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A REDUCTION OF THE MULTIDIMENSIONAL QUANTIZATION AND U(1)-COVERING

TRAN VUI

Dedicated to the memory of Professor Le Van Thiem

Abstract. *Using the metaplectic representation in terms of the Bargmann - Segal model we lift the character of the solvable radical of the stabilizer to $U(1)$ - covering. Applying the structure and pairing of vacuum states for positive polarizations suggested by P.L. Robinson and J.H.Rawnsley we present a reduction of the procedure of multidimensional quantization to the case of semi-simple groups. Our geometric construction is connected with the M.Duflo's second method by using M_p^c-structures in place of metaplectic structures. From a semi-simple datum we construct unitary induced representations of Lie groups by means of positive solvable distributions. They are then illustrated as representations obtained from the procedure of geometric multidimensional quantization.*

1. INTRODUCTION

The multidimensional quantization procedure of Do Ngoc Diep [1-3] could be viewed as a geometric version of M.Duflo's construction of the unitary dual of Lie groups (see [4]). To avoid the Mackey's obstructions when reducing Kirillov's method of orbits to special contexts, M.Duflo lifted the construction to Z_2 - covering by using metaplectic structure. Our purpose is also to eliminate the Mackey's obstructions to obtain linear representations from the projective ones.In [8] by using the technique of P.L.Robinson and J.H.Rawnsley (see [6]) we lifted the construction to $U(1)$-covering via Mp^c- structures instead of metaplectic structures. It is a reduction of the multidimensional quantization to the case of discrete groups by using $U(1)$ - covering.

In this paper we shall reduce the same problem to the case of semisimple Lie groups. This reduction is connected with the M.Duflo's second method of which we proposed a geometric version in [7]. From a semisimple datum we shall construct unitary representations of Lie groups by means of positive solvable distributions (Theorem 1). They then will be illustrated as representations obtained from a natural generalization of Kirillov-Kostant-Souriau's procedure of quantization (Theorem 2).

2. U(1) -COVERING OF THE RADICAL AND SEMI - SIMPLE DATA

Let G be a connected and simply connected real Lie group. Denote by $\mathcal{G}$ the Lie algebra of G and by $\mathcal{G}^*$ its dual space. The group G acts in $\mathcal{G}^*$ by the coadjoint representation. Let $F \in \mathcal{G}^*$ be an arbitrary point in an orbit ω, and G_F be the stabilizer of this point. Denote by $\mathcal{G}_F$ its Lie algebra, $\mathcal{R}(F)$ the solvable radical of $\mathcal{G}_F$ and R_F the corresponding analytic subgroup in G.

2.1. $U(1)$ - covering of the radical R_F of the stabilizer G_F

Denote by $\mathcal{AP}^{U(1)}(G)$ the set of all F in $\mathcal{G}^*$ which are $U(1)$-admissible (see [8]) and positive well- polarizable; $X_{irr}^{U(1)}(F)$ the set of all equivalent classes of irreducible unitary representations of $G_F^{U(1)}$ such that the restriction of each of them to $(G_F^0)^{U(1)}$ is a multiple of the character $\chi_F^{U(1)}$, then a member $\tilde{\sigma} \in X_{irr}^{U(1)}(F)$ is considered as an irreducible projective represetation of the discrete group $G_F^0 \setminus G_F$, (see [8]). We set

$$\mathcal{B}^{U(1)}(G) = \{(F,\tilde{\sigma})/F \in \mathcal{AP}^{U(1)}(G),\ \tilde{\sigma} \in X_{irr}^{U(1)}(F)\}$$

For every $(F,\tilde{\sigma}) \in \mathcal{B}^{U(1)}(G)$, in [8] we can choose a $(\tilde{\sigma},\chi_F^{U(1)})$ - polarization $(\mathcal{N},\rho,\sigma_0)$ and construct a natural unitary representation of G. Here, we reduce the same problem to the case of semi-simple Lie group. It is considered as a semi-simple datum described in detail in subsection 2.2.

Let S_F be the semi-simple component of G_F in its Cartan-Levi-Maltsev's decomposition, $G_F = S_F.R_F$. From the local triviality of the S_F - principal bundle $S_F \mapsto R_F\backslash G \xrightarrow{k} G_F\backslash G$ there exists a connection on the bundle. Then the Kirillov 2-from B_Ω of K - orbit Ω induces a nondegenerate closed G - invariant 2 - form $\tilde{B}_\Omega$ on the horizontal part $T_H(R_F \setminus G)$ defined by the formula

$$\tilde{B}_\Omega(f)(\tilde{X},\tilde{Y}) = B_\Omega(F)(k_*\tilde{X},k_*\tilde{Y}),$$

where $f \in R_F \setminus G$, $k(f) = F \in \Omega$, and k_* is the linear lifting isomorphism induced from k, (see [7]).

The symplectic group $Sp(T_{(f)H}(R_F \setminus G),\tilde{B}_\Omega(f))$ consists of all the real automorphisms which preserve the symplectic form $\tilde{B}_\Omega(f)$. Then $Sp(T_{(f)H}(R_F \setminus G))$

has an $U(1)$ - connected covering $Mp^c(T_{(f)H}(R_F \setminus G))$, (see [8]). We have a surjective group homomorphism δ from $Mp^c(T_{(f)H}(R_F \setminus G))$ to $Sp(T_{(f)H}(R_F \setminus G))$ and the following sequence is central short exact one

$$1 \to U(1) \to Mp^c(T_{(f)H}(R_F \setminus G)) \xrightarrow{\delta} Sp(T_{(f)H}(R_F \setminus G)) \to 1.$$

Proposition 2.1. *There exists a group homomorphism j from R_F to $Sp(T_{(f)H}(R_F \setminus G))$.*

P r o o f. Let $g \in R_F$ then $\widetilde{Adg}^{-1} : \mathcal{G}/\mathcal{G}_F \longrightarrow \mathcal{G}/\mathcal{G}_F$ is a real automorphism induced from $Ad\, g^{-1}$. The following diagram is commutative

$$\begin{array}{ccc} \mathcal{G}/\mathcal{G}_F & \xrightarrow{\widetilde{Adg}^{-1}} & \mathcal{G}/\mathcal{G}_F \\ \uparrow k_* & & \uparrow k_* \\ T_{(f)H}(R_F \setminus G) & \xrightarrow{j(g)} & T_{(f)H}(R_F \setminus G) \end{array}$$

Putting $j(g) = k_*^{-1}\widetilde{Adg}^{-1}k_*$, we have

$$\tilde{B}_\Omega(f)(j(g)\tilde{X}, j(g)\tilde{Y}) = B_\Omega(F)(k_*j(g)\tilde{X}, k_*j(g)\tilde{Y}) = \tilde{B}_\Omega(f)(\tilde{X}, \tilde{Y}),$$

thus $j(g) \in Sp(T_{(f)H}(R_F \setminus G))$. □

Denote by $R_F^{U(1)}$ the Lie subgroup of the Cartesian product of Lie group $R_F \times Mp^c(T_{(f)H}(R_F \setminus G))$ consisting of all pairs (g, U) such that $\delta(U) = j(g)$.

Proposition 2.2. *$R_F^{U(1)}$ is $U(1)$ - covering of R_F.*

P r o o f. We have

$$R_F^{U(1)} = \{(g, U)/\delta(U) = k_*^{-1}\tilde{A}dg^{-1}k_*\}$$

where $U \in Mp^c(T_{(f)H}(R_F \setminus G))$ has the parameters (λ, ϕ) with $\phi \in Sp(T_{(f)H}(R_F \setminus G))$ and $\lambda \in \mathrm{C}$ such that $|\lambda^2 \mathrm{Det} C_\phi| = 1$, where $C_\phi = \frac{1}{2}(\phi - i\phi i)$ commutes with $i \in \mathrm{C}, i^2 = -1$.

Since $\delta(U) = \phi$ every member of $R_F^{U(1)}$ has the form $(g; (\lambda, \tilde{A}dg^{-1}))$ such that

$$|\lambda^2 \mathrm{Det} C_{k_*^{-1}\tilde{A}dg^{-1}k_*}| = 1$$

Putting $l(g;(\lambda, k_*^{-1}\tilde{A}dg^{-1}k_*)) = (\lambda, k_*^{-1}\widetilde{Adg^{-1}}k_*) \in Mp^c(T_{(f)H}(R_F \setminus G))$, and $\sigma_{j(g;U)} = g$. Then the following diagram

$$\begin{array}{ccccccc} 1 \to U(1) \to & R_F^{U(1)} & \xrightarrow{\quad\sigma_j\quad} & R_F & \to 1 \\ \Big\downarrow id & \Big\downarrow l & & \Big\downarrow j & \\ 1 \to U(1) \to & Mp^c(T_{(f)H}(R_F \setminus G)) & \xrightarrow{\delta} & Sp(T_{(f)H}(R_F \setminus G)) & \to 1 \end{array}$$

is commutative, the proposition is proved.

2.2. $(r, U(1)$-admissible forms and semi-simple data.

We do not assume that Ω passing $F \in \mathcal{G}^*$ is an integral orbit, i.e there does not exist a unitary character χ_F of G_F.

Since $1 \to U(1) \to R_F^{U(1)} \to R_F \to 1$ is a short exact sequence we have futhermore a split short exact sequence of the corresponding Lie algebras as follows:

$$0 \to u(1) \to \mathrm{Lie}R_F^{U(1)} \to \mathcal{R}_F \to 0.$$

Thus the Lie algebra of $R_F^{U(1)}$ is $\mathcal{R}_F \oplus u(1)$, (see [6, §5]).

Definition 2.1. A point $F \in \mathcal{G}^*$ is called $(r, U(1))$ – *admissible* (r for radical, $U(1)$ for $U(1)$-covering) iff there exists a unitary character $\eta_F^{U(1)} : R_F^{U(1)} \to S^1$ such that

$$(d\eta_F^{U(1)})(X, \varphi) = \frac{i}{\hbar}(F(X) + \varphi) \text{ where } (X, \varphi) \in \mathcal{R}_F \oplus u(1).$$

Remarks

i) If $\mathcal{G}$ is solvable, we have $R_F = G_F^0$ then the notion of $(r, U(1))$-admissible form coincides with the notion of $U(1)$-admisible form that is described in [8].

ii) If F is $U(1)$-admissible (see [8]) then it is $(r, U(1))$-admissible but the reverse does not hold in general.

Denote by $Y_{irr}^{U(1)}(F)$ the set of all equivalent classes of irreducible unitary representations of $G_F^{U(1)}$ such that the restriction of each of them to $R_F^{U(1)}$ is a multiple of the character $\eta_F^{U(1)}$. If $F \in \mathcal{G}^*$ is $(r, U(1))$-admissible then it is r-admissible (see [7] and [4]), so $Y_{irr}^{U(1)}(F)$ is the set of equivalent classes of irreducible projective representations of the semi-simple group $R_F \setminus G_F$ (see [4]). If F is $(r, U(1))$-admissible and $\tau \in Y_{irr}^{U(1)}(F)$, the pair (F, τ) is called a *semi-simple datum*.

2.3. Solvable positive distribution.

Definition 2.2. A smooth complex tangent distribution $\tilde{L} \subset (T(R_F \setminus G))^{\mathbf{C}}$ is called a *positive solvable distribution iff*

i) $\tilde{L}$ is an integrable and G-invariant subbundle of $(T_H(R_F \setminus G))^{\mathbf{C}}$,

ii) $\tilde{L}$ is invariant under the action Ad of G_F,

iii) $\forall f \in R_F \setminus G$, the fibre $\tilde{L}_f$ is a positive polarization of the symplectic vector space $((T_{(f)H}(R_F \setminus G))^{\mathbf{C}}, \tilde{B}_\Omega(f))$, i.e

α) $\dim \tilde{L}_f = \frac{1}{2} \dim T_{(f)H}(R_F \setminus G)$,

β) $\tilde{B}_\Omega(f)(\tilde{X}, \tilde{Y}) = 0,$ for all $\tilde{X}, \tilde{Y} \in \tilde{L}_f$,

γ) $i\tilde{B}_\Omega(f)(\tilde{X}, \overline{\tilde{X}}) \geq 0$, for all $\tilde{X} \in \tilde{L}_f$, where $\overline{\tilde{X}}$ is the conjugation of $\tilde{X}$, we say that $\tilde{L}$ is *strictly positive* iff the inequality (γ) is strict for nonzero $\tilde{X} \in \tilde{L}_f$.

Proposition 2.3. *Suppose that $\tilde{L}$ is a positive solvable distribution then the inverse image $\mathcal{N}$ of $L_F = k_* \tilde{L}_f$ under the natural projection*

$$p : \mathcal{G}^{\mathbf{C}} \longrightarrow \mathcal{G}^{\mathbf{C}} / \mathcal{G}_F{}^{\mathbf{C}}$$

is a positive polarization in $\mathcal{G}^{\mathbf{C}}$. So F is a one dimensional representation of $\mathcal{N}$.

P r o o f. We have

$$\dim \mathcal{N} / \mathcal{G}_F = \dim L_F = \dim k_* \tilde{L}_f =$$
$$= \frac{1}{2} \dim T_{(f)H}(R_F \setminus G) = \frac{1}{2} \dim \Omega.$$

On the other hand, $\tilde{B}_\Omega(f)(\tilde{X}, \tilde{Y}) = B_\Omega(F)(k_* \tilde{X}, k_* \tilde{Y})$ then $\mathcal{N}$ is a positive polarization in $\mathcal{G}^{\mathbf{C}}$.

Suppose $\tilde{L} \subset (T_H(R_F \setminus G))^{\mathbf{C}}$ to be a positive solvable distribution such that $\tilde{L} \cap \overline{\tilde{L}}$ and $\tilde{L} + \overline{\tilde{L}}$ are the complexifications of some real distributions. In this case, the corresponding complex subalgebra $\mathcal{N} = p^{-1}(k_* \tilde{L}_f)$ satisfies the following conditions: $\mathcal{N} \cap \overline{\mathcal{N}}$ and $\mathcal{N} + \overline{\mathcal{N}}$ are the complexifications of the real Lie subalgebras $\mathcal{B} = \mathcal{N} \cap \mathcal{G}$ and $\mathcal{M} = (\mathcal{N} + \overline{\mathcal{N}}) \cap \mathcal{G}$. Denote by B_0 and M_0 the corresponding analytic subgroups.

Definition 2.3. A positive solvable distribution $\tilde{L}$ is called *closed* iff all the subgroups B_0, M_0 and the semi-direct products $B = R_F.B_0$ and $M = R_F.M_0$ are closed in G.

In what follows we assume that $\tilde{L}$ is closed. The following proposition can be proved in a similar way as Proposition 2.3 in [8].

Proposition 2.4. *In the neighbourhood of the identity of $R_F^{U(1)}$ we have*

$$\eta_F^{U(1)}(g; (\lambda, \tilde{A}dg^{-1})) = \exp(\frac{i}{\hbar}(F(X) + \varphi),$$

where $\varphi \in \mathbf{R}$ satisfying the relation $\lambda^2 DetC_{\tilde{A}dg^{-1}} = e^{\frac{i}{\hbar}\varphi}$. The integral kernel of $\eta_F^{U(1)}$ is given by the formula

$$u(z,w) = \exp\{\frac{i}{\hbar}(F(X)+\varphi) + \frac{i}{2} < z, w > - \frac{1}{4\hbar} < w, w >\},$$

where $z, w \in \mathcal{M}^{\mathbf{C}}/\mathcal{B}^{\mathbf{C}}$.

2.4. Extension of the character on $B^{U(1)} = R_F^{U(1)} \ltimes B_0$

According to the Definition 2.3., B_0 is a normal subgroup in B, and R_F has adjoint action on B_0. Moreover, we have the homomorphism $\sigma_j : R_F^{U(1)} \longrightarrow R_F$, thus $R_F^{U(1)}$ acts on B_0. Hence we can define the semi-direct product $R_F^{U(1)} \ltimes B_0$ and the following diagram is commutative

$$\begin{array}{ccccccc}
1 \rightarrow U(1) & \longrightarrow & R_F^{U(1)} & \longrightarrow & R_F & \longrightarrow & 1 \\
\uparrow \mathrm{id} & & \uparrow p_{B_0} & & \uparrow p_{B_0} & & \\
1 \rightarrow U(1) & \longrightarrow & R_F^{U(1)} \ltimes B_0 & \longrightarrow & R_F \cdot B_0 & \rightarrow & 1
\end{array}$$

where p_{B_0} is the quotient homomorphism between the corresponding groups. Then $R_F^{U(1)} \times B_0$ is the $U(1)$-covering of $B = R_F \cdot B_0$. By putting $B^{U(1)} = R_F^{U(1)} \times B_0$ we have $\mathcal{B} \oplus u(1)$ being the Lie algebra of $B^{U(1)}$. Denote by $B_0^{U(1)}$ the inverse image of B_0 in $B^{U(1)}$ under the $U(1)$-covering projection.

Let $\tilde{\sigma}$ be some fixed irreducible unitary representation of G_F in a separable Hilbert space $\tilde{V}$ such that the restriction of $\eta_F^{U(1)} \cdot (\tilde{\sigma}\sigma_j)$ to $R_F^{U(1)}$ is a multiple of the character $\eta_F^{U(1)}$, where σ_j is the homomorphism defined in Proposition 2.2 .

Definition 2.4. The triplet $(\tilde{L}, \rho, \delta_0)$ is called a *solvable* $(\tilde{\sigma}, \eta_F^{U(1)})$*- polarization*, and $\tilde{L}$ is called a *weakly Lagrangian distribution* iff

i) σ_0 is an irreducible representation of the subgroup $(B_0)^{U(1)}$ in a Hilbert space V such that

$$\sigma_0 | R_F^{U(1)} \cap B_0^{U(1)} = (\tilde{\sigma}\sigma_j)\eta_F^{U(1)}$$

ii) ρ is a representation of the complex Lie algebra $\mathcal{N} \oplus u(1)$ in V such that

$$d\sigma_0 = \rho \mid_{\mathcal{B} \oplus u(1)} .$$

Proposition 2.5. *If Ω_F is $(r, (U(1))$-admissible, $(\tilde{L}, \rho, \sigma_0)$ is a $(\tilde{\sigma}, \eta_F^{U(1)})$-solvable polarization then there exists a unique irreducible representation*

$$\sigma : B^{U(1)} \longrightarrow U(V) \textit{ such that}$$

$$\sigma\mid_{R_F^{U(1)}} = (\tilde{\sigma}\sigma_j)\eta_F^{U(1)} \textit{ and } d\sigma = \rho\mid_{B\oplus u(1)}.$$

P r o o f. Repeating the proof of Theorem 2 in [1] and using the fact that

$$\sigma_0\mid_{R_F^{U(1)}\cap H_0^{U(1)}} = (\tilde{\sigma}\sigma_j)\eta_F^{U(1)} \text{ and } d\sigma_0 = \rho\mid_{B\oplus u(1)}$$

we can extend σ_0 to σ that satisfies the conditions in the proposition.

3. INDUCED REPRESENTATION OBTAINED FROM A SEMI-SIMPLE DATUM.

3.1. Induced representation.

Let $(F, \tilde{\sigma}\sigma_j\eta_F^{U(1)})$ be a semi-simple datum.

Proposition 3.1. *With any solvable $(\tilde{\sigma}, \eta_F^{U(1)})$-polarization $(\tilde{L}, \sigma_0, \rho)$ there exists a natural construction of induced representation of G.*

P r o o f. If we choose $F_0 = 0 \in \mathcal{G}$ then $G_{F_0} = G$. According to the definition, $Mp^c(\mathcal{G}/\mathcal{G}_{F_0}, \tilde{B}_{F_0}) = U(1)$. We put $G^{U(1)} = G \times U(1)$. Fixing a connection $\overline{\Gamma}$ on $B \mapsto G \longrightarrow\!\!\!\!\!\!\!\!\longrightarrow B\setminus G$ we have the corresponding ones $\Gamma, \tilde{\Gamma}$ on $G_F \mapsto G \longrightarrow\!\!\!\!\!\!\!\!\longrightarrow G_F\setminus G$ and $R_F \mapsto G \longrightarrow\!\!\!\!\!\!\!\!\longrightarrow R_F\setminus G$. Passing to the $U(1)$-covering of leaves, i.e the structural groups are lifted to $U(1)$-covering. We obtain the following isomorphisms between the total spaces of principal bundles

$$G^{U(1)} \cong G_{\overline{\Gamma}}^{U(1)} \cong G_{\Gamma}^{U(1)} \cong G_{\tilde{\Gamma}}^{U(1)}, \text{ (see [8], Proposition 3.1).}$$

We have a principal $B^{U(1)}$-bundle on $B\setminus G$, a principal $G_F^{U(1)}$-bundle on the orbit Ω, a principal $R_F^{U(1)}$-bundle on $R_F\setminus G$ and two homomorphisms between them as follows:

$$\begin{array}{ccccc}
B^{U(1)} \mapsto G_{\overline{\Gamma}}^{U(1)} & & G_F^{U(1)} \mapsto G_{\Gamma}^{U(1)} & & R_F^{U(1)} \mapsto G_{\tilde{\Gamma}}^{U(1)} \\
\downarrow & & \downarrow & & \downarrow \\
B\setminus G & \xleftarrow{\quad\pi\quad} & G_F\setminus G & \xleftarrow{\quad k\quad} & R_F\setminus G
\end{array}$$

where π and k are the natural projections.

In view of the representation $\sigma : B^{U(1)} \to U(V)$ let us denote by $\overline{E}_{\sigma,\rho}$; $E_{\sigma|G_F^{U(1)},\rho}$; $\tilde{E}_{\sigma|R_F^{U(1)},\rho}$ the vector bundles on $B\backslash G$, $G_F\backslash G$ and $R_F\backslash G$ associated respectively with σ and its restrictions on $G_F^{U(1)}$ and $R_F^{U(1)}$. In the category of smooth vector bundles $k^*\pi^*\overline{E}_{\sigma,\rho}$; $k^*(E_{\sigma|G_F^{U(1)},\rho})$ and $\tilde{E}_{\sigma|R_F^{U(1)},\rho}$ are equivalent. The fixing connection Γ induces a connection also denoted by Γ on the bundle $G_F^{U(1)} \mapsto G^{U(1)} \to \Omega$, then we obtain an affine connection ∇^Γ on the associated bundle $\pi^*\overline{E}_{\sigma,\rho}$. At last we have an affine connection $\tilde{\nabla}^\Gamma$ on $\tilde{E}_{\sigma|R_F^{U(1)},\rho}$ (briefly $\tilde{E}_{\sigma,\rho}$). We put

$$S_{\tilde{L},S_F(\tilde{E}_{\sigma,\rho})} = \left\{ s \in S_{S_F}(\tilde{E}_{\sigma,\rho}) / \tilde{\nabla}^\Gamma_{\tilde{\xi}} s = 0, \forall \tilde{\xi} \in \tilde{L} \right\},$$

where $S_{S_F}(\tilde{E}_{\sigma,\rho})$ is the vector space of S_F-equivariant sections of the bundle $\tilde{E}_{\sigma,\rho}$. Then the natural representation of G in $S_{\tilde{L},S_F}(\tilde{E}_{\sigma,\rho})$ denoted by Ind $(G, \tilde{L}, (\tilde{\sigma}\sigma_j)\eta_F^{U(1)}, \sigma_0, \rho)$ is called the induced representation, (see [5], §13). □

3.2. Induced unitary representation.

For convenience in what follows we write P in place of the principal $R_F^{U(1)}$-bundle $R_F^{U(1)} \mapsto G^{U(1)} \twoheadrightarrow R_F \setminus G$. By using the group homomorphism $l : R_F^{U(1)} \longrightarrow Mp^c(T_{(f)H}(R_F\backslash G))$ and the metaplectic representation:

$$\mu : Mp^c(T_{(f)H}(R_F \setminus G)) \longrightarrow \mathrm{End}\mathcal{E}'(T_{(f)H}(R_F \setminus G)$$

(see [6], §2) we have then the bundle associated to P via the homomorphism $\mu \circ l$. We consider the homomorphism $(\tilde{\sigma}\sigma_j\eta_F^{U(1)})(\mu \circ l)$ defined as follows

$$(\tilde{\sigma}\sigma_j\eta_F^{U(1)})(\mu \circ l)(g, U) = (\tilde{\sigma}\sigma_j\eta_F^{U(1)})(g, U) \otimes (\mu \circ l)(g, U)$$

Denote by $\mathcal{E}'_{\sigma,\rho}(P)$ the vector bundle associated to P via the homomorphism $(\tilde{\sigma}\sigma_j\eta_F^{U(1)})(\mu \circ l)$ with $V \otimes (T_{(f)H}(R_F\backslash G))$ as the typical fibre.

Theorem 1. *With any solvable $(\tilde{\sigma}, \eta_F)$-polarization $(\tilde{L}, \rho, \sigma_0)$ there exists a natural unitary representation of G in a Hilbert space $\mathcal{H}_{\tilde{L}}$*

P r o o f. According to Proposition 3.2 in [8], and since k_* is the linear lifting isomorphism for each $f' \in R_F\backslash G$ there is a canonical linear map

$$W_{f'} : (T_{(f')H}(R_F \setminus G))^{\mathbf{C}} \to \mathrm{End}\,(\mathcal{E}'_{\sigma,\rho}(p))_{f'}$$

such that if $\tilde{X}, \tilde{Y} \in (T_{(f')H}(R_F \setminus G))^{\mathbf{C}}$ then

$$[W_{f'}(\tilde{X}), W_{f'}(\tilde{Y})] = -\frac{i}{\hbar}\tilde{B}_\Omega(f')(\tilde{X}, \tilde{Y}).$$

We assume that $\dim_R T_{(f)H}(R_F\backslash G) = 2m$, and the top exterior power K^L is a complex line bundle in $\Lambda^m(T_{(f)H}(R_F \setminus G))^{\mathbb{C}}$ with the basis vector denoted by $\mathcal{K}_{\tilde{L}}$. Putting

$$\{\mathcal{E}'_{\sigma,\rho}(P)\}^{\tilde{L}}_{f'} = \left\{\varphi \in \mathcal{E}'_{\sigma,\rho}(P)_{f'} / \tilde{X} \in \tilde{L}_{f'} \Rightarrow W_{f'}(\tilde{X})\varphi = 0\right\}$$

then $\mathcal{E}'_{\sigma,\rho}(P)^{\tilde{L}}$ is a tensor product of the complex line bundle $\mathcal{E}'(P)^{\tilde{L}}$ with the bundle $P(\tilde{\sigma}\sigma_j\eta_F^{U(1)})$ associated with the representation $\tilde{\sigma}\sigma_j\eta_F^{U(1)}$. Arguing as in Theorem 6.9 in [6] and Proposition 3.3 in [8], there exists a canonical isomorphism of complex bundles:

$$\mathcal{E}'_{\sigma,\rho}(P)^{\tilde{L}} \otimes \mathcal{E}'(P)^{\tilde{L}} \otimes K^{\tilde{L}} \longrightarrow P(\tilde{\sigma}\sigma_j\eta_F^{U(1)}).$$

By putting $Q(P)^{\tilde{L}} = \mathcal{E}'_{\sigma,\rho}(P)^{\tilde{L}} \otimes K^{\tilde{L}}$, we have $[Q(P)^{\tilde{L}}]^2 = [P(\tilde{\sigma}\sigma_j\eta_F^{U(1)})]^2 \otimes K^{\tilde{L}}$. As in the above section, $\tilde{\nabla}^\Gamma$ is the connection in $P(\tilde{\sigma}\sigma_j\eta_F^{U(1)})$, it uniquely defines a connection denoted by $\tilde{\nabla}^{\tilde{L}}$ in $Q(P)^{\tilde{L}}$. Denote by $S_{\tilde{L},S_F}(Q(P)^{\tilde{L}})$ the space of all the sections of $Q(P)^{\tilde{L}}$ for which $\tilde{\nabla}^{\tilde{L}}_{\tilde{\xi}} s = 0$, $\forall \tilde{\xi} \in \tilde{L}$ and S_F-equivariant. Put $H_{\tilde{L}} = \{s \in S_{\tilde{L},S_F}(Q(P)^{\tilde{L}}) /$ the density $\langle s, s\rangle_{\tilde{L}}$ has compact support $\}$, denote by $\mathcal{H}_{\tilde{L}}$ the Hilbert space which is the complection of the space $H_{\tilde{L}}$. The natural unitary representation of G in $\mathcal{H}_{\tilde{L}}$ is the required representation. □

4. A REDUCTION OF THE PROCEDURE OF MULTIDIMENTSIONAL QUANTIZATION

We use the bundle $Q(P)^{\tilde{L}} = P(\tilde{\sigma}\sigma_j\eta_F^{U(1)}) \otimes \mathcal{E}'(P)^{\tilde{L}} \otimes K^{\tilde{L}}$ to construct the procedure of quantization that is defined as follows:

$$\widehat{(\,)} : C^\alpha(\Omega) \longrightarrow \mathcal{L}(\mathcal{H}_{\tilde{L}}),$$

$$f \mapsto \hat{f} = f + \frac{\hbar}{i}\tilde{\nabla}^{\tilde{L}}_{\tilde{\xi}_f},$$

where $\mathcal{L}(\mathcal{H}_{\tilde{L}})$ is the space of all (un) bounded Hermitian operators on $\mathcal{H}_{\tilde{L}}$ and $\tilde{\nabla}^{\tilde{L}}_{\tilde{\xi}}$ is the covariant derivation associated with the connection $\tilde{\nabla}^{\tilde{L}}$ on the G-bundle $Q(P)^{\tilde{L}}$. We recall that $\tilde{\nabla}_{\tilde{\xi}}$ is defined by the formula:

$$\tilde{\nabla}^{\tilde{L}}_{\tilde{\xi}_f} = L_{\tilde{\xi}_f} + \frac{i}{\hbar}\alpha(\tilde{\xi}_f),$$

where α is 1-form of connection $\nabla^{\tilde{L}}_{\tilde{\xi}_f}$, $L_{\tilde{\xi}_f}$ is the Lie derivation along $\tilde{\xi}_f$ which is the horizontal lifting of strictly hamiltonian vector field corresponding to f.

Remark. As discussed above if $\tilde{\Gamma}$ denotes the connection on $R_F^{U(1)} \to G^{U(1)} \to R_F \setminus G$, then the connection form α can be computed from the relation $\frac{i}{\hbar}\alpha(\tilde{\xi}_f) = \rho(p_{\tilde{\Gamma}}, \tilde{\xi}_f)$, where $p_{\tilde{\Gamma}}$ is the projection onto the vertical directions. These operators are given by the integral kernel from Proposition 2.4. in view of Definition 2.4, Proposition 2.5.

By a similar argument as in [3] and [8] we obtain the follwing result.

Proposition 4.1. *The following three conditions are equivalent:*

i) The application $f \mapsto \hat{f}$ is a procedure of quantization,

ii) curv $\tilde{\nabla}^{\tilde{L}}(\tilde{\xi}, \tilde{\eta}) = -\frac{i}{\hbar}\tilde{B}_\Omega(\tilde{\xi}, \tilde{\eta}) \cdot I$,

iii) $d_{\tilde{\nabla}^{\tilde{L}}}\alpha(\tilde{\xi}, \tilde{\eta}) = -\tilde{B}_\Omega(\tilde{\xi}, \tilde{\eta}) \cdot I$,

where $\tilde{\xi}, \tilde{\eta}$ are the horizontal lifting of strictly hamiltonian field ξ, η on Ω

Theorem 2. *If curv $\tilde{\nabla}^{\tilde{L}} = -\frac{i}{\hbar}\tilde{B}_\Omega$ then with any solvable $(\tilde{\sigma}, \eta_F^{U(1)})$-polarization $(\tilde{L}, \rho, \sigma_0)$ the multidimensional quantization procedure gives us a representation ψ which lifts to the representation Ind $(G; \tilde{L}, (\tilde{\sigma}\sigma_j)\eta_F^{U(1)}, \sigma_0, \rho)$.*

P r o o f. Suppose that curv$\tilde{\nabla}^{\tilde{L}} = -\frac{i}{\hbar}\tilde{B}_\Omega$. From Proposition 4.1. we obtain the following representation of Lie algebra in the space $\mathcal{L}(\mathcal{H}_{\tilde{L}})$

$$\Lambda : \mathcal{G} \to \mathcal{L}(\mathcal{H}_{\tilde{L}});\ X \mapsto \Lambda(X) = \frac{i}{\hbar}\hat{f}_X,$$

where $f_X \in C^\infty(\Omega)$ is the generating function of the hamiltionian field ξ_X corresponding to $X \in \mathcal{G}$. If G is connected and simply connected we obtain a unitary representation T of G defined by

$$T(\exp X) = \exp(\Lambda(X)), X \in \mathcal{G}.$$

On the other hand, we define the representation ψ as follows

$$\psi : \mathcal{G} \to \mathcal{L}(\mathcal{H}_{\tilde{L}}),$$

$$X \mapsto \psi(X) = L_{\tilde{\xi}_X} + \frac{i}{\hbar}\{\varphi_X + \alpha(\tilde{\xi}_X)\},$$

where α is the 1-form of $\tilde{\nabla}^{\tilde{L}}$ such that $\frac{\hbar}{i}\alpha$ is the connection form of $\tilde{\nabla}^{\tilde{L}}$, and $\varphi_X : \Omega \longrightarrow \mathbf{R}$ is defined by $\varphi_X(F) = F(X)$. In the group level this representation

ψ lifts to the representation Ind $(G, \tilde{L}, (\tilde{\sigma}\sigma_j)\eta_F^{U(1)}, \sigma_0, \rho)$. Therefore, it suffices to prove that the representation Λ coincides with the representation ψ. We have

$$-\frac{i}{\hbar}\hat{\varphi}x = \frac{i}{\hbar}(\varphi x + \frac{\hbar}{i}\tilde{\nabla}^{\tilde{L}}_{\tilde{\xi}x}) = \frac{i}{\hbar}\varphi x + \tilde{\nabla}^{\tilde{L}}_{\tilde{\xi}x}$$

$$= L_{\tilde{\xi}x} + \frac{i}{\hbar}\{\varphi x + \alpha(\tilde{\xi}x)\}$$

$$= \psi(X),$$

the Theorem 2 is proved. □

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Deparment of Mathematics,
Pedagogical Institute, Hue,
Vietnam.

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