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SOME RESULTS ON QUASI–FROBENIUS RINGS

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Abstract. *Quasi–Frobenius rings (briefly, QF–rings) form an important class of non-semisimple artinian rings. Many characterizations of QF–rings by means of small and non-small idempotents in a left and right artinian ring were obtained by Harada [7], [8]. The purpose of this paper is to give similar characterizations of a QF–ring R satisfying weaker assumptions. We also obtain a result of QF–quotient rings which extends Theorem 6.18 of [2].*

1. DEFINITIONS AND NOTATIONS

We assume throughout that all rings are associative with identity and all modules are unitary. We first recall some notions used in the paper. For a module M we denote by $E(M)$, $J(M)$, $Z(M)$ and $S(M)$ the injective hull, the Jacobson radical, the singular submodule and the socle of M, respectively. For a subset A of a ring $R, r(A)$ and $l(A)$ denote the right and left annihilators of A in R, respectively. A module M is called a CS–module if for every submodule A of M (denoted by $A \hookrightarrow M$) there exists a direct summand A^* (denoted by $A^* \overset{\oplus}{\hookrightarrow} M$) containing A such that A is essential in A^* (denoted by $A \underset{e}{\hookrightarrow} A^*$). M is called a continuous module if M is a CS–module and for every submodule A and B of M with $A \cong B$ and $B \overset{\oplus}{\hookrightarrow} M$ implies $A \overset{\oplus}{\hookrightarrow} M$. A ring R is called left (right) continuous if R is as a left (right, respectively) R–module continuous.

A module M is called a small module if M is small in $E(M)$, i.e. for any proper submodule H of $E(M)$, $H + M \neq E(M)$. If M is not small, M is called non–small. Let e be an idempotent of R, then e is called non–small if eR_R is a non-small module. Dually, M is called a cosmall module if for any projective module I and any epimorphism $f : P \longrightarrow M$, $ker(f)$ is essential in P. i.e. for each non–zero submodule H of P, $ker(f) \cap H \neq 0$. If M is not cosmall, M is called non–cosmall module (see e.g. [7], [13]). M is called hollow if every proper submodule of M is small in M. If for any direct decomposition of $S(M) : S(M) = \sum_I^{\oplus} A_\alpha$ there exists a direct decomposition $M = \sum_I^{\oplus} M_\alpha$ of M such that $S(M_\alpha) = A_\alpha$ for all $\alpha \in I$, then we say that M has the extending property of direct decomposition of socle $S(M)$.

Let M be an R-module and I a right ideal of R. We take an R-homomorphism f of I to M. Consider a diagram:

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, "i"] \arrow[d, "f"'] & R \arrow[dl, dashed, "h"] \\ & M & \end{tikzcd}$$

where i is the inclusion. M is called a mini-injective (uni-injective) module if there exists $h \in Hom_R(R, M)$ such that $hi = f$ for every minimal right ideal of R (every uniform right ideal I of R, respectively). It is clear that injective $\rightarrow$ uni-injective $\Rightarrow$ mini-injective. The converse is not true in general (see [8, Example 5]).

In [7], Harada studied the following conditions:

$(*)$ Every non-small right R-module contains a non-zero injective submodule.

$(*)^*$ Every non-cosmall right R-module contains a non-zero projective direct summand.

In [10], Oshiro defined H-rings and *co*-H-rings related to $(*)$ and $(*)^*$ respectively. A ring R is called a right H-ring if R is right artinian and R satisfies $(*)$. Dually, R is called a right *co*-H-ring if R satisfies $(*)^*$ and the ACC on right annihilators.

Let R be a ring. R is said to be right QF-2 if $S(eR)$ is simple for every primitive idempotent e, and R is called right QF-3 if the injective hull $E(R_R)$ of R_R is projective. Left QF-n $(n = 2, 3)$ are defined similarly. N denotes the nilpotent radical of R. For an ideal I of R we write:

$C(I) = \{c \in R : c + I \text{ is a regular element of } R/I\}$. Hence $C(0)$ is the set of regular elements of R. Right (left) reduced rank of R is denoted by $\rho(R_R)(\rho({}_RR)$, respectively) (see [2], [6] for the definition)

We refer to the books [5] and [14] for other interesting properties of QF-rings and their generalizations.

2. WHEN ARE RIGHT H-RINGS AND RIGHT CO-H-RINGS QUASI-FROBENIUS?

Assume R is right perfect. Then there exists a complete set $\{g_i\}$ of mutually orthogonal primitive idempotents such that $1 = \sum g_i$. We can devide $\{g_i\}$ into two parts $\{e_i\}_{i=1}^n \cup \{f_j\}_{j=1}^m$, where each e_iR is non–small and each f_jR is small. Now if we denote the primitive idempotents by e and f, we mean that e is non–small and f is small.

First we prove the following Lemma:

Lemma 2.1. *If R is a right perfect ring satisfying $(*)$ such that $e_iRf_j = 0$ for every non–small e_i and small f_j and R has ACC for right annihilators, then R is QF.*

P r o o f. Since R satisfies $(*)$, to prove that R is right self–injective it suffices to show that R does not contain any small primitive idempotent.

Assume on the contrary that f is a small primitive idempotent. Then by [7, Lemma 1.2], we have an exact sequence

$$\sum{}^{\oplus} e_kR \longrightarrow E(fR) \longrightarrow 0$$

for some e_k

Consider the following diagram:

$$\begin{tikzcd} & fR \arrow[dl, dashed, "k"'] \arrow[d, "i"] & \\ \sum{}^{\oplus} e_kR \arrow[r] & E(fR) \arrow[r] & 0, \end{tikzcd}$$

where i is the inclusion. Since fR is projective, there exists a homomorphism h such that the diagram is commutative. Howewer, because i is monomorphic, so is h.

By assumption, $h(fR)fR \hookrightarrow (\sum^{\oplus} e_iR)fR = 0$. It follows that $h[(fR)^2] = 0$, hence $(fR)^2 = 0$. Thus $f^2 = f = 0$. By [5, Theorem], R is then QF. The proof of Lemma 2.1 is complete.

Harada [7, Proposition 2.6] characterized QF–rings by right H–rings and the two–sided artinian rings. This can be extended as follows:

Theorem 2.2. *For a ring R, the following conditions are equivalent:*

1) R is a QF–ring

2) R is a right perfect ring satisfying $(*)$ such that $e_iRf_j = 0$ for every non–small e_i and small f_j and R has ACC or DCC for right annihilators.

3) R is a right noetherian ring satisfying $(*)$ and $l(J) \hookrightarrow r(J)$.

4) R is a left and right perfect ring such that e_iR is injective and $e_iR/e_il(J)$ is small whenever $e_il(J) \neq 0$ for every non–small e_i.

P r o o f. 1) $\Rightarrow$ 3) and 1) $\Rightarrow$ 4) see [7, Proposition 2.6].

1) $\Leftrightarrow$ 2). By Lemma 2.1.

3) $\Rightarrow$ 1). Since R is a right noetherian and R satisfies $(*)$, R is right artinian by [7, Proposition 2.1]. Note that R satisfies $(*)$, it follows that e_iR is injective for every non-small primitive idempotent e_i. Thus R is right QF-3 by using [7, Theorem 1.3]. By [11, Lemma 3.4], R is a QF-ring.

4) $\Rightarrow$ 1). Assume 4). Then R is right artinian by [10, Theorem 2.11]. Assume that $J^n = 0$ and $J^{n-1} \neq 0$. By condition 4) we see that $tR/tl(J^k)$ is small, $k = 1, ..., n-1$, where t is a primitive idempotent. In fact, if $t = e$, then $eR/el(J)$ is smalll and $(eR/el(J).r(J) = 0$ by [13, Proposition 4.8], hence $eR.r(J) \hookrightarrow el(J) \hookrightarrow el(J^k), k = 1, ..., n-1$; it follows that $eR/el(J^k)$ is small. Similarly, if $t = f$, then $fR/fl(J^k)$ is small, where $k = 0, 1, ..., n-1$. Hence the claim is verified.

Now, let f be a non-zero small idempotent. Since R is right artinian; $E := E(fR)$ can be expressed as $E = E_1 \oplus \cdots \oplus E_h$ with each E_i indecomposable cyclic hollow by [10, Proposition 2.10]. Also by [10, Proposition 2.10], there exist primitive idempotents $t_1, ...t_h$ and integers $k_1, ..., k_h \in \{0, 1, ..., n-1\}$ such that $E_i \cong t_iR/t_il(J^{k_i})$. Now if there is an i with $k_i = 0$, then $E_i \cong t_iR$, therefore t_i is non-small. But the injectivity of t_iR yields $E(fR) \cong t_iR$, a contradiction since fR is small in $E(fR)$. Thus $k_i \neq 0$ for every i. However as we showed above, each $t_iR/t_il(J^{k_i})$ is small, a contradiction to the injectivity of E_i. Thus $f = 0$.

Using condition 4) and the above claim we see that R is right self-injective, hence R is QF. Thus the proof is complete.

In terms of continuous rings, H-or $co-H$-rings, we have other characterizations of QF-rings.

Theorem 2.3. *For a ring R the following assertions are equivalent:*

1) R is QF.

2) R is right continuous ring such that R has ACC on right annihilators and $R_R \oplus R_R$ is a right CS-module.

3) R is a right continuous, right $co-H$-ring.

4) R is a right continuous, right H-ring.

P r o o f. 1) $\Rightarrow$ 2) is obvious.

2) $\Leftrightarrow$ 3). By [9, Theorem 3.4], R is semiprimary. By [12, Theorem II], 2) $\Leftrightarrow$ 3).

3) $\Rightarrow$ 1). Assume 3). Then $J = Z(R_R)$ follows from [15, Theorem 4.6]. Hence R is QF by [10, Theorem 4.3].

1) $\Rightarrow$ 4) is obvious.

4) $\Rightarrow$ 1) is similar to 3) $\Rightarrow$ 1).

Note that if R is a right CS-ring, then every cyclic right R-module is a direct sum of a singular module and a projective module. However every 2-generated right R-module does not have this property, in general. Since, if not we can derive from this that $R_R \oplus R_R$ is a CS–module. Then by Theorem 2.3 we see that a right continuous ring satisfying ACC on right annihilators is QF. But this is imposible by Example 3.11 of [9]. If follows that a right continuous ring need not satisfy condition $(*)^*$.

The following lemma is essentially stated in [8, Theorem 13.7].

Lemma 2.4. *If R is a right QF-2 ring and R_R is uni–injective, then R is right self–injective.*

Harada [8] characterized QF–rings by using the concept of mini–injectivity and uni–injectivity for right and left artinian rings R. We extend his results as follows.

Theorem 2.5. *Let R be a ring. Then the following conditions are equivalent:*

1) R is a QF–ring.

2) R is a right artinian ring and every projective right R–module and every projective left R–module has the extending property of direct decomposition of the socle.

3) R is a right and left QF-2, right artinian ring and right self–mini–injective.

4) R is a right self–mini–injective, right H–ring.

5) R is a right noetherian ring satisfying $(*)^*$ and R is right self–mini–injective.

6) R is right QF-2 ring satisfying ACC on right or left annihilators and R_R is uni–injective.

P r o o f. 1) $\Rightarrow$ 2) see [8, Theorem 13].

2) $\Rightarrow$ 3). By [8, Theorem 3], R is right QF-2 and right self–mini–injective. Since every projective left R–module has the extending property of direct decomposition of the socle, R is left QF-2.

3) $\Rightarrow$ 1). If R_R is artinian and mini–injective, then by [8, Theorem 5], $l(J) \hookrightarrow r(J)$. Since R is right artinian and R is left and right QF-2 we see that both R_R and ${}_RR$ are direct sum of uniform modules. This together with $l(J) \hookrightarrow r(J)$ and [11, Theorem 3.5] shows that R is QF.

1) $\Rightarrow$ 4) and 1) $\Rightarrow$ 5) are obvious.

4) $\Rightarrow$ 1). By [10, Proposition 2.10], R is semiprimary QF-3. Moreover, $l(J) \hookrightarrow r(J)$ follows from [8, Theorem 5]. Hence R is QF by [11, Lemma 3.4].

5) $\Rightarrow$ 1). By [10, Theorem 3.18], R is semiprimary QF-3 ring. Hence R is right artinian because R is right noetherian by assumption of 5). Moreover, from [8, Theorem 5] it follows $l(J) \hookrightarrow r(J)$. Hence R is QF by [11, Lemma 3.4].

1) $\Rightarrow$ 6) is obvious.

6) $\Rightarrow$ 1). By Lemma 2.4 and [5, Theorem 24.20], R is then QF.

Note that the "two-sided QF-2" condition in 3) of Theorem 2.5 cannot be reduced to a one-sided QF-2 ring. Harada [8, Example 5.2] gave an example of an artinian, right self-mini-injective, left QF-2 ring R which is however not right QF-2. This means that R is not QF.

3. QF-QUOTIENT RINGS

Following [2, Theorem 6.18] we consider the following property (P) of a ring R. A ring R is said to satisfy (P), if

(i) R contains a direct sum of uniform right ideals which contains a right regular element.

(ii) R contains a direct sum of uniform left ideal which contains a left regular element.

(iii) $l(N) \hookrightarrow r(N)$, where N is the nilpotent radical of R.

Theorem 3.1. *Let R and R/N be right and left Goldie rings such that $\rho(R_R)$ is finite and R satisfies left Ore condition. Then R has QF-quotient ring if and only if R satisfies (P).*

P r o o f. If R has QF-quotient ring then R satisfies (P) can be proved easily. Conversly, assume R satisfies (P). Since R and R/N are right Goldie together with i) and by [2, Lemma 6.14], it follows that the elements of $C(N)$ are right regular. But right-left symmetry we also obtain that elements of $C(N)$ are left regular. Thus $C(N) \subseteq C(O)$, where $C(N)$ is the set of regular elements modulo N. By [6, Exercise 10.F], $C(O) \subseteq C(N)$. Hence $C(O) = C(N)$. Also by [6, Exercise 10. G], R has a right artinian right quotient ring Q. However Q is also a left quotient ring of Q because R satisfies left Ore condition.

What properties does Q have ?. First note that (i) and (ii) also hold in Q. But right and left regular elements of Q are units of Q, therefore Q_Q and ${}_QQ$ are direct sums of uniform right ideals and uniform left ideals, respectively. Moreover, let J' be the Jacobson radical of Q. Then by [2, Theorem 9.2], $J' = NQ = QN$. We want to prove that $l_Q(J') \hookrightarrow r_Q(J')$. In fact, if $q \in Q, qJ' = 0$, then $qN = 0$. Write $q = ac^{-1} = d^{-1}b$ for some regular elements c, d, then $bN = 0$, i.e. $b \in l(N) \hookrightarrow r(N)$. Thus $Nb = 0$ or $QNb = J'b = 0$. It follows that $J'dac^{-1} = 0$, hence $J'da = 0$ or $NQda = 0$. Thus $NQd^{-1}da = NQa = 0$. Then $NQac^{-1} = NQq = J'q = 0$. This shows $q \in r_Q(J')$. Since Q is semiprimary, $l_Q(J') = Soc(Q_Q) \hookrightarrow r_Q(J') = Soc({}_QQ)$.

Now, by [11, Theorem 3.5] R is then QF, completing the proof of Theorem 3.1.

Corollary 3.2. *Let R have right and left Krulll dimension and satisfy ACC and DCC on right annihilators. Then R has a QF-quotient ring if and only if R satisfies (P).*

Corollary 3.3.([2, Theorem 6.18]). *Let R be a left and right noetherian ring then R has a QF-quotient ring if and only if R satisfies (P).*

By the same argument of proving Theorem 3.1, we obtain:

Proposition 3.4. *Let R and R/N be right non-singular ring having finite Goldie dimension. Moreover, $\rho(R_R) < \infty$ and R satisfies left Ore condition, then R has QF-quotient ring if and only if R satisfies (P).*

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