A FORMULA FOR LOJASIEWICZ NUMBERS AND A NEW CHARACTERIZATION OF THE IRREGULARITY AT INFINITY OF ALGEBRAIC PLANE CURVES

HA HUY VUI

Dedicated to the memory of Professor Le Van Thiem

1.

According to the work [2]-[4], the Lojasiewicz number is an useful invariant of the singularities at infinity of polynomials. The aims of this paper are following.

(i) To relate the Lojasiewicz number at infinity of an algebraic plane curve to the Lojasiewicz numbers of the compactification of the curve in the projective plane.

(ii) Using the result of (i), to give a new characterization of the irregularity at infinity, different from the characterizations, given in the previous works [1]-[4], [5], [6], [9], [10], [11]. Besides these results, this paper contains also the description of the Newton-Puiseux expansions at infinity of affine curves.

2.

Let $P(x,y) \in \mathbf{C}[x,y]$ be a polynomial of two complex variables.

2.1. Definition. *(i) The value $t_0 \in \mathbf{C}$ is called regular at infinity, if there are $\delta > 0, r \gg 1$, such that*

$$P : P^{-1}(D_\delta) - B_r \longrightarrow D_\delta$$

is a locally trivial C^∞-fibration, where

$$D_\delta = \{t \in \mathbf{C}, |t - t_0| < \delta\}$$

and

$$B_R = \{z = (x, y) \in \mathbf{C}^2, |z| \leq r\}.$$

(ii) If the value t_0 is not regular at infinity, it is called critical value, corresponding to the singularities at infinity of P.

(iii) The curve $V = P^{-1}(t_0)$, where t_0 is a critical value, corresponding to the singularities at infinity of P, is called irregular at infinity.

Let $0 \in \mathbf{C}$ be a critical value, corresponding to the singularities at infinity of P. For $\delta > 0$ and $r \gg 1$ we define

$$\varphi_\delta(r) = \inf_{\substack{|(x,y)|=r \\ (x,y)\in P^{-1}(\overline{D}_\delta)}} |gradP(x, y)|,$$

where $\overline{D}_\delta = \{t \in \mathbf{C},\ |t| \leq \delta\}$.

Let

$$\mathcal{L}_\infty(V) = \lim_{\delta \to 0} \lim_{r \to \infty} \frac{\ln \varphi_\delta(r)}{\ln r}. \tag{2.1}$$

2.2. **Definition** [**2**]. *Number $\mathcal{L}_\infty(V)$, defined by (2.1), is called Lojasiewicz's number at infinity of V.*

Let $\overline{V} = \{(x : y : z) \in \mathbf{CP}^2,\ z^d P(\frac{x}{z}, \frac{y}{z}) = 0\}$ be the compactification of V in $\mathbf{CP}^2$.

Let $\overline{P} = z^d P(\frac{x}{z}, \frac{y}{z})$.

Let $A_1, \cdots, A_k$ be the points of intersection of $\overline{V}$ with the "line at infinity" $z = 0$ of $\mathbf{CP}^2$. Each point A_i belongs to one of two open set $U_1 = \{(x : y : z), x = 1\}$ and $U_2 = \{(x : y : z), y = 1\}$. Let, for example, $A_i \in U_1$. Let $\mathcal{L}_{A_i}(\overline{V})$ be the Lojasiewicz number of the germ at A_i of the analytical function $\overline{P}(1, y, z)$, defined by C.T.Kuo and Y.U.Lu in [7].

We define the number

$$\mathcal{L}(\overline{V}) = \max_{i=1,\dots,k} \mathcal{L}_{A_i}(\overline{V}).$$

Our results are

2.3. **Theorem.** *Let $P^{-1}(0)$ be an irregular at infinity algebraic affine plane curve. Ler d be a degree of $P(x, y)$. Then we have*

$$\mathcal{L}_\infty(V) + \mathcal{L}(\overline{V}) = d - 2.$$

2.4. Theorem. *The curve $V = P^{-1}(0)$ is irregular at infinity if and only if*

$$\mathcal{L}(\overline{V}) > d - 1.$$

In order to prove these theorems, we need a version at infinity of the Newton-Puiseux theorem, which will be given in the next section.

3.

3.1 Definition. *The polynomial $P(x,y)$ is called convenient at infinity, if $P_d(0,y) \not\equiv 0$ and $P_d(x,0) \not\equiv 0$, where $P_d(x,y)$ is a homogeneous part of the highest degree of $P(x,y)$.*

3.2.*Remark.* If $P(x,y)$ is convenient at infinity, then

$$P_d(x,y) = \prod_{i=1}^{k} (p_i x = q_i y)^{s_i}, \quad \sum_{i=1}^{k} S_i = d,$$

where $P_i \neq 0$ and $q_i \neq 0$; $i = 1, \ldots k$.

3.3. **Proposition** (Newton-Puiseux expansions at infinity). *Suppose that $P(x,y)$ is convenient at infinity and let $V = P^{-1}(0)$. Then there exist d roots $y = a_i(x), i = 1, \ldots d$, verifying the equation*

$$P(x, a_i(x)) \equiv 0.$$

Each such a root $a_i(x)$ has the form

$$a_i(x) = c_i x + \sum_{j=-1}^{k_0} C(i)_{0,j} x^j + c(i)_1 x^{m_1(i)/n_1(i)} + \sum_{j=-1}^{-k_1} C(i)_{1,j} x^{\frac{m_1(i)+j}{n_1(i)}} + \cdots +$$

$$+ C(i)_{g(i)} x^{m(i)_{g(i)}/n_1(i)\cdots n_{g(i)}(i)} + \sum_{j=-1}^{-\infty} C(i)_{g(i),j} x^{m(i)_{g(i)}+j}/n_1(i)\cdots n_{g(i)}(i),$$

where $c_i \in \{-\frac{p_i}{q_i}, j = 1, \cdots k\}$ and the numbers $m_j(i), n_j(i)$ are integers. Moreover, the series

$$a_i(\tau^{n_1(i)\cdots n_{g(i)}(i)}), i = 1, \cdots, d$$

converge for $|\tau| \geq r_0$, *where* r_0 *is some positive number.*

P r o o f. Let $P_d(x,y) = \prod_{i=1}^{k}(p_i x + q_i y)^{s_i}$, and $P = P_d + Q$. Since $P(x,y)$ is convenient at infinity, we have

$$\overline{V} \cap \{z = 0\} = \bigcup_{i=1}^{k} A_i,$$

where $A_i = (1, -\frac{p_i}{q_i}, 0) \in \mathbf{CP}^2$. Let us consider the equation $z^d P(\frac{1}{z}, \frac{y}{z}) = 0$, locally at A_i. We make a change of variables

$$u_i = y + \frac{p_i}{q_i}.$$

Let

$$f_i(u_i, z) = z^d P(\frac{1}{z}, \frac{u_i - \frac{p_i}{q_i}}{z}).$$

Then

$$f_i(u_i, z) = [\prod_{j \neq i}^{k}(p_j + q_j u_i - q_j \frac{p_i}{q_i})^{s_i}] - q_i^{s_i}.u_i^{s_i} +$$
$$+ z^d Q(\frac{1}{z}, \frac{u_i - \frac{p_i}{q_i}}{z}).$$

One puts

$$g_i(u_i) = [\prod_{j \neq i}^{k}(p_j + q_j u_i - q_j \frac{p_i}{q_i})^{s_i}] - q_i^{s_i}.u_i^{s_i}$$

and

$$h_i(u_i, z) = z^{d-1} Q(\frac{1}{z}, \frac{u_i - \frac{p_i}{q_i}}{z}),$$

then the multiplicity of $g(u_i)$ at $u_i = 0$ is s_i and $h_i(u_i, z)$ is a polynomial of variables u_i and z. Thus $f_i(u_i, z) = g_i(u_i) + z h_i(u_i, z)$. Applying a version of Newton's algorithm, described by R.Walker in $[W]$ for the germ of $f_i(u_i, z)$ at the point A_i, we have exactly s_i solutions $u_{i_1}(z), ..., u_{is_i}(z)$ of the equation $f(u_i, z) = 0$, satisfying the condition

$$u_{ij}(z) \to 0 \text{ as } z \to 0, \; j = 1, ..., s_i.$$

Each of these solutions has the form

$$u(z) = \sum_{i=1}^{k_0} C_{0,j} z^j + C_1 z^{\frac{m'_1}{n'_1}} + \sum_{i=1}^{k_1} C_{1,j} z^{\frac{m'_1+j}{n_1}} + \cdots +$$
$$+ C_g z^{\frac{m'_g}{n'_1 \cdots n'_g}} + \sum_{i=1}^{\infty} C_{g,j} z^{\frac{m'_g+j}{n'_1 \cdots n'_g}}.$$

Moreover, the series $u(\tau^{n'_1 \cdots n'_g})$ converges for $|\tau| < \varepsilon$ for some $\varepsilon > 0$ sufficiently small.

Let

$$a'(z) = -\frac{p_i}{q_i} + u(z),$$

then $a'(z)$ is one of solutions of $z^d P(\frac{1}{z}, \frac{y}{z}) = 0$, locally at A_i, and $a'(z) \to -\frac{p_i}{q_i}$ as $z \to 0$.

Now, the proof of Proposition 3.3 follows from the following

3.4. **Lemma.** *There exist one-to-one correspondence between the solutions $a'(z)$ of the equation $z^d P(\frac{1}{z}, \frac{y}{z}) = 0$ and the solutions $a(x)$ of the equation $P(x, y) = 0$. More precisely, if $y = a'(z)$ is a solution of $z^d P(\frac{1}{z}, \frac{y}{z}) = 0$ then $y = a(x) = xa'(\frac{1}{x})$ is a solution of $P(x, y) = 0$. Inversely, if $y = a(x)$ is a solution of $P(x, y) = 0$, then $y = za(\frac{1}{z})$ is a solution of $z^d P(\frac{1}{z}, \frac{y}{z}) = 0$.*

P r o o f. Evident. Thus, the Proposition 3.3 is proved.

4.

We recall some results of [7] and [3]

4.1. Let A_i be, as in Section 2, a point of intersection of $\overline{V}$ with the "line at infinity" $\{z = 0\} \subset \mathbf{CP}^2$. Let $a'_{i1}(z), ..., a'_{is_i}(z)$ be the Puiseux expansions of $\overline{V}$ at the point A_i. Let $\mathcal{L}_{A_i}(\overline{V})$ be the Lojasiewicz number of the germ at A_i of the function $\overline{P} = z^d P(\frac{1}{z}, \frac{y}{z})$ For each $a'_{ij}(z)$ we construct the series $\varphi'_{ij}(z)$ as follows. Let

$$\zeta'_j = \max_{j \neq k} v(a'_{ij}(z) - a'_{iK}(z)),$$

where $v(.)$ is a valuation of fractional series. The series $\varphi'_{ij}(z)$ is the series $a'_{ij}(z)$ with its term of degree $\zeta'^{(i)}_j$ replaced by $z^{\zeta'_j{}^{(i)}}$, ξ is a generic coefficient, and all higher order terms omitted. Then, according to Theorem A of [7], we have

$$\mathcal{L}_{A_i}(\overline{V}) = \max_{j=1,\ldots,s_i} l_j'^{(i)} - 1,$$

where $l_j'^{(i)} = v(\overline{P}(z, \varphi_{ij}'(z))$.

4.2. Let $a_1(x), \ldots, a_d(x)$ be the Puiseux expansions at infinity of V. For any $i = 1, \ldots, d$, one puts

$$\varsigma_i = \min_{j \neq i} v(a_i(x) - a_j(x)).$$

One denotes by $\varphi_i(x)$ the series $a_i(x)$ with its term of degree ς_i replaced by ξx^{ς_i}, ξ is a generic coefficient, and all lower order terms omitted. Let $l_i = v(P(x, \varphi_i(x)))$. According to Theorem 1.3.2. of [3], if V is an irregular at infinity algebraic curve, then

$$\mathcal{L}_\infty(V) = \min_{i=1,\ldots,d} l_i - 1.$$

5.

P r o o f of Theorem 2.3

Let M_i' be the set of Puiseux expansions of $\overline{V}$ at the point A_i (with the notations of Sections 2.4). Let $a_i'(z) \in M_i'$ and $\varphi_i'(z)$ be the fractional series, obtained from $a_i'(z)$ as in Section 4. Let

$$l'(a_i') = v(\overline{P}(z, \varphi_i'(z))).$$

Then

$$l'(a_i') = \sum v(\varphi_i'(z) - a_k'(z)),$$

where $a_R'(z)$ runs through all $a_k'(z) \in M_i'$. By Lemma 3.4

$$l'(a_i') = \sum_{q_k(x) \in M_i} v(x\varphi_i(\frac{1}{x}) - xa_k(\frac{1}{x})),$$

where $a_k(x)$ runs through all elements of the set

$$M_i = \{a_k(x), \exists a_k'(z) \in M_i', a_k(x) = xa_k'(\frac{1}{z})\}$$

and $\varphi_i(x) = x\varphi_i'(\frac{1}{x})$. Thus

$$l'(a_i') = \#M_i + \sum_{a_k(x)\in M_i} v(\varphi_i(\frac{1}{x}) - a_k(\frac{1}{x}))$$
$$= s_i + \sum_{a_k(x)\in M_i} v(\varphi_i(\frac{1}{x}) - a_k(\frac{1}{x})). \quad (5.1)$$

We observe that

$$\sum_{a_k(x)\in M_i} v(\varphi_i(\frac{1}{x}) - a_k(\frac{1}{x})) = -\sum_{a_j(x)\in M_i} v(\varphi_i(\frac{1}{x}) - a_j(\frac{1}{x})) +$$
$$+ \sum_{k=1}^{d} v(\varphi_i(\frac{1}{x}) - a_j(\frac{1}{x})). \quad (5.2)$$

It follows from the proof of Proposition 3.3 that

$$v(\varphi_i(\frac{1}{x}) - a_k(\frac{1}{x})) = -1$$

for $a_k(x) \notin M_i$. Therefore

$$\sum_{a_k(x)\notin M_i} v(\varphi_i(\frac{1}{x}) - a_k(\frac{1}{x})) = -(d - \#M_i) = -d + s_i. \quad (5.3)$$

On the other hand,

$$\sum_{j=1}^{d} v(\varphi_i(\frac{1}{x}) - a_j(\frac{1}{x})) = -\sum_{j=1}^{d} v(\varphi_i(x) - a_j(x)) = -l(a_i), \quad (5.4)$$

where $l(a_i) = v(P(x, \varphi_i(x)))$.

It follows from (5.1)-(5.4) that

$$l'(a_i') = s_i + [-l(a_i) - (-d + s_i)] = d - l'(a_i). \quad (5.5)$$

By Secton 4.1

$$\mathcal{L}_{A_i}(\overline{V}) = \max_{a_j'\in M_i'} (l'(a_j') - 1$$

and by (5.5)

$$\mathcal{L}_{A_i}(\overline{V}) = \max_{a_j \in M_i} (d - l(a_j) - 1$$
$$= d - \min_{a_j \in M_i} l(a_j) - 1.$$

Thus

$$\mathcal{L}(\overline{V}) = \max_{i=1,\dots,k} \mathcal{L}_{A_i}(\overline{V}) = (d-1) - \min_{i=1,\dots,k} \min_{a_j \in M_i} l(a_j)$$
$$= (d-1) - \min_{j=1,\dots d} l(a_j).$$

Since V is irregular at infinity, by Section 4.2, we have

$$\min_{j=1,\dots,d} l(a_j) - 1 = \mathcal{L}_\infty(V).$$

Then

$$\mathcal{L}(\overline{V}) = d - 1 - (\min_{i=1,\dots,d} l(a_j) - 1) - 1,$$

or

$$\mathcal{L}(\overline{V}) + \mathcal{L}_\infty(\overline{V}) = d - 2.$$

Theorem 2.3 is proved.

6.

P r o o f of Theorem 2.4

Suppose that V is an irregular at infinity curve. Then, by Section 4.2, $\mathcal{L}_\infty(V) < -1$. It follows from Theorem 2.3 that $\mathcal{L}(\overline{V}) > d - 1$.

Conversely, supposing that $\mathcal{L}(\overline{V}) > d - 1$, we have to show that V is an irregular at infinity curve.

6.1. **Lemma.** *Let $b_1(x), \dots, b_{d-1}(x)$ be the Puiseux expansions at infinity of the curve $W = P_y^{-1}(0)$. Suppose that $\mathcal{L}(\overline{V}) > d - 1$. Then there exist b_{s_0}, $s_0 \in \{1, \dots, d-1\}$, s.t.*

$$v(P(x, b_{s_0}(x))) < 0.$$

P r o o f. Let $\mathcal{L}(\overline{V}) = l'(a'_{i_0}) - 1 > d - 1$ and let

$$a_{i_0}(x) = xa'_{i_0}(\frac{1}{x}).$$

Then

$$l(a_{i_0}) = d - l'(a'_{i_0}) < 0.$$

Let us denote

$$\varsigma_{i_0} = \min_{i \neq i_0} v(a_{i_0}(x) - a_i(x))$$

and let

$$\varsigma_{i_0} = v(a_{i_0}(x) - a_{j_0}(x)).$$

By Lemma 3.14 of [3], there exists $b_{s_0}(x) \in \{b_1(x), ..., b_{d-1}(x)\}$ such that

$$\varsigma_{i_0} = v(a_{i_0}(x) - a_{j_0}(x)) = v(a_{i_0}(x) - b_{s_0}(x)) = v(a_{j_0}(x) - b_{s_0}(x)).$$

We obrerve that

$$v(a_r(x) - b_{s_0}(x)) \leq v(a_r(x) - a_{i_0}(x)) \tag{6.1}$$

for all $r \in \{1, ..., d\} - \{i_0\}$.

In fact

$$v(a_r(x) - a_{i_0}(x)) \geq v(a_{i_0}(x) - a_{j_0}(x)) = v(a_{i_0}(x) - b_{s_0}(x))$$

and therefore, applying Lemma 3.12 of [3] for three series $a_r(x), a_{i_0}(x), b_{s_0}(x)$, we get (6.1).

We see then

$$\begin{aligned} v(P(x, b_{s_0}(x))) &= \sum_{r=1}^{d} v(a_r(x) - b_{s_0}(x)) = \\ &= \sum_{r \neq i_0} v(a_r(x) - b_{s_0}(x)) + \varsigma_{i_0} \leq \\ &\leq \sum_{i \neq i_0} v(a_r(x) - a_{i_0}(x)) + \varsigma_{i_0} = l(a_{i_0}) < 0 \end{aligned}$$

and hence Lemma (6.1) is proved.

We continue the proof of Theorem 2.4. Let denote by $t(x) = P(x, b_{s_0}(x))$. Then, by Lemma 6.1,

$$t(x) \to 0 \text{ as } x \to \infty$$

Since $b_{s_0}(x)$ is a Puiseux expansion at infinity of $W = P_y^{-1}(0)$, the points $(x, b_{s_0}(x))$ are nothing else but the points of ramification of the Riemann surface $V_{t(x)} = P^{-1}(t(x))$. Then, by using the arguments of the proof of Theorem 2.2.1 in [3], we get, for $r \gg 1$

$\lfloor(V - B_r) = \lfloor(V_{t(x)} - B_r) +$ number of points of ramification of $V_{t(x)}$ in $V_{t(x)} - B_r$,

where $\lfloor(.)$ is the Euler characteristic.

Since $(x, b_{s_0}(x)) \in V_{t(x)} - B_r$, we get

$$\lfloor(V - B_r) \neq \lfloor(V_{t(x)} - B_r)$$

which shows that 0 is a critial value, coresponding to the singularities at infinity of P. The theorem is proved.

7.

7.1 Recently, Professor P. Casson-Nogeies communicated me that she has received also Theorem 2.3, independently.

7.2. As we know, if a germ of an analytical function satisfies some conditions of nondegeneracy, then its local number of Lojasiewicz can be computed via its Newton's diagram ([8]). In the light of Theorem 2.3., it would be nice to have a formula expessing the number $\mathcal{L}_\infty(V)$ in terms of Newton's diagram at infinity of V.

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Institute of Mathematics,
P.O.Box 631 Boho
10000 Hanoi, Vietnam

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