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THE METHOD OF ORIENTING CURVES AND ITS APPLICATION TO AN OPTIMAL CONTROL PROBLEM OF HYDROELECTRIC POWER PLANTS

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In the memory of Professor Hj. Wacker

Abstract. *In this paper, the Method of Orienting Curves for solving a class of linear optimal control problems is described and then is applied to find an optimal control of a hydroelectric power plant.*

1. INTRODUCTION

In the recent years, a so-called Method of Orienting Curves was developed for solving optimal control problems with state constraints (cf. [2], [5], [10]). Some practical problems were solved by means of this method, e. g., the navigation problem of Zermelo (cf. [6]), Steiner's problem of finding inpolygons of a given convex polygon with minimal circumference (cf. [7]), and an inventory problem (cf. [10]).

The aim of this paper is to describe the Method of Orienting Curves (MOC) for solving a class of optimal control problems which are linear in the control variable, and to show how these results can be applied to find an optimal control of a hydroelectric power plant.

The problem of optimal control of hydroelectric power plants was investigated by many authors (cf. [1], [8], [9]). But there are many still unsolved questions in practice. In this paper, we will concern with such a problem which can be described as follows.

The volume of the storage lake of the plant at the time t is denoted by $x(t)[m^3]$. The flow through the turbines and the natural influx to the reservoir at the time t are $\omega(t)[m^3/s]$ and $b(t)[m^3/s]$, respectively. Then the variation of the storage volume can be represented by

$$\dot{x}(t) = b(t) - \omega(t).$$

Let ℓ be the head, i.e., the height difference between the surface of the reservoir and the turbines. Obviously, ℓ depends on the volume x. Then the energy produced at the time t is equal to $gc\ell(x(t))w(t)$, where g is the gravitation constant, and c is the efficiency of the plant. It is possible that c depends on $x(t)$ and $w(t)$ but for the sake of simplicity, here c is assumed to be a positive constant.

If $a(.)$ represents a positive so-called valuation function, then the total rated energy in the time interval $[0,T]$ is given by

$$\int_0^T gca(t)\ell(x(t))w(t)dt.$$

This intergral delivers the electric energy produced in $[0,T]$ if $a(t) \equiv 1$. If $a(t)$ is a piecewise-constant tariff function, then this intergral is the total profit obtained in $[0,T]$. Here, we confine ourselves to the case where $a(.)$ is a constant as well as the case where $a(.)$ is continuously differentiable on $[0,T]$, e.g., $a(.)$ may be proportional to the national demand for electric energy.

Our aim is to maximize the total rated energy within the fix time interval $[0,T]$. Let β^- and β^+ be the lower and upper bounds of the flow through the turbines. Let x_0 and x_T be the given values of the volume at the start and the end, and $\alpha_1(t)$ and $\alpha_2(t)$ be the minimal and maximal allowed level of the reservoir volume at the time t, respectively. Using the transformation $u(t) := -w(t)$, $\beta_1 := -\beta^+$ and $\beta_2 = -\beta^-$, we get the following optimal control problem

$$\text{Minimize} \int_0^T a(t)\ell(x(t))u(t)dt, \tag{1}$$

subject to

$$\begin{aligned} &\dot{x}(t) = b(t) + u(t), \quad x(0) = x_0, \quad x(T) = x_T, \\ &\alpha_1(t) \le x(t) \le \alpha_2(t), \quad t \in [0,T], \\ &\beta_1 \le u(t) \le \beta_2. \end{aligned} \tag{2}$$

The set $G := \{(t,\xi) \in \mathbf{R}^2 \mid 0 \le t \le T,\ \alpha_1(t) \le \xi \le \alpha_2(t)\}$ is called the *state region*, and any open set G^{ex} of $\mathbf{R}^2$ containing G is an *extended state region.* It is not unrealistic to assume that

$$\{t \in [0,T] \mid \exists i, j \in \{1,2\} \;:\; b(t) + \beta_i = \dot{\alpha}_j(t)\} \tag{3}$$

is a finite set, and that

$$\begin{aligned} &b(.) \text{ is continuous on } [0,T], \\ &b(t) > 0,\; \ell(\xi) > 0, \text{ and } \ell'(\xi) > 0 \text{ for } (t,\xi) \in G^{ex}. \end{aligned} \tag{4}$$

The problem (1)-(4) will be called (P^*), and only the follwing case is considered

$$-\frac{\ell'(\xi)}{\ell(\xi)} < \frac{\dot{a}(t)}{a(t)b(t)} \text{ for all } (t,\xi) \in G^{ex}. \tag{5}$$

Obviously, (5) holds if $a(.)$ is a constant.

2. METHOD OF ORIENTING CURVES

The problem (P^*) is a special case of the following optimal control problem (P):

$$\text{Minimize} \int_0^T L(t,x(t),u(t))dt, \quad L(t,\xi,v) = L_1(t,\xi) + L_2(t,\xi)v,$$

under the constraints

$$\begin{aligned} &\dot{x}(t) = f(t,x(t),u(t)), \quad f(t,\xi,v) = f_1(t,\xi) + f_2(t,\xi)v, \\ &\alpha_1(t) \le x(t) \le \alpha_2(t), \\ &x(0) = x_0, \quad x(T) = x_T, \\ &\beta_1 \le u(t) \le \beta_2. \end{aligned}$$

Here, the functions $L_1(.,\xi)$ and $f_1(.,\xi)$ are continuous, the functions $L_1(t,.)$, $f_1(t,.)$, and $L_2(.,.)$ are continuously differentiable for $(t,\xi) \in G^{ex}$. Besides $\alpha_1(.)$ ans $\alpha_2(.)$ are piecewise continuously differentiable on $[0,T]$.

For our further investigation, the following hypotheses are inserted

$$\beta_1 < \beta_2, \quad \alpha_1(t) < \alpha_2(t), \quad \text{and } f_2(t,\xi) > 0 \tag{6}$$

for $(t,\xi) \in G^{ex}$, and

$$f(t,\alpha_i(t),\beta_i) \ne \dot{\alpha}_i(t), \quad \text{a. e. in } [0,T], \quad i = 1,2 \text{ and } j = 1,2. \tag{7}$$

We are interested in normal optimal procsses of (P), i. e., optimal processes which fullfil Pontryagin's maximum principle (PMP) given in [3, p. 234] for the Lagrange multiplier $\lambda_0 > 0$.

Let $(x^*(.), u^*(.))$ be an optimal process of (P). A subinterval $I \subset [0, T]$ will be called a contact interval of $x^*(.)$ with $\alpha_i(.)$, $i = 1$ or 2, if $x^*(t) = \alpha_i(t)$ for all $t \in I$, and every interval I' with $I' \supset I$ and $I' \setminus I \neq \emptyset$ contains at least one $t \in I' \setminus I$ such that $x^*(t) \neq \alpha_i(t)$.

From now on , S^* stands for the set of all triple $(x^*(.), u^*(.), p^*(.))$ where $(x^*(.), u^*(.))$ is a normal optimal process of (P), and $p^*(.)$ is the corresponding solution of the adjoint equation in Pontryagin's maximum principle. Furthermore, $x^*(.)$ has only finitely many contact intervals with $\alpha_i(.)$, $i = 1, 2$.

It is well-known that the state constraints cause many difficulties in solving (P). The state constraints imply the existence of the measures μ_1, μ_2 as Lagrange multipliers in PMP. These measures determine how long the optimal trajectory keeps staying on the state boundaries after it meats these boundaries. Unfortunately, these measures are still unknown. The purpose of the MOC is to overcome these difficulties by taking advantage of some special kinds of solutions of the following system:

$$\begin{aligned}
&p(z) = q, \quad x(z) = y, \\
&\dot{p}(t) = -p(t)[f_{1\xi}(t, x(t)) + f_{2\xi}(t, x(t))u(t)] + \\
&\qquad + L_{1\xi}(t, x(t)) + L_{2\xi}(t, x(t))u(t), \\
&\dot{x}(t) = f_1(t, x(t)) + f_2(t, x(t))u(t), \\
&u(t) \in \arg\max_{v \in [\beta_1, \beta_2]} H(t, x(t), v, p(t), 1) \text{ a. e. in } [0, T],
\end{aligned} \tag{8}$$

where $q \in \mathbf{R}$ and $(z, y) \in G$ while H is defined by $H(t, \xi, v, q, \lambda) := qf(t, \xi, v) - \lambda L(t, \xi, v)$. In fact, (8) is a local version of PMP without state constraints.

Let us denote the set of all solutions $(x(.), u(.), p(.))$ of (8) for some certain parameters z, y, q by $S_{z,y,q}$. If $(x(.), u(.), p(.)) \in S_{z,y,q}$, then $x(.)$ is said to be a stationary function through (z, y). We are intereted in the following kinds of them.

Definition 2.1. *If for some $q \in \mathbf{R}$, there exists $(x(.), u(.), p(.)) \in S_{z,y,q}$ such that*

$$x(T) = x_T \text{ and } (t, x(t)) \in G \text{ for all } t \in [z, T], \tag{9}$$

then $x(.)$ is called a final function and its graph is said to be a final curve through $(x, u) \in G$.

To continue, we need some further notations. The lower and upper boundary B_1 and B_2 of the state region G are the graphs of $\alpha_1(.)$ and $\alpha_2(.)$, respectively.

I.e.,

$$B_1 = \{(t,\xi) \mid t \in [0,T],\ \xi = \alpha_1(t)\},$$
$$B_2 = \{(t,\xi) \mid t \in [0,T],\ \xi = \alpha_2(t)\}.$$

The sets

$$B_1^{ex} = B_1 \cup \{(0,\xi) \mid \alpha_1(0) \le \xi < x_0\} \cup \{(T,\xi) \mid \alpha_1(T) \le \xi < x_T\},$$
$$B_2^{ex} = B_2 \cup \{(0,\xi) \mid x_0 < \xi \le \alpha_2(0)\} \cup \{(T,\xi) \mid x_T < \xi \le \alpha_2(T)\}$$

are called the *extended lower* and *extended upper boundary* of G, respectively.

Definition 2.2. *If for some $q \in \mathbf{R}$ there is $(x(.), u(.), p(.))$ belonging to $S_{z,y,q}$ such that there exist ρ and ν with*

$$\begin{aligned} &0 \le z < \nu < \rho \le T,\\ &(t, x(t)) \in G \text{ for } t \in [z,\rho],\\ &(\nu, x(\nu)) \in B_i \text{ for } i = 1 \text{ or } 2,\\ &(\rho, x(\rho)) \in B_j^{ex} \text{ for } j \ne i, \end{aligned} \tag{10}$$

then $x(.)$ is said to be an orienting function and its graph is an orienting curve through the initial point $(z,y) \in G$

In this case $(\rho, x(\rho))$ is the *terminal point* of this orienting curve if it holds, additionally to (10), that

$$\text{for all } \delta > 0, \text{ there exists } t \in (\rho, \rho+\delta) \text{ with } (t, x(t)) \notin G. \tag{11}$$

$(\nu, x(\nu))$ is called the *transfer point* of this curve if, additionally to (10)-(11),

$$(t, x(t)) \notin B_i \text{ for all } t \in (\nu, \rho).$$

Definition 2.3. *Let $q \in \mathbf{R}$ and $(x(.), u(.), p(.)) \in S_{z,y,q}$. Suppose that $i \in \{1,2\}$ and that*

$$\begin{aligned} &z \in [z_1, z_2] \subset [0,T], \quad z_1 < z_2,\\ &(z_k, x(z_k)) \in B_i^{ex} \cup \{(0,x_0),(T,x_T)\}, \quad k = 1,2,\\ &(t, x(t)) \in G, \quad t \in [z_1, z_2], \end{aligned} \tag{12}$$

then $x(.)$ is called a barrier function through (z,y).

If, additionally to (12)

$$(z, y) \in B_j, \quad j \neq i$$

i. e., $y = x(z) = \alpha_j(z)$, then $(z, \alpha_j(z))$ is called a ***narrow pass point.***

Next, we state the MOC which delivers the optimal state function $x^*(.)$ of (P) (cf. [2], [10]).

Algorithm

Step 1. Begin in $(0, x_0)$. Set $\ell = 0$, $t_0 = 0$, and $x^*(t_\ell) = x_0$.

Step 2. Consider $(t_\ell, x^*(t_\ell))$,

– If there is a final curve through $(t_\ell, x^*(t_\ell))$, go to Step 5.

– If there is an orienting curve through $(t_\ell, x^*(t_\ell))$, go to Step 3.

– If neither of the above cases appear, then it is guaranteed that (under the assumption $S^* \neq \emptyset$) $x^*(t_\ell) = \alpha_{i_\ell}(t_\ell)$, $i_\ell = 1$ or 2, and there exists $t'_\ell > t_\ell$ such that $(i, \alpha_{i_\ell}(t))$ is a narrow pass point for all $t \in [t_\ell, t'_\ell]$. Then go to Step 4.

Step 3. Let $x(.)$ be an orienting function through $(t_\ell, x^*(t_\ell))$ with $(t_{\ell+1}, x(t_{\ell+1}))$ as its transfer point. Then we have $x^*(t) = x(t)$ for all $t \in [t_\ell, t_{\ell+1}]$. Set $\ell := \ell + 1$ and go to Step 2.

Step 4. Determine $t_{\ell+1}$ with:

– $(t, \alpha_{i_\ell}(t))$ is a narrow pass point for all $t \in [t_\ell, t_{\ell+1}]$,

– Either $t_{\ell+1} = T$ or there exists an orienting curve or a final curve through $(t_{\ell+1}, \alpha_{i_\ell}(t_{\ell+1}))$.

Then $x^*(t) = \alpha_{i_\ell}(t)$ for all $t \in [t_\ell, t_{\ell+1}]$. Set $\ell := \ell + 1$ and go to Step 2.

Step 5. Let $x(.)$ be a final function through $(t_\ell, x^*(t_\ell))$. Then $x^*(t) = x(t)$ for all $t \in [t_\ell, T]$. STOP.

It was shown in [10] that the optimal state function $x^*(.)$ of (P) can be determined completely after realizing a finite number of steps of the above algorithm if $S^* \neq \emptyset$. In this case, S^* has only one element. Consequently, the problem (P) possesses at most one normal optimal process which has finitely many contact intervals with the boundary of the state region G. Moreover, if all three cases mentioned in Step 2 do not appear, then $S^* = \emptyset$.

3. SOLVING (P^*) BY THE METHOD OF ORIENTING CURVES

We now try to apply the results just stated in the previous section to get the optimal process $(x^*(.), u^*(.))$ of (P^*). For this aim, we need a further investigation. First, the structure of final and orienting curves are studied. This information helps us to know more about the variation of the optimal volume of the storage lake during the time interval in which the volume is less than the maximal level and greater than the minimal level (see Theorem 5.1 in [10]). Second, we give the way to check if a point on the boundary of the state region is a narrow pass point, i.e., when the optimal water volume attains the maximal or the minimal level.

3.1. Stationary, Orienting, and Final Functions

For (P^*), the system (8) can be written as follows

$$p(z) = q, \quad x(z) = y, \tag{13}$$

$$\dot{p}(t) = -a(t)\ell'\big(x(t)\big)u(t). \tag{14}$$

$$\dot{x}(t) = b(t) + u(t), \tag{15}$$

$$u(t) \in \arg\max_{v\in[\beta_1,\beta_2]} \big\{ \big[p(t) - a(t)\ell\big(x(t)\big)\big]v \big\} \text{ a. e. in } [0,T]. \tag{16}$$

Suppose that $\big(x(.), u(.), p(.)\big)$ is a triple of functions which satisfies (13)-(15). Define

$$\phi(t) := p(t) - a(t)\ell\big(x(t)\big). \tag{17}$$

Moreover, for (P^*), the function $h(.,.)$ defined in [4] is

$$h(t,\xi) = -a(t)b(t)\ell'(\xi) - \dot{a}(t)\ell(\xi), \quad (t,\xi) \in G^{ex}.$$

Note that under the condition (5), $h(t,\xi) < 0$ for all $(t,\xi) \in G^{ex}$.

Theorem 3.1. *Let $z' \in [0,T]$ and $z' \geq z$. Let further that $\big(x(.), u(.), p(.)\big)$ be an element of $S_{z,y,q}$ which satisfies $\big(t, x(t)\big) \in G$ for all $t \in [z, z']$. Then*

$$\phi(t) = \phi(z) + \int_z^t h\big(\tau, x(\tau)\big)d\tau,$$

$$u(t) = \beta_1 \text{ a. e. in } \big\{t \in [0.T] \mid \phi(t) < 0\big\},$$

$$u(t) = \beta_2 \text{ a. e. in } \big\{t \in [0,T] \mid \phi(t) > 0\big\}.$$

Theorem 3.1 is a direct consequence of Lemma 3.1 in [2]. It implies the following

Theorem 3.2. *Suppose that* $[z_1, z_2] \subset [0,T]$ *and* $z \in [z_1, z_2]$. *Suppose further that* $(x(.), u(.), p(.)) \in S_{z,y,q}$ *and* $(t, x(t)) \in G^{ex}$ *for all* $t \in [z_1, z_2]$. *Then there exists* $s \in [0,T]$ *such that*

$$\begin{aligned} &\phi(t) > 0 \text{ for all } t \in (z_1, s), \\ &\phi(t) < 0 \text{ for all } t \in (s, z_2). \end{aligned} \tag{18}$$

Theorem 3.3. *Let* $z \in [z_1, z_2] \subset [0,T]$ *and* $(z, y) \in G$. *Suppose that* $(x(.), u(t))$ *satisfies (15) with* $x(z) = y$ *and* $(t, x(t)) \in G^{ex}$ *for all* $t \in [z_1, z_2]$. *Then* $x(.)$ *is a stationary function through* (z, y) *if and only if there exists* $s \in [z_1, z_2]$ *with*

$$\begin{aligned} &u(t) = \beta_2 \text{ for } t \in (z_1, s), \\ &u(t) = \beta_1 \text{ for } t \in (s, z_2). \end{aligned} \tag{19}$$

P r o o f. Clearly, the "only if" part follows from Theorem 3.1 and Theorem 3.2. For the "if" part, it suffices to prove the existence of a parameter q and a function $p(.)\colon [z_1, z_2] \to \mathbf{R}$ such that

$$(x(.), u(.), p(.)) \in S_{z,y,q}. \tag{20}$$

Let us set $q := a(z)\ell(y) - \int_z^s h(\tau, x(\tau))d\tau$, and denote by $p(.)$ the solution of (14) with $p(z) = q$. By virtue of (19) and Theorem 3.1, to prove (20) we only have to verify (18).

By taking Remark 3.1 in [2] into account, we get

$$\phi(t) = \phi(z) + \int_z^t h(\tau, x(\tau))d\tau.$$

Since $h(t, \xi) < 0$ for all $(t, \xi) \in G^{ex}$, the function $\phi(.)$ is decreasing on $[z_1, z_2]$. The inequality (18) follows from the fact that $\phi(s) = 0$. □

Hence, a given triple (z, y, s), where $(z, y) \in G$ and $s \in [0,T]$, completely determines a stationary function $x(.)$ through (z, y) by means of Theorem 3.3. In this situation, $x(.)$ will be called the stationary function defined by (z, y, s). Moreover, s is said to be the *switching time*, and $(s, x(s))$ is the *switching point* of this curve.

Since a final or an orienting function is a stationary function wich satisfies (9) or (10), respectively, the above observation gives us the structure of a final or of an orienting curve, too. Concretely, if $(z, y) \in G$, $[z, z'] \subset [0,T]$, and if $x(.)$ is a final or an orienting function through (z, y) defined on $[z, z']$, then there exists $s \in [0,T]$ such that the control function $u(.)$ corresponding to $x(.)$ can be determined by (19) (here $[z,z']$ plays the role of $[z_1,z_2]$).

For simplicity, denote by $\eta(.;z,y,\beta_i)$ the solution of the differential equation

$$\dot{x}(t) = b(t) + \beta_i, \quad i = 1,2 \tag{21}$$

with the initial condition $x(z) = y$. The following is useful for our further consideration. Its proof is simple and will be omitted here.

Lemma 3.1. *Suppose that (z,y) is an arbitrary point of G. Suppose further that there exist $z_1, z_2 \in [0,T]$, $z_1 \le z \le z_2$ with $(z_1, \eta(z_1;z,y,\beta_1)) \in B_1^{ex}$, $(z_2, \eta(z_2;z,y,\beta_1)) \in B_2^{ex}$, and $(t, \eta(t;z,y,\beta_1)) \in G$ for all $t \in [z_1,z_2]$. If one of the followings holds*

(i) $z_2 = T$, or

(ii) $z_2 < T$ and for every $\varepsilon > 0$, there exists $t_\varepsilon \in (z_2, z_2+\varepsilon)$ such that

$$\eta(t_\varepsilon;z,y,\beta_1) > \alpha_2(t_\varepsilon), \tag{22}$$

the $S^* = \emptyset$.

Lemma 3.1 gives a conclusion which is rather obvious in practice. In the case where $\alpha_1(.)$ and $\alpha_2(.)$ are constant, suppose that there is a time subinterval $[z_1,z_2]$ such that the natural influx comes so much that the increasing water volume still overreaches the level $\alpha_2 - \alpha_1$ even the flow through the turbines is kept at the maximal control value β^+ during this time subinterval, i.e., $\int_{z_1}^{z_2} (b(t) - \beta^+)dt > \alpha_2 - \alpha_1$. Then the mentioned optimal control problem has no solution.

The following delivers a similar meaning.

Lemma 3.2. *Suppose that $(z,y) \in G$. Suppose further that there exist $z_1, z_2 \in [0,T]$, $z_1 \le z \le z_2$ with $(z_1, \eta(z_1;z,y,\beta_2)) \in B_2^{ex}$, $(z_2, \eta(z_2;z,y,\beta_2)) \in B_1^{ex}$, and $(t, \eta(t;z,y,\beta_2)) \in G$ for all $t \in [z_1,z_2]$. If one of the followings holds*

(i) $z_2 = T$, or

(ii) $z_2 < T$ and for every $\varepsilon > 0$ there exists $t_\varepsilon \in (z_2, z_2+\varepsilon)$ satisfying

$$\eta(t_\varepsilon;z,y,\beta_2) < \alpha_1(t_\varepsilon), \tag{23}$$

then $S^* = \emptyset$.

We are now in a position to give the way how to contruct a final or an orienting curve through a given point $(z,y) \in G$. For our purpose, we only need to consider the case where (z,y) belongs to the optimal trajectory, i.e. $y = x^*(z)$.

Consider $\eta(.;z,y,\beta_1)$ and $\eta(.;z,y,\beta_2)$.

If there exists $z' \geq z$ such that either

$$- \quad \eta(z'; z, y, \beta_1 = \alpha_2(z') \quad \text{and for all } \varepsilon > 0, \quad \text{there exists } t_\varepsilon \in (z', z' + \varepsilon) \text{ satisfying (22), or} \tag{24}$$

$$- \quad \eta(z'; z, y, \beta_2) = \alpha_1(z') \quad \text{and for all } \varepsilon > 0, \quad \text{there exists } t_\varepsilon \in (z', z' + \varepsilon) \text{ satisfying (23)} \tag{25}$$

holds, then it is easy to see that $S^* = \emptyset$.

If there is $i \in \{1,2\}$ such that $\eta(t; z, y, \beta_i) \in G$ for all $t \in [z,T]$ and $\eta(T; z, y, \beta_i) = x_T$, then $\eta(.; z, y, \beta_i)$ is a final function through (z, y).

Otherwise, there is $\tilde{z} \in [z,T]$ such that one of the followings occurs

(i) $\tilde{z} = T$, $(\tilde{z}, \eta(\tilde{z}; z, y, \beta_2)) \in B_2^{ex}$, and $(t, \eta(t; z, y, \beta_2)) \in G$, $t \in [z, \tilde{z}]$,

(ii) $\tilde{z} < T$, $\eta(\tilde{z}; z, y, \beta_2) = \alpha_2(\tilde{z})$, $(t, \eta(t; z, y, \beta_2)) \in G$, $t \in [z, \tilde{z}]$, and for any $\varepsilon > 0$ there exists $t_\varepsilon \in (t_2, t_2 + \varepsilon)$ such that $\eta(t_\varepsilon; z, y, \beta_2) > \alpha_2(t_\varepsilon)$.

We consider (ii). The case (i) can be treated similarly

Let us confine our attention to the function $\eta(.; \tilde{z}, \alpha_2(\tilde{z}), \beta_1)$.

– If there exists $z'' \in (\tilde{z}, T]$ such that

$$(z'', \eta(z''; \tilde{z}, \alpha_2(\tilde{z}), \beta_1)) \in B_1^{ex} \text{ and } (t, \eta(t; \tilde{z}, \alpha_2(\tilde{z}), \beta_1)) \in G, \ t \in [\tilde{z}, z''],$$

then the stationary function $x(.)$ defined by $(z, y, \tilde{z})$ is an orienting function through (z, y).

– If $\eta(T; \tilde{z}, \alpha_2(\tilde{z}), \beta_1) = x_T$ and $(t, \eta(t; \tilde{z}, \alpha_2(\tilde{z}), \beta_1)) \in G$ for $t \in [\tilde{z}, T]$, then the stationary function $x(.)$ defined by $(z, y, \tilde{z})$ is a final function through (z, y).

– If there is $\hat{z} \in [\tilde{z}, T]$ such that $(t, \eta(t, \tilde{z}, \alpha_2(\tilde{z}), \beta_1)) \in G$, $t \in [\tilde{z}, \hat{z}]$, and for all $\varepsilon > 0$ there is $t_\varepsilon \in (\hat{z}, \hat{z} + \varepsilon)$ with $\eta(t_\varepsilon; \tilde{z}, \alpha_2(\tilde{z}), \beta_1) > \alpha_2(t_\varepsilon)$, then we know that $\tilde{z}$ cannot be a switching time of any final or any orienting function which we are looking for. This situation means to the regulator that it is too late for him to change the flow through the turbines from the minimal to the maximal level. He should have been changing the flow to the maximal level at a certain time before $\tilde{z}$. In this case, we look for another switching time $s \in [z, \tilde{z}]$ and a time $\nu \in [z, T]$ which satisfy one of the following systems

$$\begin{aligned} \eta(s; z, y, \beta_2) &= y', \\ \eta(T; s, y', \beta_1) &= x_T, \quad \textit{or} \end{aligned} \tag{26}$$

$$\begin{aligned} \eta(s; z, y, \beta_2) &= y', \\ \eta(\nu; s, y', \beta_1) &= \alpha_2(\nu). \end{aligned} \tag{27}$$

If there is an $s \in [z, \tilde{z}]$ which satisfies (26) and $(t, \eta(t; z, y', \beta_1)) \in G$ for all $t \in [s, T]$, then the stationary function $x(.)$ defined by (z, y, s) is a final function through (z, y).

If there exist $s \in [z, \tilde{z}]$, $\nu \in [z, T)$ which satisfies (27) and there is $\rho \in (\nu, T]$ such that

$$(\rho, \eta(\rho; s, y', \beta_1)) \in B_1^{ex},$$
$$(t, \eta(t; s, y', \beta_1)) \in G \text{ for all } t \in [s, \rho],$$

then the stationary function $x(.)$ defined by (z, y, s) is an orienting function through (z, y).

3.2. Barrier functions and narrow pass points

Let $(x^*(.), u^*(.))$ be an optimal prorcess of (P^*). Then by virtue of Lemma 3.4 in [2] and Theorem 4 in [4] we can conclude that $x^*(.)$ has no contact interval with $\alpha_1(.)$ which contains more than one point. Moreover, through an arbitrary point of $\text{gr}x^*$ there is no orienting curve with "real" transfer point on B_1. Concretely, we have

Lemma 3.3. *Suppose that $S^* \neq \emptyset$, $(x^*(.), u^*(.), p^*(.)) \in S^*$, and $z \in [0, T)$. Suppose further that $x(.)$ is an orierting function through $(z, x^*(z))$ with transfer point $(\nu, x(\nu)) \in B_1$. Then there is a final or an orienting function $x'(.)$ through $(z, x(z))$ with transier point $(\nu', x'(\nu')) \in B_2$ where $\nu' > \nu$.*

Lemma 3.3 is a consequence of Theorem 5.2 in [10] and Lemma 3.1.

By virtue of Lemma 3.3 for any orienting curve through a point on the optimal trajectory we can assume that its transfer point belongs to the upper boundary B_2 of the state region G. Hence, when realizing the algorithm given in Section 2, we can set $i_\ell = 2$ for all ℓ.

As we knew, from a point $(z, x^*(z))$ on B_2, $\text{gr}x^*$ keeps remaining on B_2 during the time subinterval $[z, z']$ as long as $(t, \alpha_2(t))$ keeps being a narrow pass point for all $t \in [z, z']$. This is the reason why narrow pass points are considered. But how can we check if a point on B_2 is a narrow pass point? The following gives an answer for this question.

Theorem 3.4. *Suppose that z is an arbitrary point of $[0, T]$. Then $(z, \alpha_2(z))$ is a narrow pass point if and only if the stationary function $\tilde{x}(.)$, defined by $(z, \alpha_2(z), z)$ is a barrier function (i.e., z is the switching time of $\tilde{x}(.)$).*

P r o o f. Obviously, we only have to prove the "only if" part.

Suppose that $(z, \alpha_2(z))$ is a narrow pass point and $x(.)$ is a barrier function (through $(z, \alpha_2(z))$) which satisfies (12) with $i = 1$. Let $\tilde{u}(.)$ and $u(.)$ be the control functions which exist corresponding to $\tilde{x}(.)$ and $x(.)$, respectively. We get

$$u(t) \le \tilde{u}(t) = \beta_2 \quad \text{for } t \in (z_1, z),$$
$$u(t) \ge \tilde{u}(t) = \beta_1 \quad \text{for } t \in (z, z_2).$$

Hence

$$x(t) \ge \tilde{x}(t) \quad \text{for all } t \in [z_1, z_2].$$

Together with (12), this equality ensures the existence of z_1', $z_2' \in [z_1, z_2]$ with $z_1' < z_2'$, and

$$(z_k', \tilde{x}(z_k')) \in B_i^{ex} \cup \{(0, x_0), (T, x_T)\}, \; k = 1, 2,$$
$$(t, \tilde{x}(t)) \in G \quad \text{for } t \in [z_1', z_2']$$

which proves $\tilde{x}(.)$ to be a barrier function through $(z, \alpha_2(z))$. The proof is complete. □

In practice, the conclusion means that a point on B_2 belongs to the optimal trajectory (i.e., $(z, \alpha_2(z))$ is a narrow pass point) if there exist $z_1 \in [0, z)$, $z_2 \in (z, T]$ such that the followings hold.

(i) If the flow through the turbines is kept at the minimal level β^-, then the storage volume will arise from $\alpha_1(z_1)$ at the time z_1 to the maximal level $\alpha_2(z)$ at the time z.

(ii) If the flow through the turbines is kept at the maximal level β^+, then the storage volume will fall from $\alpha_2(z)$ at the time z to the minimal level $\alpha_1(z_2)$ at the time z_2.

3.3. Some remarks on the optimal trajectory of (P^*)

The condition (3) ensures that an optimal state function $x^*(.)$ of (P^*) has at most finitely many contact intervals with the boundaries $\alpha_1(.)$ and $\alpha_2(.)$. Therefore the algorithm described in section 2 can deliver the optimal solution after finitely many steps. As result, apart from the trivial cases that (P^*) has no solution or there is a final curve through $(0, x_0)$, we can point out the times $t_0, t_1, \ldots, t_{2n+1}$ which satisfy

(i) $t_0 = 0 \le t_1 \le t_2 \le \cdots \le t_{2n} \le t_{2n+1} = T$,

(ii) There exists an orienting function $x_i(.)$ through (t_0, x_0) with transfer point $(t_1, \alpha_2(t_1))$.

(iii) There is an orienting function $x_{k+1}(.)$ through $(t_{2k}, \alpha_2(t_{2k}))$ with transfer point $(t_{2k+1}, \alpha_2(t_{2k+1}))$, $k = 1, 2, \ldots, n$.

(iv) For all $t \in [t_{2k+1}, t_{2k+2}]$, $(t, \alpha_2(t))$ is a narrow pass point, $k = 1, 2, \ldots, n$.

(v) Through $(t_{2n}, \alpha_2(t_{2n}))$ there is a final function $x_{n+1}(.)$.

Then the optimal state function $x^*(.)$ is defned by

$$x^*(t) = \begin{cases} x_{k+1}(t) & \text{if } t \in [t_{2k}, t_{2k+1}],\ k = 0, 1, 2, \ldots, n. \\ \alpha_2(t) & \text{if } t \in \bigcup_{k=0}^{n} [t_{2k+1}, t_{2k+2}], \end{cases}$$

By the algorithm, $(x^*(.), u^*(.))$ will be the unique optimal process of (P^*) where $u^*(.)$ is the optimal control function defined by

$$u^*(t) = \dot{x}^*(t) - b(t), \quad t \in [0, T].$$

Consequently, the optimal flow through the turbines is given by

$$w^*(t) = b(t) - \dot{x}^*(t)$$

for all $t \in [0, T]$.

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