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ON A GENERALIZATION OF A HARDY-LITTLEWOOD THEOREM

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Dedecated to the memory of Professor Le Van Thiem

Abstract. *The Hardy-Littlewood theorem on a characterization of moduli of continuity and an extimation of derivatives of holomorphic Lipschitz functions in the unit disk are generalized for the case of more general domains.*

1. INTRODUCTION

Let $f(x)$ and $g(x)$ be nonnegative functions defined on an abstract set X. By $f(x) << g(x), x \in X$, we mean that there exists a constant $C > 0$, which is independent on x, and for which the inequality $f(x) \leq Cg(x)$ holds for all $x \in X$. In the case $f(x) << g(x)$ and $g(x) << f(x), x \in X$, we write $f(x) \asymp g(x)$, $x \in X$.

Modulus of continuity, by definition, is a continuous function $s(t) : [0, \infty) \to [0, \infty)$ satisfying the following conditions: $s(0) = 0$ and $0 \leq s(t) - s(t') \leq s(t - t')$, $0 \leq t' \leq t$. We denote by S the set of all differentiable moduli of continuity $s(t)$ such that $s'(t)$ are decreasing.

Let G be a domain on the complex plane $\mathbf{C}$ and $s \in S$.

$H(G)$ denotes the space of all functions holomorphic in G with the topology of uniform convergence on compact subsets of G; $\mathrm{Lip}_s(G)$ denotes the class of all functions $f(z) : G \to \mathbf{C}$ for each of which there exists a constant $M < \infty$ such that

$$|f(z_1) - f(z_2)| \leq M.s(|z_1 - z_2|) \text{ for all } z_1, z_2 \in G.$$

One denotes by $||f||_s$ the infimum of all such M. Note that if $s(t) = t^\alpha, 0 < \alpha \leq 1$, then the classes $\mathrm{Lip}_s(G)$ are usual Lipschitz classes.

The following theorem is due to Hardy and Littlewood [1]. See also [2, p.74].

Theorem A. *Let D be the unit disk and $s(t) = t^\alpha$, $0 < \alpha \leq 1$. If $f \in H(D) \cap Lip_s(D)$ then*

$$|f'(z)| \ll s'(1-|z|),\ z \in D.$$

In [3] this result is generalized for the case of functions $s \in S$, namely

Theorem B. *Let D be the unit disk and $s \in S$. The following statements are equivalent:*

(i) $f \in H(D) \cap Lip_s(D)$, *then*

$$|f'(z)| \ll s'(1-|z|),\ z \in D.$$

(ii) $\limsup\limits_{t \to 0^+} \dfrac{s(t)}{ts'(t)} < \infty.$

The following question arises: are there results of a such sort in a general case, i.e. in the case of arbitrary domains?. In this note we shall show that for certain (sufficiently general) domains the answer will be affirmative. The methods are similar to those used in [3].

2. DEFINITIONS

Let G be an arbitrary bounded domain with boundary ∂G. From compactness of ∂G it follows that if $w \in \mathbf{C}$ then the quantity $d_w := \sup\limits_{z \in G} |z - w|$ is finite and positive, moreover the supremum is attained on ∂G, i.e. the set $P_w := \{z \in \partial G : \ |z - w| = d_w\}$ is nonempty. This means that for every $w \in \mathbf{C}$, $\partial U(w, d_w) \cap \partial G \neq \emptyset$, where $U(w, d_w)$ is the open disk of radius d_w and with centre w. The points in $\partial U(w, d_w) \cap \partial G$ are called convex-points of G with respect to w.

For every $z \in G$ one denotes by $\rho(z, \partial G)$ the distance of z to ∂G. We say that G is a (ρ)-domain if there exists a point $w_0 \in \mathbf{C}$ for which there is a convex-point $z_0 \in P_{w_0}$ such that for some $r_1 \in (0,1)$ the following conditions hold:

(C_1) all points $z \in I_{z_0} := \{z = rw_0 + (1-r)z_0,\ 0 < r < r_1\}$ belong to G;

(C_2) $\rho(z, \partial G) \gg d_{w_0} - |z - w_0|,\ z \in I_{z_0}$.

Geometrically, condition (C_1) means that on the segment joining w_0 and z_0 there is a point z_1 such that the open segment joining z_0 and z_1 belongs to G. The (C_2) says that for all points z from the segment joining z_0 and z_1 the distance of z to ∂G and the distance of z to z_0 are equivalent (in the sense $\asymp$). Here we notice that for all $z \in I_{z_0}$ the inequality $\rho(z, \partial G) \leq d_{w_0} - |z - w_0|$ always holds.

3. MAIN RESULTS

Theorem. *Let G be a bounded domain and $s \in S$. Consider the following two statements*

(a) $\limsup\limits_{t \to 0^+} \dfrac{s(t)}{ts'(t)} < \infty$;

(b) If $f \in H(G) \cap Lip_s(G)$ *then*

$$|f'(z)| \ll s'(\rho(z, \partial G)),\ z \in G.$$

Then we have

1) Statement (a) implies statement (b);

2) If G is a (ρ)-domain then statement (b) implies statement (a)

P r o o f. Suppose that (a) holds. Then there exist positive constants t_0 and C_0 such that $s(t)/t < C_0 s'(t)$ for all $t \in (0, t_0]$. Since $s(t)/t$ and $s'(t)$ are decreasing we get $s(t)/ts'(t) < s(t_0)/t_0 s'(d_{w_0})$ for all $t \in [t_0, d_{w_0}]$.

Hence

$$\frac{s(t)}{t} \leq C_1.s'(t) \text{ for all } t \in (0, d_{w_0}],$$

where $C_1 = \max\{C_0; s(t_0)/t_0 s'(d_{w_0})\}$.

Now let $f \in H(G) \cap \mathrm{Lip}_s(G)$. Fix $z \in G$. One has for $0 < R < \rho(z, \partial G)$

$$|f'(z)| = |2\pi R \int_0^{2\pi} \frac{f(z + Re^{i\theta}) - f(z)}{e^{i\theta}} d\theta| \leq$$

$$\leq \|f\|_s.s(R)/R \leq C_1.\|f\|_s.s'(R)$$

from which (b) follows by letting R tend to $\rho(z, \partial G)$.

Now, we suppose that (b) holds and that G is a (ρ)-domain. We shall show that if (a) fails then one can construct a power series in $H(G) \cap \mathrm{Lip}_s(G)$ for which (b) fails, a contradiction. Indeed, assuming (a) fails, we can choose a rapidly decreasing sequence of positive numbers $\{t_k\}$ satisfying the folowing conditions

$$\frac{s(t_k)}{t_k s'(t_k)} \geq e^{2k},\ k = 1, 2, \ldots$$

Let z_0 be a convex-point satisfying the conditions (C1) and (C2) and let $z_0 - w_0 = |z_0 - w_0|.e^{i\theta_0}$, $0 \leq \theta_0 < 2\pi$. Define $n_k = [\frac{1}{t_k}], c_k = s(t_k)e^{-i(n_k - 1)\theta_0}/e^k d_{w_0}^{n_k}$, where $[x]$ denotes the integer part of x. Then a power series $f(z) = \sum\limits_{k=1}^{\infty} c_k(z - w_0)^{n_k}$ converges in the disk $\{z : |z - w_0| < d_{w_0}\}$.

Now for $z_1, z_2 \in G$ and $q \in \mathbf{N}$ one has

$$|(z_1 - w_0)^q - (z_2 - w_0)^q| \leq |(z_1 - w_0)|^q + |(z_2 - w_0)|^q \leq 2d_{w_0}^q$$

and

$$|(z_1 - w_0)^q - (z_2 - w_0)^q| = |z_1 - z_2| . |(z_1 - w_0)^{q-1} + \ldots +$$

$$+ (z_2 - w_0)^{q-1}| \leq |z_1 - w_2| q d_{w_0}^{q-1}$$

Consequently,

$$|(z_1 - w_0)^q - (z_2 - w_0)^q| \leq \min\{d_{w_0}^{q-1}.q.|z_1 - z_2|, 2d_{w_0}^{q-1}\}.$$

Hence,

$$\sup_{z_1, z_2 \in G} \frac{|(z_1 - w_0)^q - (z_2 - w_0)^q|}{s(|z_1 - z_2|)} \leq$$

$$\leq \max\{\sup_{0<t<1/q} \frac{q d_{w_0}^{q-1} t}{s(t)}; \sup_{1/q<t<2d_{w_0}} \frac{2d_{w_0}^q}{s(t)}\}$$

(for $q \geq [1/2d_{w_0}]$)

$$= \max\{\frac{d_{w_0}^{q-1}}{s(1/q)}; \frac{2d_{w_0}^q}{s(1/q)}\} =$$

$$= \max\{1; 2d_{w_0}\} \cdot \frac{d_{w_0}^{q-1}}{s(1/q)} = C_{w_0} \frac{d_{w_0}^{q-1}}{s(1/q)}.$$

We have

$$\sup_{z_1, z_2 \in G} \frac{|f(z_1) - f(z_2)|}{s(|z_1 - z_2|)} = \sup_{z_1, z_2 \in G} \frac{\left|\sum_{k=1}^{\infty} c_k (z_1 - w_0)^{n_k} - (z_2 - w_0)^{n_k}\right|}{s(|z_1 - z_2|)} \leq$$

$$\leq \sum_{k=1}^{\infty} |c_k| \cdot \sup_{z_1, z_2 \in G} \frac{|(z_1 - w_0)^{n_k} - (z_2 - w_0)^{n_k}|}{s(|z_1 - z_2|)} \leq \sum_{k=1}^{\infty} |c_k| \frac{c_{w_0} d_{w_0}^{n_k - 1}}{s(1/n_k)} =$$

$$= \sum_{k=1}^{\infty} \frac{s(t_k) C_{w_0}}{e^k d_{w_0} s(1/n_k)} = \frac{C_{w_0}}{d_{w_0}} \sum_{k=1}^{\infty} \frac{1}{e^k} = \frac{C_{w_0}}{d_{w_0}(e-1)} < \infty.$$

On the other hand, since for $z \in I_{z_0}, \rho(z, \partial G) \asymp d_{w_0} - |z - w_0|$, it follows that

$$\sup_{z\in G} \frac{|f'(z)|}{s'(\rho(z,\partial G))} \geq \sup_{z\in I_{z_0}} \frac{|f'(z)|}{s'(\rho(z,\partial G))} \asymp$$

$$\asymp \sup_{z\in I_{z_0}} \frac{|f'(z)|}{s'(d_{w_0} - |z-w_0|)} = \sup_{z\in I_{z_0}} \frac{|\sum_{k=1}^{\infty} c_k n_k (z-w_0)^{n_k-1}|}{s'(d_{w_0} - |z-w_0|)}.$$

Note that if $z \in I_{z_0}$ then $z = rw_0 + (1-r)z_0, 0 < r < r_1 < 1$. One has $z - w_0 = (1-r)(z_0 - w_0)$. Hence

$$\sup_{z\in I_{z_0}} \frac{|\sum_{k=1}^{\infty} c_k n_k (z-w_0)^{n_k-1}|}{s'(d_{w_0} - |z-w_0|)} = \sup_{0<r<r_1} \frac{\sum_{k=1}^{\infty} s(t_k) n_k (1-r)^{n_k-1}/e^k d_{w_0}}{s'(rd_{w_0})} \geq$$

$$\geq \sup_k \sup_{0<r<r_1} \frac{s(t_k) n_k (1-r)^{n_k-1}}{e^k d_{w_0} s'(rd_{w_0})} \geq \sup_k \frac{s(t_k) n_k (1 - \frac{1}{n_k} d_{w_0})^{n_k-1}}{e^k d_{w_0} s'(1/n_k)} \gg$$

$$\gg \sup_k \frac{s(t_k)\cdot 1/et_k \cdot 1/e^{1/d}}{e^k d_{w_0} s'(1/n_k)} \gg \sup_k \frac{s(t_k)}{e^k s'(t_k) t_k} \geq \sup_k \frac{e^{2k}}{e^k} = \infty.$$

This completes the proof of the theorem.

4. EXAMPLES.

a) The simplest examples of (ρ)-domains are disks. Moreover it is worth noticing the following fact:

Let G be a bounded domain satisfying the condition (C_1). If G contains an image of I_{z_0} with respect to a rotation around z_0 by some angle φ_0, then G satisfies the condition (C_2), i.e. G is a (ρ)-domain.

Indeed, in this case $\rho(z, \partial G) = (d_{w_0} - |z - w_0|) \cdot \sin\varphi_0$ for all $z \in I_{z_0}$. Thus, convex and starlike domains are (ρ)-domains.

In view of this fact we can construct general examples of (ρ)-domains by the following way: Consider an arbitrary disk $U(w_0, d)$ and fix a point $z_0 \in \partial U$. Take an angle $\widehat{Az_0B}$, where A and B lie on different sides of the segment joining z_0 and w_0 being in $U(w_0, d)$. Then all domains G which are contained in U such that two segments Az_0, Bz_0 are on ∂G will be (ρ)-domains. It is clear that the set $G \setminus \widehat{Az_0B}$ can be modified arbitrarily.

b) Now we give a method to construct domains, which are not (ρ)-domains. Take, for example, a regular triangle ABC. Put it in the coordinate system so that $A(0;b), B(-a;0)$ and $C(a;0)$, where $b = a\sqrt{3}, a > 0$. We adjoin sequentially by segments the points $C(a;0) \to C_1(0;0) \to C_2(a/2; b - b/2) \to C_3(0; b - b/2) \to \cdots \to C_{2k}(a/2^k; b - b/2^k) \to C_{2k+1}(0; b - b/2^k) \to \cdots$ and the points $B(-a;0) \to B_1(0;b/4) \to B_2(-a/2; b - b/2) \to B_3(0; b - 3b/8) \to \cdots \to B_{2k}(-a/2^k; b - b/2^k) \to B_{2k+1}(0; b - 3b/2^{k+2}) \to \cdots$, then we obtain the infinite zigzag lines $CC_1C_2C_3 \cdots C_{2k}C_{2k+1} \cdots$ and $BB_1B_2B_3 \cdots B_{2k}B_{2k+1} \cdots$. One calls described process a "zigzag transform" of the triangle ABC with respect to the side CB.

Now consider a regular triangle HKL. Let M, N, P be middle points of the sides KL, LH, HK. We do "zigzag transform" for triangles HPN, KMP and LMN with respect to the sides NP, PM and MN respectively. It is easy to verify that the obtained domains is not (ρ)-domain.

5. OPEN PROBLEMS.

1) Is there a domain, which is not a (ρ)-domain but for which the Theorem is true ?

2) Is it true that if two statements (a) and (b) in the Theorem are equivalent then the domain G must be (ρ)-domain ?

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