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ON THE OPENNESS OF THE LOCUS OF POINTS HAVING POLYNOMIAL TYPE BOUNDED ABOVE BY A CONSTANT

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Dedicated to the memory of Professor Le Van Thiem

Abstract. *The aim of this note is to examine necessary and sufficient conditions for the locus of points having the polynomial type bounded above by a constant to be an open set in the Zariski topology.*

1. INTRODUCTION

U. Daepp and A. Evans [D-E] have given a criterion for the openness of the Generalized Cohen- Macaulay (abbr. GCM) loci of a factor ring of a Cohen-Macaulay (abbr. CM) ring. In particular, from this paper one can easily give examples showing that Nagata criterion is not valid for the Buchsbaum or GCM property (cf. [K]). Inspired of their results we study the openness of loci of points having *polynomial type* bounded above by a constant. The *polynomial type* of a local ring was introduced first in $[C_1]$ and $[C_3]$ as follows:

Let (A, m) be a local Noetherian ring with $\dim A = d$, $x = \{x_1, ..., x_d\}$ a system of parameters of A and $n = (n_1, ..., n_d)$ a d-tuple of positive integers. We set

$$I_A(n,x) = l(A/(x_1^{n_1}, ..., x_d^{n_d})A) - n_1...n_d e(x, A).$$

In answering a question of R. Y. Sharp, the first author has shown in $[C_1]$ and $[C_3]$ that the least degree of all polynomials in n bounding above $I_A(n,x)$ is a finite number and independent of the choice of x . We call this new invariant the *polynomial type* of A and denote it by $p(A)$.

This invariant shows how far a local ring is from being a CM ring.

Example 1. i) A is CM iff $p(A) = -1$.
ii) A is GCM iff $p(A) \leq 0$.

Let R be a Noetherian ring. We call the polynomial type of R the number $p(R) := \sup\{p(R_Q); Q \in \operatorname{Spec}(R)\}$.

For any integer $k \geq -1$ we set

$$p_{\leq k}(R) = \{P \in \operatorname{Spec}(R); p(R_P) \leq k\}.$$

The main result of this paper is the following theorem.

Theorem 1.1. *Let R be a homomorphic image of a CM ring. If $p(R) \leq 1$, then $p_{\leq 0}(R)$ (the set of GCM points of R) is open in Spec(R) if and only if every prime ideal contained in $p_0(R)$ has only finitely many over prime ideals.*

If $p(R) \geq 2$, then $p_{\leq k}(R)$ is never open in Spec(R) for every integer k with $0 \leq k \leq p(R) - 2$.

To prove this, in Section 2 we will study a more general situation, namely the openness of loci defined by a function $f : \operatorname{Spec}(R) \longrightarrow Z$, where Z is the set of all integers. In the last section we apply the results of Section 2 to the function defined by the polynomial types in order to prove the main Theorem.

2. OPENNESS OF LOCI

Let $f : \operatorname{Spec}(R) \longrightarrow Z$ be a function satisfying the following properties :

(i) f is bounded below,

(ii) $f(P) \leq f(Q)$ for any $P \subsetneq Q$ from Spec(R). Moreover, if $f(Q) > n_f := \min\{f(Q');\ Q' \in \operatorname{Spec}(R)\}$, then $f(P) < f(Q)$.

For any integer $k \geq n_f$ we set

$$f_{\leq k}(R) = \{P \in \operatorname{Spec}(R); f(P) \leq k\},$$

$$f_{>k}(R) = \{P \in \operatorname{Spec}(R); f(P) > k\}$$

and

$$f_k(R) = \{P \in \operatorname{Spec}(R); f(P) = k\}.$$

For any ideal J of R, we denote by $V(J)$ (resp. $V_m(J)$) the set of prime (resp. minimal prime) ideals lying over J. Let $\Omega(R) \subseteq \operatorname{Spec}(R)$ denote the subspace of maximal ideals of R with the induced topology.

Theorem 1. *Assume that $f_{>k}(R)$ is closed in Spec(R). Then $f_{>k+1}(R) \cap \Omega(R)$ is closed in $\Omega(R)$.*

P r o o f. By the assumption, there exists an ideal J of R such that $f_{>k}(R) = V(J)$. We set

$$\aleph = V_m(J) \setminus \big(f_{k+1}(R) \cap \Omega(R)\big).$$

Of course, we can assume that $f_{>k+1}(R) \neq \emptyset$. Then $\aleph \neq \emptyset$. Let

$$a = \bigcap_{P \in \aleph} P.$$

In oder to prove the theorem we only need to show that

$$f_{>k+1}(R) \cap \Omega(R) = V(a) \cap \Omega(R).$$

The inclusion $\subseteq$ is clear.

To prove the inclusion $\supseteq$, let $P \in V(a) \cap \Omega(R)$. Take $Q \in \aleph$ such that $Q \subseteq P$. Since $Q \in V(J) = f_{>k}(R)$, then $f(Q) > k$. If $Q \neq P$, from the property (ii) of f we get $f(P) > k+1$. Now let $Q = P$. Since $Q \in \aleph$ and $Q \in \Omega(R)$, then $Q \notin f_{k+1}(R)$. Therefore $f(P) > k+1$. Also, in both cases we have $P \in f_{>k+1}(R) \cap \Omega(R)$. This completes the proof.

Theorem 2. *Let $k > n_f$ be an integer. If $f_{\leq k}(R)$ is open in Spec(R), then every $P \in f_k(R)$ has only finitely many over prime ideals. The converse is true under the assumption that $f_{\leq k-1}(R)$ is open in Spec(R).*

P r o o f. Let D denote the set of all prime ideals having only finitely many over prime ideals. For an arbitrary $P \in \mathrm{Spec}(R) \setminus \Omega(R)$, let $I(P)$ denote the intersection of all minimal prime ideals properly containing P.

To prove the first statement we will show that $f_{>k}(R)$ is not closed if $f_k(R) \not\subseteq D$. Take a prime $P \in f_k(R) \setminus D$. Let U be an arbitrary open neighbourhood of P. Then $\mathrm{Spec}(R) \setminus V(x) \subseteq U$ for some element $x \in R \setminus P$. Note that for any prime ideal P' with $P' \neq I(P')$ each minimal over prime of P' is an associated prime ideal of $I(P')$. In particular, it follows that P' has only finitely many minimal over primes. On the other hand, if $\dim R/P' > 1$, it is easily seen that P' has infinitely many over prime ideals. Since $P \notin D$ it follows that $P = I(P)$. Hence there exists a prime $Q \supseteq P$ such that $x \notin Q$. Since $P \in f_k(R)$, we get from the property (ii) of f that $f(Q) \geq k+1$ which implies that $Q \in f_{>k}(R) \cap \mathrm{Spec}(R) \setminus V(x)$ .This shows that $(U \cap f_{>k}(R)) \neq \emptyset$, i.e. $f_{>k}(R)$ is not closed.

To prove the second statement we may assume that $f_{>k-1}(R)$ is closed and $f_k(R) \subseteq D$. Let $f_{>k-1} = V(J)$. We set

$$b_1 = \bigcap_{P \in V_m(J) \setminus f_k(R)} P,$$

$$b_2 = \bigcap_{Q \in f_k(R) \setminus \Omega(R)} I(Q)$$

and

$$b = b_1 \cap b_2. \tag{*}$$

It suffices to prove that $f_{>k}(R) = V(b)$. To prove the inclusion $\subseteq$, let $P \in f_{>k}(R)$. Since $P \in V(J)$, there exists $Q \in V_m(J)$ such that $Q \subseteq P$. If $Q \notin f_k(R)$, then $b \subseteq Q \subseteq P$. If $Q \in f_k(R)$, then $Q \subsetneq P$ and $Q \in f_k(R) \setminus \Omega(R)$. Hence $b \subseteq I(Q) \subseteq P$. Also, in both cases we get that $P \in V(b)$.

Now we prove the inclusion $\supseteq$. Let $P \in V(b)$. First note that $f_k(R) \subseteq V_m(J)$. Thus there are only finitely many prime ideals in the right side of (*). If there exists $Q \in V_m(J) \setminus f_k(R)$ such that $Q \subseteq P$, then we are done by using the property (ii) of f . Otherwise, we can find $Q' \in f_k(R) \setminus \Omega(R)$ such that $Q' \subseteq I(Q') \subset P$. By the assumption, $Q' \in D$. From this it follows that $Q' \neq I(Q')$. Hence $Q' \subsetneq P$ which implies that $f(P) > f(Q) = k$, as required.

3. OPENNESS OF LOCI VIA POLYNOMIAL TYPE

We begin this section with a lemma showing that the function

$$p : \mathrm{Spec}(R) \longrightarrow Z : Q \mapsto p(R_Q)$$

satisfies the conditions of functions considered in the Section 2.

Lemma 3. *Let $P \subsetneq Q$ be two prime ideals of R . Assume that R_Q is a non- CM ring. Then $p(R_P) < p(R_Q)$.*

P r o o f. If $\dim(R_Q/PR_Q) > p(R_Q)$, then by $[C_2$, Corollary 3.6], $(R_Q)_{PR_Q} \simeq R_P$ is a CM ring. Since R_Q is a non-CM ring, $p(R_P) = -1 < p(R_Q)$. If $\dim(R_Q/PR_Q) \leq p(R_Q)$, then by $[C_2$, Corollary 5.2], we get

$$p(R_P) \leq p(R_Q) - \dim(R_Q)/PR_Q) < p(R_Q).$$

Lemma 4. *($[C_2$, Theorem 4.1]). Let (A, m) be a homomorphic image of a CM ring and k a positive integer. Then the following conditions are equivalent:*

i) $p(A) \leq k$;

ii) For any $P \in SpecA$ with $\dim(A/P) > k$, A_P is a CM ring.

The following lemma extends [C-S-T, Satz 3.8]. It also gives an inductive definition for $p(A)$ when A is a homomorphic image of a CM ring.

Lemma 5. *Let (A, m) be as in Lemma 4. Then*

$$p(A) \leq \sup \{p(A_Q); Q \neq m\} + 1.$$

Moreover, if A is non- CM , then the equality holds.

P r o o f. We set $k = \sup\{p(A_Q); Q \neq m\} + 1$. The case $k = 0$ has been proved in [C-S-T, Satz 3.8]. Now we assume that $k > 0$. We will use Lemma 4 to show that $p(A) \leq k$. Let $P \in \mathrm{Spec}(R)$ with $\dim(A/P) > k$. Choose a prime ideal Q such that $P \subsetneq Q \subsetneq m$ and $\dim(A_Q/PA_Q)) > p(A_Q)$. Applying Lemma 4 to A_Q we get that A_P is CM. Hence, again by Lemma 4, $p(A) \leq k$.

The second statement of the lemma follows from Lemma 3.

Now we can prove the main result.

Theorem 6. *Let R be a homomorphic image of a CM ring . If $p(R) \leq 1$, then $p_{\leq 0}(R)$ is open in Spec(R) if and only if every prime contained in $p_0(R)$ has only finitely many over prime ideals. If $p(R) \geq 2$, then $p_{\leq k}(R)$ is never open in Spec(R) for every integer k with $0 \leq k \leq p(R) - 2$.*

P r o o f. It is well- known that $p_{-1}(R)$, the set of CM points, is open in Spec(R). Therefore the first statement is deduced from Lemma 3 and Theorem 2. To show the second one it should be noted that any prime ideal having only finitely many over prime ideals must have *coheight* ≤ 1 . Thus, if $p_{\leq k}(R)$ is open in Spec(R) , then by Theorem 2 and Lemma 5 we must have $p(R) \leq k + 1$. The proof is complete.

Theorem 6 and Lemma 4 give a criterion for the openness of the set of GCM points of a factor ring of a CM ring as follows:

Corollary 7. *Let (A, m) be as in Lemma 4. Then the set of GCM points of A is open in SpecA if and only if $p(A) \leq 1$.*

Example 2. For $d > 1$, let $B_d = k[[Y_1, ..., Y_{d+1}]]/(Y_1Y_{d+1}, ..., Y_dY_{d+1})$, where k is a field and $Y_1, ..., Y_{d+1}$ are indeterminates. We denote by x_i the natural image of $Y_i + Y_{d+1}$ in B_d, $i = 1, ..., d$, then $x = \{x_1, ..., x_d\}$ forms a system of parameters for B_d. It can be verified that

$$I_{B_d}(n, x) = \min\{n_1, ..., n_d\}.$$

Therefore $p(B_d) \leq 1$. Hence the set of GCM points of B_d is open in SpecB_d.

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