \ \theta \in [-h, 0]\},$$

$$c = \sup\{\|x\|, x \in B\},$$

and

$$x^{g,\varphi}(t) = \begin{cases} \varphi(t), \text{ for } \ t \in [-h, 0], \\ \varphi(0) + \int_0^t g(s)ds, \ \text{for} \ \ t \in [0, T]. \end{cases}$$

It follows that X is equicontinuous in $C_E(-h, T)$ and hence $\overline{X}$ (the closure of X in C_E) is also equicontinuous in C_E. Since the function $\psi(\varphi^0, .) : X \to R$ is convex and continuous, this implies that for each $\lambda \in R$ the set

$$Z(\lambda) = \{x(.) \in \overline{X} : \psi(\varphi^0, x(.)) \leq \lambda\}$$

is equicontinuous, closed, convex subset of C_E. Thus, by Corollary 4.3, $Z(\lambda)$ is closed in C_{E_σ}. This means that $\psi(\varphi^0, .)$ is u.s.c on $\overline{X}$ when regarded as a function from $C_{E_\sigma}(-h, T)$ into R. Hence, the set of admissible trajectories $N_f(\varphi^0)$ being a compact subset of C_{E_σ} must admit a minimizing point for $\psi(\varphi^0, .)$ which is an optimal solution of the problem (P). Now, since M is convex and compact in $C_E(-h, 0)$, the same reasoning as above shows that the convex function $\psi(., x)$ is l.s.c. on M when regarded as a function from $C_{E_\sigma}(-h, 0)$ to R. Consequently, we can apply the Theorem 1 of [1] (in §2.5, Chapter 2) to conclude that the Bellman (or marginal) function

$$J(\varphi^0) = \min\{\psi(\varphi^0, x(.)), x(.) \in N_f(\varphi^0)\}$$

is l.s.c on M. The theorem is proved.

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