$$\mu_{a_1,\dots,a_n}(B) = \mu\{x \in V : [(x,a_1),\dots,(x,a_n)] \in B\}.$$

The family $\{\mu_{a_1,\dots,a_n}\}$ is consistent. Conversely, each consistent family of probability measures on spaces R^n $(n = 1, 2, \dots)$ defines a c.m. An important method to produce c.m.'s is as follows: Let M be a linear mapping from V' into $L_0(\Omega)$. It is easy to verify that the family $\{\mu_{a_1,\dots,a_n}\}$ given by $\mu_{a_1,\dots,a_n} = \mathcal{L}(Ma_1,\dots,Ma_n)$ is consistent so it generates a c.m. on V. Conversely, every c.m. on V is generated by some linear mapping from V' into $L_0(\Omega_1)$ where $(\Omega_1, \mathcal{F}_1, P_1)$ is an appropriate probability space. The characteristic function of a c. m. is defined by $\hat{\mu}(a) = \int_R \exp\{it\} d\mu_a(t)$, $a \in V'$.

If μ is a c.m. on V then a Radon measure $\overline{\mu}$ on V is said to be a Radon extension of μ if $\overline{\mu}(Z) = \mu(Z)$ for each $Z \in \mathcal{C}$. In this case we say that μ is a Radon measure.

Theorem 1.1 *(see [9] p. 296) The c.m. μ on V is a Radon measure if and only if the linear mapping M generating μ is decomposed by a V-valued random vector f in the sense that*

$$\forall a \in V' \qquad P\{\omega : Ma(\omega) = \langle f(\omega), a\rangle\} = 1.$$

The following lemma is generally known but for lack of a suitable reference we prove it here

Lemma 1.2. *Let $f \in L^0_V(\Omega)$. Then the linear mapping $a \to \langle f, a\rangle$ from V' into $L_0(\Omega)$ is continuous w.r.t. the Mackey topology on V'.*

P r o o f. Given $t > 0$ and $\varepsilon > 0$. Let μ be the distribution of f. Since μ is a Radon measure there exists an absolutely convex and compact set K such that $\mu(K) \geq 1 - \varepsilon$. Put $U = \{a \in V' : \sup_{b \in K} |\langle b, a\rangle| \leq t\}$. U is a neighbourhood of the Mackey topology. Then for $a \in U$ we have $P\{\omega : |\langle f(\omega), a\rangle| \leq t\} = \mu\{b : |\langle b, a\rangle| \leq t\} \geq \mu(K) \geq 1 - \varepsilon$ i.e. $P\{\omega : |\langle f(\omega), a\rangle| > t\} < \varepsilon$. The Lemma is proved.

II. RANDOM OPERATORS WITH SAMPLE PATHS IN A SUBSET OF $L(X,Y)$

Recall that by a random operator A from X into Y we mean a linear mapping A from X into $L^0_Y(\Omega)$ such that $\forall \varepsilon > 0$, $\lim_{x_n \to 0} P\{\|Ax_n\| > \varepsilon\} = 0$. $L(\Omega, X, Y)$ denotes the set of all random operators from X into Y. Two random operators A and $B \in L(\Omega, X, Y)$ are called a modification of one another if for each $x \in X$,

$P\{\omega : Ax(\omega) = Bx(\omega)\} = 1$. For each fixed ω, the mapping $x \to Ax(\omega)$ is called a sample path of A.

In what follows, V denotes a Banach space whose members are linear continuous operators from X into Y and it is equipped with a norm stronger than the usual operator norm.

Example 1 (The class of p-summing operators)

An operator $T \in L(X,Y)$ is said to be p-summing $(0 < p < \infty)$ if for each sequence (x_n) in X such that $\sum_{n=1}^{\infty} |(x_n, a)|^p < \infty$ for all $a \in X'$ we have $\sum_{n=1}^{\infty} \|Tx_n\|^p < \infty$. Alternatively, T is p-summing if and only if there exists a constant C such that for each finite sequence (x_n) in X

$$\sum_n \|Tx_n\|^p \leq C^p \sup_{\|a\|\leq 1} \Big\{ \sum_n |(x_n, a)|^p \Big\}.$$

The minimal C for which the above inequality holds is denoted by $\|T\|_\Pi$.

An operator $T \in L(X,Y)$ is said to be completely summing if T is p-summing for all p. We denote by $\Pi_p(X,Y)$ and $\Pi_0(X,Y)$ the class of all p-summing operators from X into Y and the class of all completely summing operators from X into Y, respectively. For $p \geq 1$, $\Pi_p(X,Y)$ is a Banach space under the norm $\|T\|_\Pi$. In the case X and Y are Hilbert spaces $\Pi_p(X,Y)$ is exactly the class of H-S operators. For more information about p-summing operators, see [9].

Example 2. Let (S, Σ, λ) be a finite measurable space. The symbol L_p will always mean $L_p(S, \Sigma, \lambda)$. Following Linde [7] an operator $T \in L(X', L_p)$ $(1 < p \leq 2)$ is said to be a Λ_p - operator if the function $g : X' \to R$ given by $g(a) = \exp\{-\|Ta\|^p\}$ is a characteristic function of a Radon measure μ_T on X. The class of all Λ_p - operators in $L(X', L_p)$ is denoted by $\Lambda_p(X', L_p)$. The set $\Lambda_p(X', L_p)$ becomes a Banach space under the norm defined by $\|T\|_\Lambda = \Big\{ \int \|x\|^r d\mu_T \Big\}^{1/r}$ where $1 \leq r < p$.

Now we are going to introduce the class of H-S type operators and the class of H-S type dual operators which are natural generalizations of the class of H-S operators between Hilbert spaces.

Definition 2.1. *1) An operator $T \in L(\ell_p, Y)$ $(1 < p < \infty)$ is said to be a H-S type operator if $\sum_n \|Te_n\|^{p'} < \infty$, where (e_n) is the sequence of standard unit vectors and p' is the conjugate number of p $(1/p + 1/p' = 1)$. The set of all H-S type operators from ℓ_p into Y is denoted by $S(\ell_p, Y)$.*

2) An operator $T \in L(X, \ell_r)$ $(1 < r < \infty)$ is said to be a H-S type dual operator if the dual operator $T^* : \ell'_r \to X'$ is a H-S type operator. The set of all H-S type dual operators from X into ℓ_r is denoted by $U(X, \ell_r)$.

For $T \in S(\ell_p, Y)$ we put $\|T\|_s^{p'} = \sum_n \|Te_n\|^{p'}$ By the Holder's inequality we get

$$\|Tx\| \leq \|x\|.\|T\|_s \quad \text{and} \quad \|T\| \leq \|T\|_s \tag{2.1}$$

It is easy to check that the function $T \to \|T\|_s$ is a norm on $S(\ell_p, Y)$. Moreover $S(\ell_p, Y)$ is a Banach space under the norm $\|T\|_s$.

For $T \in U(X, \ell_r)$ we put $\|T\|_u = \|T^*\|_s$. Again, $U(X, \ell_r)$ is a Banach space under the norm $\|T\|_u$.

Theorem 2.2. *1) If $T \in S(\ell_p, Y)$ then T is a compact operator and for each $R \in L(Y, Z)$ we have $HT \in S(\ell_p, Y)$. If Y is separable, so is $S(\ell_p, Y)$.*

2) If $T \in U(X, \ell_r)$ then T is a compact operator and for each $R \in L(Z, X)$ we have $TR \in U(Z, \ell_r)$. If X' is separable so is $U(X, \ell_r)$.

P r o o f. 1) For each n we define a continuous finite rank operator T_n by $T_n x = \sum_{k=1}^{n} (x, e_k) Te_k$. Then by (2.1) we have $\|T - T_n\|^{p'} \leq \sum_{k=n+1}^{\infty} \|Te_k\|^{p'} \to 0$ as $n \to \infty$. Hence T is a compact operator. For $R \in L(Y, Z)$ we have

$$\sum_n \|RTe_n\|^{p'} \leq \|R\|^{p'} \sum_n \|Te_n\|^{p'} < \infty \quad \text{so} \quad RT \in S(\ell_p, X).$$

Now let Y be separable and D be a countable set dense in Y. For each finite subset $I = \{d_1, \ldots, d_n\} \subset D$ we define an operator $T_I \in S(\ell_p, Y)$ by $T_I x = \sum_{k=1}^{n} (x, e_k) d_k$. The family $\{T_I\}$ is countable. Given $\varepsilon > 0$. There exists n such that $\|T - T_n\|_s^{p'} = \sum_{k=n+1}^{\infty} \|Te_k\|^{p'} < \varepsilon$. Then for each k we can find $d_k \in D$ such that $\|Te_k - d_k\| < \varepsilon / 2^{k/p'}$. Then for $I = \{d_1, \ldots, d_n\}$ we have

$$\|T_n - T_I\|_s = \left\{ \sum_{k=1}^{n} \|Te_k - d_k\|^{p'} \right\}^{1/p'} < \varepsilon.$$

Consequently $\|T - T_I\|_s \leq \|T - T_n\|_s + \|T_n - T_I\|_s < 2\varepsilon$

2) The proof is similar to that of the assertion 1).

Theorem 2.3. *Let either $1 < p' < r < 2$ or $1 < p' < \infty$, $r = 2$. Then we have $\Lambda_r(\ell_p, L_r) = \Pi_0(\ell_p, L_r) = S(\ell_p, L_r)$.*

P r o o f. By Theorem 3.1 [13] we have $\Lambda_r(\ell_p, L_r) \subset \Pi_0(\ell_p, L_r)$. The inclusion $\Pi_0(\ell_p, L_r) \subset S(\ell_p, L_r)$ is clear. Now it is enough to prove that $S(\ell_p, L_r) \subset \Lambda_r(\ell_p, L_r)$. Let $T \in S(\ell_p, L_r)$ and μ be the c.m. on ℓ_p' with the characteristic function $\exp\{-\|Ta\|^r\}$ $a \in \ell'_p$. Suppose that $M : \ell_p \to L_0(\Omega_1)$ is the linear mapping generating μ. We have $E\exp\{iMe_n\} = \exp\{-\|Te_n\|^r\}$. Hence $\sum_n E|Me_n|^{p'} = K\sum_n \|Te_n\|^{p'} < \infty$ where K is a constant. From this we get $\sum_n |Me_n|^{p'} < \infty$ a.s.. Put $\overline{M}(\omega) = \{Me_n(\omega)\}_1^\infty$. $\overline{M}$ is a random vector with values in ℓ'_p and $Ma(\omega) = (\overline{M}(\omega), a)$ a.s. for each $a \in \ell_p$. In view of Theorem 1.1 the c.m. μ is a Radon measure i.e. $T \in \Lambda_r(\ell_p, L_r)$.

Theorem 2.4. *Let $1 < r < p' < 2$ Then we have*

$$\Lambda_r(L_p, \ell_r) = \Pi_0(L_p, \ell_r) = U(L_p, \ell_r).$$

P r o o f. At first, let us remark that since $1 < r < p' < 2$ the space L'_p is of stable type r and embeds into an appropriate space $L_r(S_1, \Sigma_1, \lambda_1)$. By Theorem 3.1 and Theorem 3.5 in [13] we have $\Pi_0(L_p, \ell_r) = \Lambda_r(L_p, \ell_r)$. Hence it sufficies to prove that $\Lambda_r(L_p, \ell_r) = U(L_p, \ell_r)$. A straightforward application of Ito-Nisio's theorem yields that $T \in \Lambda_r(L_p, \ell_r)$ if and only if the series $\sum_n T^* e_n \Theta_n$ converges a.s. in L'_p where $\{\Theta_n\}$ are i.i.d real-valued random variables with characteristic function $\exp\{-|t|^r\}$. Since $r < 2$ this property is equivalent to $\sum_n \|T^* e_n\|^r < \infty$ i.e. $T \in U(L_p, \ell_r)$.

Corollary 2.5. *Let $1 < r < 2 < p$. A linear continuous operator from ℓ_p into ℓ_r is completely summing if and only if it is a $H-S$ type operator in the case $p' < r$ or it is a $H-S$ type dual operator in the case $p' > r$.*

We return to the problem of sample path properties of random operator.

Lemma 2.6. *1) Let $A \in L(\Omega, X, Y)$. Then A admits a modification with sample paths in V if and only if there is a mapping $\overline{A} : \Omega \to V$ such that $\forall x \in X \quad P\{\omega : Ax(\omega) = \overline{A}(\omega)x\} = 1$.*

2) In this case, if V is separable then $\overline{A}$ is $(\mathcal{F}, \mathcal{B})$ - measurable.

P r o o f. The assertion 1) is obvious. In order to prove 2) for $x \in X$ and $y \in Y'$ we denote by $x \otimes y$ the linear continuous functional on V defined by $\langle b, x \otimes y\rangle = (bx, y)$. Clearly, the family $F = \{x \otimes y, x \in X, y \in Y'\}$ separates the points of V. Further, for each $x \otimes y$ the function $\omega \to \langle \overline{A}(\omega), x \otimes y\rangle = (Ax(\omega), y)$ is measurable. Hence $\overline{A}$ is $(\mathcal{F}, \mathcal{A}(F))$ - measurable. Thus the assertion 2) follows from the fact that $\mathcal{A}(F) = \mathcal{B}$ if V is separable.

Theorem 2.7. *Let $A \in L(\Omega, \ell_p, Y)$ where $1 < p < \infty$. Then A admits a modification with sample paths in $S(\ell_p, Y)$ if and only if the series $\sum_n \|Ae_n\|^{p'}$ converges a. s.*

P r o o f. Suppose that A admits a modification with sample paths in $S(\ell_p, Y)$. By Lemma 2.6 there is a mapping $\overline{A} : \Omega \to S(\ell_p, Y)$ and a set D of probability 1 such that $Ae_n(\omega) = \overline{A}(\omega)e_n$ for all n and all $\omega \in D$. Then for each $\omega \in D$ we have

$$\sum_n \|Ae_n(\omega)\|^{p'} = \sum_n \|\overline{A}(\omega)e_n\|^{p'} < \infty.$$

Conversely, suppose that $\sum_n \|Ae_n\|^{p'} < \infty$ a.s. Put

$$D = \Big\{\omega \ : \ \sum_n Ae_n(\omega)\|^{p'} < \infty\Big\}$$

For each n define a mapping $B_n : \Omega \to S(\ell_p, Y)$ by

$$B_n(\omega)x = \sum_{k=1}^{n} (x, e_k)Ae_k(\omega).$$

We have

$$\|B_{n+m}(\omega) - B_n(\omega)\|_s^{p'} = \sum_{k=n+1}^{n+m} \|Ae_k(\omega)\|^{p'}$$

Consequently, for each $\omega \in D$, $\{B_n(\omega)\}$ is a Cauchy sequence in $S(\ell_p, Y)$ so it converges in $S(\ell_p, Y)$ to some $B(\omega)$. Consider a mapping $\overline{A} : \Omega \to S(\ell_p, Y)$ defined by $\overline{A}(\omega) = B(\omega)$ if $\omega \in D$ and $\overline{A}(\omega) = 0$ otherwise. Then for each $x \in X$ $\overline{A}(\omega)x = \sum_{k=1}^{\infty} (x, e_k)Ae_k(\omega)$ for all $\omega \in D$. On the other hand, the series $\sum_k (x, e_k)Ae_k(\omega)$ converges to $Ax(\omega)$ in $L_Y^0(\Omega)$. Hence $P\{\omega : Ax(\omega) = \overline{A}(\omega)x\} = 1$.

Theorem 2.8. *Let X be a Banach space with the Schauder basis (h_n) and (h_n^*) be the Schauder corresponding basis for X'. A random operator $A \in L(\Omega, X, \ell_r)$ $(1 < r < \infty)$ admits a modification with sample paths in $U(X, \ell_r)$ if and only if for each k the series $\sum_i (Ah_i, e_k)h_k^*$ is convergent a.s. in X' and $\sum_k \|\sum_i (Ah_i, e_k)h_k^*\|^r < \infty$ with probability one.*

P r o o f. The necessity: By Lemma 2.6 there is a mapping $\overline{A}$ from Ω into $U(X, \ell_r)$ and a set D of probability one such that $Ah_i(\omega) = \overline{A}(\omega)h_i$ for all i and all $\omega \in D$. Then for each k and each $\omega \in D$ we have

$$\sum_i (Ah_i(\omega), e_k)h_i^* = \sum_i (\overline{A}(\omega)h_i, e_k)h_i^* = \sum_i (h_i, \overline{A}^*(\omega)e_k)h_i^* = \overline{A}^*(\omega)e_k.$$

Moreover since $\overline{A}(\omega) \in U(X, \ell_r)$ we have $\sum_k \|\overline{A}^*(\omega)e_k\|^r < \infty$ for all $\omega \in D$.

The sufficiency: Assume that $\xi_k = \sum_i (Ah_i, e_k)h_k^* \in L^0_{X'}(\Omega)$ and $\sum_k \|\xi_k(\omega)\|^r < \infty$ a.s. Then there is a set D of probability 1 such that $\sum_k \|\xi_k(\omega)\|^r < \infty$ for all $\omega \in D$. From this it follows that the series $\sum_k (x, e_k)\xi_k(\omega)$ converges in X' for each $x \in \ell'_r$ and each $\omega \in D$. Define a linear operator $B(\omega)$ for each $\omega \in D$ by $B(\omega)x = \sum_{k=1}^{\infty} (x, e_k)\xi_k(\omega)$ and let $\overline{A}(\omega)$ be a restriction of $B^*(\omega)$ to X. We have

$$\sum_k \|\overline{A}^*(\omega)e_k\|^r = \sum_k \|B^{**}(\omega)e_k\|^r = \sum_k \|B(\omega)e_k\|^r = \sum_k \|\xi_k(\omega)\|^r < \infty.$$

Hence $\overline{A}(\omega) \in U(X, \ell_r)$. By putting $\overline{A}(\omega) = 0$ if $\omega \notin D$ we define a mapping $\overline{A} : \Omega \to U(X, \ell_r)$. It remains for us to prove that $Ax(\omega) = \overline{A}(\omega)x$ a.s. Indeed for each i, k and $\omega \in D$ we have

$$(\overline{A}(\omega)h_i, e_k) = (B^*(\omega)h_i, e_k) = (h_i, B(\omega)e_k) = (h_i, \xi_k(\omega)) = (Ah_i(\omega), e_k).$$

So $Ah_i(\omega) = \overline{A}(\omega)h_i$ for each i and each $\omega \in D$. It follows that

$$\overline{A}(\omega)x = \sum_i (x, h_i^*)\overline{A}(\omega)h_i = \sum_i (x, h_i^*)Ah_i(\omega)$$

for each $\omega \in D$. But for each $x \in X$, the series $\sum_i (x, h_i^*)Ah_i(\omega)$ converges to $Ax(\omega)$ in probability. Hence $Ax(\omega) = \overline{A}(\omega)x$ a.s. as required.

III. (V, p) - REPRESENTATION OF RANDOM OPERATORS

A Radon measure μ on V is said to be of order p ($p > 0$) if $\int \|x\|^p d\mu(x) < \infty$. For simplicity of writing we refer to a Radon measure of order 0 as an arbitrary Radon measure.

Definition 3.1. *A random operator $A \in L(\Omega, X, Y)$ is said to be (V, p)-representable ($p \geq 0$) if there exists a Radon measure μ of order p on V such that for all $x_1, \ldots, x_n \in X, y_1, \ldots, y_n \in Y'$ ($n = 1, 2, \ldots$) and for all Borel sets $B \subset R^n$ we have*

$$P\Big\{\omega : [(Ax_1, y_1), \ldots, (Ax_n, y_n)] \in B\Big\} = \mu\Big\{b : [(bx_1, y_1), \ldots, (bx_n, y_n)] \in B\Big\}.$$

In this case μ will be termed the representing measure of A.

Remark 1) If V is separable and there exists a modification of A having sample paths in V then A is $(V,0)$ - representable. Indeed, by Lemma 2.6 there is a V-valued random vector $\overline{A} \in L_V^0(\Omega)$ such that $\forall x \in X$, $P\{\omega : Ax(\omega) = \overline{A}(\omega)x\} = 1$. Clearly, the distribution of $\overline{A}$ is the representing measure of A.

2) If V is nonseparable then the fact that A admits a modification with sample paths in V does not imply the $(V,0)$ - representation of A as the following example shows

Counter example: Let H be a separable Hilbert space with the orthonormal basis (e_n) and (γ_n) be a sequence of independent random variables such that γ_n obeys to the normal law of $N(0, 1/\sqrt{\log n})$.

Let μ be the distribution of the sequence (γ_n). It is well known that μ is concentrated on ℓ_∞ and the restriction of μ to ℓ_∞ is not a Radon measure for the norm topology of ℓ_∞ (it is only a Radon measure for the w^* - topology $\sigma(\ell_\infty, \ell_1)$. Further, ℓ_∞ can be identified as a subset of $L(H,H)$. Namely, every sequence (c_k) in ℓ_∞ corresponds to an operator $T \in L(H,H)$ defined by $Tx = \sum_{k=1}^{\infty}(x, e_k)c_k e_k$. Define a random operator $A \in L(\Omega, H, H)$ by $Ax = \sum_k (x, e_k)\gamma_k e_k$. It is easy to verify that there exists a modification of A whose sample paths belong to ℓ_∞. However, A is not $(V,0)$ - representable.

Theorem 3.2. *Assume that X and Y are separable. Then a random operator $A \in L(\Omega, X, Y)$ is (V,p) - representable $(p \geq 0)$ if and only if there exists a V-valued random vector $\overline{A} \in L_V^p(\Omega)$ such that*

$$\forall x \in X \quad P\{\omega : Ax(\omega) = \overline{A}(\omega)x\} = 1.$$

In particular, if A is (V,p) - representable then it is admits a modification with sample paths in V.

P r o o f. The part 'if' is clear. Indeed, the distribution of $\overline{A}$ is the representing measure of A. Now suppose that A is (V,p) - representable. Let V_0 be the support of the representing measure μ. Since μ is a Radon measure, V_0 is separable (see [7] Proposition 1.1.1). Further, we can find an increasing sequence of compact sets $(K_n) \subset V_0$ such that $\lim_n \mu(K_n) = 1$. Put $K = \bigcup_{n=1}^{\infty} K_n$. Then K is a Borel set with $\mu(K) = 1$. Because X and Y are separable there exist a sequence (x_n) in X and a sequence (y_n) in Y' with $\sup_n \|x_n\| \leq 1$ and $\sup_n \|y_n\| \leq 1$ such that $b = 0$ whenever $(bx_n, y_n) = 0$ for all n. Consider a mapping $\theta : V_0 \to R^\infty$ defined by

$$\theta(b) = \left[(bx_n, y_n)\right]_1^\infty.$$

θ is continuous and one-one. Hence $\Theta(K_n)$ is compact and $\Theta(K) = \bigcup_{n=1}^{\infty} \Theta(K_n)$ is a separable Borel set. By the Kuratowski theorem ([15] p.15) the inverse mapping $\phi = \Theta^{-1} : \Theta(K) \to V_0$ is measurable. Put $\Omega_0 = \left\{\omega : [(Ax_n, y_n)]_1^{\infty} \in \Theta(K)\right\}$. We have

$$P(\Omega_0) = \mu\left\{b : [(bx_n, y_n)]_1^{\infty} \in \Theta(K)\right\} = \mu\left\{b : \Theta b \in \Theta(K)\right\} \geq \mu(K) = 1.$$

Consider a mapping $G : \Omega \to \Theta(K)$ defined by $G(\omega) = [(Ax_n, y_n)]_1^{\infty}$ if $\omega \in \Omega_0$ and $G(\omega) = k_0$ otherwise, where k_0 is a fixed element in $\Theta(K)$. Put $\overline{A}(\omega) = \phi[G(\omega)]$. $\overline{A}$ is measurable and has separable range $\overline{A}(\Omega) \subset V_0$. Hence $\overline{A}$ is a V-valued random vector. For each $x \in X$ and each $y \in (y_n)$ we have

$$P\left\{(Ax, y) = (\overline{A}(\omega)x, y)\right\} = P\left\{(Ax, y) = (\phi[G(\omega)]x, y)\right\} =$$
$$= \mu\left\{b : (bx, y) = (\phi[\Theta b]x, y)\right\} \geq \mu\left\{b : b = \phi[\Theta b]\right\} \geq \mu(K) = 1.$$

Consequently, $\forall x \in X$, $P\{\omega : Ax(\omega) = \overline{A}(\omega)x\} = 1$. Finally

$$\int \|\overline{A}(\omega)\|^p dP = \int \|\phi[\Theta b]\|^p d\mu(b) = \int \|b\|^p d\mu(b) < \infty.$$

Theorem is fully proved.

By $L^{(p)}(\Omega, X, Y)$ we denote the set of random operators A from X into Y for which $E\|Ax\|^p < \infty$ for all $x \in X$. By the closed graph theorem $L^{(p)}(\Omega, X, Y)$ can be identified with the space of all linear continuous operators from X into $L_Y^p(\Omega)$.

Now to each $A \in L^{(p)}(\Omega, X, Y)$ $(p > 1)$ we associate a linear mapping S_A from $L'_p(\Omega)$ into the set of mappings from X into Y by the following formula.

$$(S_A h)(x) = \int h(\omega) Ax(\omega) dP \tag{3.1}$$

Here the Bochner integral (3.1) exists since $Ax \in L_Y^p(\Omega)$. By the Hölder inequality we have

$$\|(S_A h)(x)\| \leq v(A)\|h\| \|x\| \tag{3.2}$$

where $v(A) = \sup_{\|x\| \leq 1} \{E\|Ax\|^p\}^{1/p}$. From (3.2) it follows easily that S_A is a linear continuous operator from $L'_p(\Omega)$ into $L(X, Y)$ and $\|S_A\| \leq v(A)$.

Let γ be a cylindrical measure on $L'_p(\Omega)$ defined by

$$\gamma\Big\{h \in L'_p(\Omega) : [(h, g_n)]_1^N \in B\Big\} = p\Big\{\omega : [g_n(\omega)]_1^N \in B\Big\}.$$

In other words, γ is the c.m. on $L'_p(\Omega)$ generated by the identical mapping $Id : L_p(\Omega) \to L_p(\Omega)$. It is easy to see that γ is a cylindrical measure of type p i. e. $\sup\limits_{\|g\|\leq 1} \int\limits_r |t|^p d\gamma_g(t) < \infty$.

Theorem 3.3. *Let $A \in L^{(p)}(\Omega, X, Y)$. Then A is (V,p) - representable $(p > 1)$ if and only if S_A is an operator from $L'_p(\Omega)$ into V and the image of γ by S_A is a Radon measure of order p on V.*

P r o o f. Suppose that A is (V,p) - representable. By Theorem 3.2 there exists a V-valued random vector $\overline{A} \in L_V^p(\Omega)$ such that $\forall x \in X$, $Ax(\omega) = \overline{A}(\omega)x$ a.s. . From (3.1) we have

$$(S_A h)x = \int h(\omega)\overline{A}(\omega)x dP$$

Since the Bochner integral $\int h(\omega)\overline{A}(\omega)dP$ exists, this identity shows that $S_A h = \int h(\omega)\overline{A}(\omega)dP \in V$. Moreover, for each $u \in V'$ and $h \in L'_p(\Omega)$ we have

$$(S_A^* u, h) = \langle S_A h, u\rangle = \int h(\omega)\langle \overline{A}(\omega), u\rangle dP.$$

Hence $S_A^* u(\omega) = \langle \overline{A}(\omega), u\rangle$ a.s.

Denote by $S_A(\gamma)$ the image of γ by S_A. The characteristic function of the c.m. $S_A(\gamma)$ is equal to

$$\widehat{S_A(\gamma)}(u) = \int \exp\{iS_A^* u(\omega)\} dP = \int \exp\{i\langle \overline{A}(\omega), u\rangle dP.$$

This equality shows that $S_A(\gamma)$ coincides with the distribution of $\overline{A}$. Thus $S_A(\gamma)$ is a Radon measure of order p on V.

Conversely, suppose that $S_A(\gamma)$ is a Radon measure of order p on V. Then for all $x_1, \dots, x_n \in X, y_1, \dots, y_n \in Y'$ $(n = 1, 2, \dots)$ and for all Borel sets $B \in R^n$ we have

$$S_A(\gamma)\Big\{b \in V : [(bx_k, y_k)]_1^n \in B\Big\} = \gamma\Big\{h \in L'_p(\Omega) : [((S_A h)x_k, y_k)]_1^n \in B\Big\} =$$

$$= \gamma\Big\{h \in L'_p(\Omega) : [(h, (Ax_k, y_k))]_1^n \in B\Big\} = P\Big\{\omega : [(Ax_k, y_k)]_1^n \in B\Big\}.$$

This proves that A is (V,p) - representable. Theorem is proved.

The following theorem gives a sufficient condition for the (V,p) - representation of a random operator A in terms of its associated operator S_A.

Theorem 3.4. *Let $A \in L^{(p)}(\Omega, X, Y)$ and $p > 1$. If the operator S_A is p-summing operator from $L'_p(\Omega)$ into V then A is (V, p) - representable.*

P r o o f. Because γ is a cylindrical measure of type p and S_A is p-summing ($p > 1$) by Schwartz's Radonification Theorem [9], $S_A(\gamma)$ is a Radon measure of order p on V. Hence A is (V, p) - representable by Theorem 3.3.

Let $X \otimes Y'$ be the tensor product of X and Y' and α be a reasonable crossonorm on $X \otimes Y'$. Its completion under this norm is denoted by $X\widehat{\otimes}_\alpha Y'$. Every member ϕ of the dual space $[X\widehat{\otimes}_\alpha Y']'$ corresponds to some $T \in L(X, Y'')$ such that $\langle x \otimes y, \phi\rangle = (Tx, y)$ (see [1], p.230).

Definition 3.5. *We say that an operator $T \in L(X, Y)$ belongs to the class V_α if the linear functional $\hat{T}$ on $X\widehat{\otimes}_\alpha Y'$ defined by*

$$\hat{T}\Big(\sum_{i=1}^{n}(x_i \otimes y_i)\Big) = \sum_{i=1}^{n}(Tx_i, y_i)$$

is continuous i.e. $\hat{T}$ is a member of $[X\widehat{\otimes}_\alpha Y']'$.

The norm of $T \in V_\alpha$ is defined by $\|T\|_\alpha = \|\hat{T}\|$. It is easy to check that V_α is a Banach space under the norm $\|T\|_\alpha$.

Examples 1) If $\alpha = \pi$, the greatest reasonable crossnorm, then V_π consists of all linear continuous operators from X into Y.

2) If $\alpha = \varepsilon$, the least reasonable crossnorm, then V_ε is called the class of integral operators in the sense of Grothendieck. For the properties of integral operators see [1], Chapter VIII.

3) If α is the norm d_k $(1 < k < \infty)$ introduced by Saphar [10] then in view of Theorem 3.2 in [10] V_{d_k} is precisely the space of k'-summing operators from X into Y.

Let $A \in L(\Omega, X, Y)$. By the property of the tensor product A induces an unique linear mapping $T_A \ : \ X \otimes Y' \to L_0(\Omega)$ with the property $T_A(x \otimes y) = (Ax, y)$. If $A \in L^{(p)}(\Omega, X, Y)$ then $T_A u \in L_p(\Omega)$ for all $u \in X \otimes Y'$. The following theorems give conditions for the (V_α, p) - representation of A in terms of the operator T_A.

Theorem 3.6. *Assume that X and Y are separable. For a random operator $A \in L^{(p)}(\Omega, X, Y)$ to be (V_α, p) - representable $(p \geq 1)$ it is necessary that*

1) $T_A \ : \ X \otimes_\alpha Y' \to L_p(\Omega)$ is continuous

2) The extension of T_A to $X\widehat{\otimes}_\alpha Y'$, which is also denoted by T_A, is a p-summing operator.

P r o o f. By Theorem 3.2 there is a V_α - valued random vector f with $E\|f\|_\alpha^p < \infty$ such that

$$\forall x \in X \qquad Ax(\omega) = f(\omega)x \qquad \text{a.s.} \tag{3.3}$$

For each $b \in V_\alpha$, $\hat{b}$ stands for the corresponding element in $[X\hat{\otimes}_\alpha Y']'$. From (3.3) we have

$$\forall u \in X \otimes_\alpha Y' \qquad T_A u = \langle u, \hat{f}(\omega)\rangle \quad \text{in } L_p(\Omega). \tag{3.4}$$

Hence $\int |T_A u|^p dP \leq \left\{ \int \|\hat{f}\|^p dP \right\} \|u\|^p$ which shows the assertion 1).

Now from (3.4) we obtain $T_A u = \langle u, \hat{f}(\omega)\rangle$ in $L_p(\Omega)$ for all u in $X\hat{\otimes}_\alpha Y'$. For $u_1, \ldots, u_n \in X\hat{\otimes}_\alpha Y'$ we have

$$\sum_i \|T_A u_i\|^p = \int \Big\{ \sum_i |\langle u_i, \hat{f}(\omega)\rangle|^p \Big\} dP \leq \Big\{ \int \|\hat{f}\|^p dP \Big\} \sup_{\|\phi\| \leq 1} \Big\{ \sum_i |\langle u_i, \phi\rangle|^p \Big\}$$

which shows the assertion 2).

Theorem 3.7. *Assume that X, Y are separable. For a random operator $A \in L^{(1)}(\Omega, X, Y)$ to be $(V_\alpha, 1)$ - representable it is necessary and sufficient that*

1) $T_A \; : \; X \otimes_\alpha Y' \to L_1(\Omega)$ is continuous

2) The extension of $T_A \; : \; X\hat{\otimes}_\alpha Y' \to L_1(\Omega)$ is a nuclear operator.

P r o o f. The necessity: By Theorem 3.6 $T_A \; : \; X\hat{\otimes}_\alpha Y' \to L_1(\Omega)$ is continuous and there is a Bochner integrable function $f \; : \; \Omega \to V_\alpha$ such that

$$\forall u \in X\hat{\otimes}_\alpha Y', \quad T_A u = \langle u, \hat{f}(\omega)\rangle \quad \text{in } L_1(\Omega). \tag{3.5}$$

By Corollary to Pettis's Measurability Theorem [1] there exists a sequence (g_n) of countably valued functions such that $\lim g_n(\omega) = f(\omega)$ uniformly on a set of probability. Accordingly we shall assume that $\lim g_n(\omega) = f(\omega)$ uniformly on Ω. Moreover by discarding (if necessary) some of the members of the sequence (g_n) we can assume that $\|f(\omega) - g_1(\omega)\| < 1/2$ and $\|g_n(\omega) - g_{n-1}(\omega)\| < 1/2^n$ for all ω and all $n \geq 2$. Next write $g_0 = 0$ and

$$g_n - g_{n-1} = \sum_m b_{n,m} I_{A_{n,m}}$$

where $(b_{n,m})$ is a sequence in V_α, $(A_{n,m})$ is a disjoint sequence in Ω $(n \geq 1)$ and I_A indicates the characteristic function of the set A.

Since $f(\omega) = \sum_n [g_n(\omega) - g_{n-1}(\omega)]$ uniformly we have

$$\forall u \in X\hat{\otimes}_\alpha Y', \quad \langle u, \hat{f}(\omega)\rangle = \sum_n \sum_m \langle u, \hat{b}_{n,m}\rangle I_{A_{n,m}}(\omega) \quad \text{in } L_1(\Omega).$$

From this and (3.5) we get

$$\forall u \in X\hat{\otimes}_\alpha Y', \quad T_A u = \sum_n \sum_m \langle u, \hat{b}_{n,m}\rangle I_{A_{n,m}}(\omega) \quad \text{in } L_1(\Omega). \tag{3.6}$$

On the other hand

$$\sum_n \sum_m \|\hat{b}_{n,m}\| P(A_{n,m}) = \sum_n E\|g_n - g_{n-1}\| \le E\|g_1\| + \frac{1}{2} < \infty. \tag{3.7}$$

The representation (3.6) with the property (3.7) shows that T_A is a nuclear operator .

The sufficiency: Suppose that T_A is a nuclear operator. Then there exist a sequence (ϕ_n) in $[X\widehat{\otimes}_\alpha Y']'$ and a sequence (f_n) in $L_1(\Omega)$ such that $\sum_n \|\phi_n\| E|f_n| < \infty$ and for each $u \in X\widehat{\otimes}_\alpha Y'$

$$T_A u = \sum_n \langle u, \phi_n \rangle f_n(\omega) \qquad \text{in } L_1(\Omega). \tag{3.8}$$

Put $S_n(\omega) = \sum_{k=1}^{n} \phi_k f_k(\omega)$. We have $E\|S_n - S_{n+m}\| \le \sum_{k=n+1}^{n+m}{}' \|\phi_k\| E|f_k|$ so it converges to 0 as $n, m \to \infty$. Hence, there exists a Bochner integrable function $S : \omega \to [X\widehat{\otimes}_\alpha Y']'$ such that $E\|S - S_n\| \to 0$. This implies that for each $u \in [X\widehat{\otimes}_\alpha Y']'$ $\langle u, S_n(\omega)\rangle$ converges to $\langle u, S(\omega)\rangle$ in $L_1(\omega)$. In virtue of (3.8) we get

$$\forall u \in X \otimes_\alpha Y' \qquad T_A u = \langle u, S(\omega)\rangle \quad \text{a.s.} \tag{3.9}$$

Now for each ω, $S(\omega)$ corresponds to an operator $B(\omega) \in L(X, Y'')$ such that $\langle x \otimes y, S(\omega)\rangle = (B(\omega)x, y)$ for $x \in X$, $y \in Y'$. Taking $u = x \otimes y$ from (3.9) we obtain $(Ax, y) = (B(\omega)x, y)$ a.s. Since Y is separable from this it follows that $Ax(\omega) = B(\omega)x$ a.s. for each $x \in X$. To complete the proof we have to show that $B(\omega)$ maps X into Y for almost all ω. Indeed, let (x_n) be a countable set dense in X. Then there is a set D of probability 1 such that $Ax_n(\omega) = B(\omega)x_n$ for all $\omega \in D$ and all n. Thus $B(\omega)x \in Y$ for all $x \in X$ and all $\omega \in D$. This completes the proof.

For the case $p = 0$ we have the following theorem

Theorem 3.8. *Assume that X, Y are separable and two random operators $A, B \in L(\Omega, X, Y)$ are symmetric, independent in the sense that for all $x_1, \dots, x_n, x'_1, \dots, x'_m$ in $X, y_1, \dots, y_n, y'_1, \dots, y'_m$ in Y' two random vectors $[(Ax_i, y_i)]_1^n$ and $[(Bx'_i, y'_i)]_1^m$ are symmetric and independent.*

Then if $C = A + B$ is $(V, 0)$ - representable so are A and B.

P r o o f. Let us remark that each $u \in X \otimes Y'$ may be regarded as a member of V' (namely, $u = \sum_1^n (x_i \otimes y_i)$ corresponds to the linear continuous functional

$b \to \sum_1^n (bx_i, y_i)$. We claim that $X \otimes Y'$ is dense in V' with respect to (w.r.t) the Mackey topology. Indeed if this is not true then by Hahn-Banach theorem and the fact that the Mackey topology is compatible with the duality (V', V) there is $b \in V$, $b \neq 0$ such that $\langle b, u\rangle = 0$ for all $u \in X \otimes Y'$. A contradiction. Now by Theorem 3.2 there exists a V-valued random vector $\overline{C}$ such that $\forall x \in X \quad Cx(\omega) = \overline{C}(\omega)x$ a.s. From this we have $\forall u \in X \otimes Y' \ T_C u = \langle \overline{C}(\omega), u\rangle$ a.s. In view of Lemma 1.2 the linear mapping $T_C : X \otimes Y' \to L_0(\Omega)$ is continuous w.r.t. the Mackey topology (induced on $X \otimes Y'$). Because for each u in $X \otimes Y'$ two random variables $T_A u$ and $T_B u$ are symmetric and independent, by Levy's inequality we have

$$P\Big\{|T_A u| > \varepsilon\Big\} \leq 2P\Big\{|T_A u + T_B u| > \varepsilon\Big\} = 2P\Big\{|T_C u| > \varepsilon\Big\}.$$

From this it follows that T_A is continuous w.r.t the Mackey topology. Similarly, this holds for T_B. Consequently, T_A, T_B and T_C admit extensions to the entire V'. Moreover for each $u \in V'$, $T_A u$ and $T_B u$ are symmetric, independent and

$$T_A u + T_B u = T_C u. \tag{3.10}$$

Let μ, λ and ν are cylindrical measures on V generated by T_A, T_B and T_C, respectively. In view of (3.10) we have $\nu = \mu * \lambda$. Because $\nu = \mathcal{L}(\overline{C})$ is a Radon measure by Proposition 3 in [4] both μ and λ are Radon measures. Now it is easy to verify that A and B are $(V, 0)$ - representable with the representing measures μ and λ respectively.

Theorem 3.9. *Let $A \in L^{(p')}(\Omega, \ell_p, Y)$ and Y be separable where $1 < p < \infty$ Then A is $[S(\ell_p, Y), p']$ - representable if and only if $\sum_n E\|Ae_n\|^{p'} < \infty$.*

P r o o f. Suppose that A is $[S(\ell_p, Y), p']$ - representable. According to Theorem 3.2 there exist a random vector $\overline{A}$ with values in $S(\ell_p, Y)$ of the strong p'-order such that $\forall x \in \ell_p, \ Ax(\omega) = \overline{A}(\omega)x$ a.s. Consequently

$$\sum_n E\|Ae_n\|^{p'} = \sum_n E\|\overline{A}(\omega)e_n\|^{p'} = E\sum_n \|\overline{A}(\omega)e_n\|^{p'} = E\|\overline{A}\|_s^{p'} < \infty.$$

Conversely, suppose that $\sum_{n=1}^{\infty} E\|Ae_n\|^{p'} < \infty$. For each n we define a $S(\ell_p, Y)$ - valued random vector B_n by $B_n(\omega)x = \sum_{k=1}^{n} (x, e_k)Ae_k(\omega)$. Then $E\|B_{n+m} - B_n\|_s^{p'} = \sum_{k=n+1}^{n+m} E\|Ae_k\|^{p'}$ tends to 0 as $n, m \to \infty$. Hence there exists a $S(\ell_p, Y)$

- valued random vector $\overline{A}$ with $E\|A\|_s^{p'} < \infty$ such that $E\|B_n - \overline{A}\|_s^{p'}$ tends to 0 as $n \to \infty$. This implies that for each $x \in \ell_p$, $B_n(\omega)x$ converges to $\overline{A}(\omega)x$ in probability. Since $B_n(\omega)x$ converges to $Ax(\omega)$ in probability we conclude that for each $x \in \ell_p$, $Ax(\omega) = \overline{A}(\omega)x$ a.s. Thus A is $[S(\ell_p, Y), p']$ - representable by Theorem 3.2.

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