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AN EXTENSION OF FILIPPOV-GRONWALL INEQUALITY TO NON-LIPSCHITZIAN DIFFERENTIAL INCLUSIONS

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Dedicated to Professor Imre Bihari on his 75th birthday

Abstract. *We extend the Filippov-Gronwall inequality from a Lipschitzian to a non-Lipschitzian differential inclusion and get a nonlinear inequality of Filippov-Peano type. As an application of our result we show that the set of solutions to the non-Lipschitzian differential inclusion is Hausdorff-continuously dependent on the initial data.*

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1. INTRODUCTION

The purpose of this work is to extend the Filippov-Gronwall inequality [6], [2, p. 120] from a Lipschitzian to a non-Lipschitzian differential inclusion. We obtain a nonlinear inequality which we call the inequality of Filippov-Peano type. As an application of our result we show that the set of solutions to the non-Lipschitzian differential inclusion is Hausdorff-continuously dependent on the initial values.

Our method is based on the techniques introduced in the fundamental works by Filippov [6], Antosiewicz-Cellina [1], Pianigiani [14] and on a Carathéodory version of the Peano differential inequality [8, 12].

As well known, under some monotony condition Peano differential inequalities have "integrated" analogues of Bihari-Viswanatham type [3, 4, 17] which are nonlinear generalizations of Gronwall-Bellman inequality [2, 8, 12]. These inequalities play a prominent role in the qualitative theory of nonlinear differential equations, integral equations, difference equations, functional equations, etc.

In what follows $|x|$ is the Euclidean norm of $x \in R^n$, $d(x,y) = |x-y|$, $d(x,Y) = \inf_{y\in Y}|x-y|$ and $D(X,Y) = \max\{\sup_{x\in X} d(x,Y), \sup_{y\in Y} d(X,y)\}$, the Hausdorff distance of the compact subsets X and Y of R^n. The Banach space of continuous functions from $I = [0,T]$ into R^n with the usual supremum norm $|\cdot|_C$ is denoted by $C(I)$. By $AC(I)$ we mean the Banach space of absolutely continuous functions from I into R^n with the norm

$$|x|_{AC} = |x(0)| + \int_0^T |\dot{x}(t)|dt.$$

Filippov [6] considered the Cauchy problem

$$\dot{x} \in F(t,x), \quad x(0) = x_0, \tag{1.1}$$

where $F : [0,T] \times R^n \to R^n$ is a continuous multifunction Lipschitzian in x with closed values. He proved that for each given approximate solution $y(t)$, i.e. an absolutely continuous function $y : [0,T] \to R^n$ such that the distance $\rho(t) := d(\dot{y}(t), F(t,y(t)))$ is integrable, there exists a true solution $x(t)$ to (1.1) verifying the inequalities of Gronwall type

$$|x(t) - y(t)| \le \eta(t) \tag{1.2}$$

and

$$|\dot{x}(t) - \dot{y}(t)| \le k(t)\eta(t) + \rho(t) \quad \text{a.e.} \tag{1.3}$$

where $k(t)$ is the Lipschitz constant of $F(t,\cdot)$,

$$\eta(t) = \delta \exp\left(\int_0^t k(s)ds\right) + \int_0^t \exp\left(\int_s^t k(u)du\right)\rho(s)ds \tag{1.4}$$

and $\delta = |x(0) - y(0)|$ (see also Aubin-Cellina [2, p. 120]).

This result of Filippov was extended by Himmelberg-Van Vleck [11] from a jointly continuous to a jointly measurable Lipschitzian multifunction F. Modifying Filippov's successive approximation [6] and applying the techniques of [1, 7]. Ornelas [13] found a continuous version of the Filippov-Gronwall inequality for Lipschitzian differential inclusions. Solutions depending continuously on the initial conditions were also obtained in [5] by a careful use of Liapunov's Theorem on the range of vector measures [7] and the Filippov-Gronwall inequality.

In the paper by Pianigiani [14] the fundamental theory of differential inclusions satisfying the conditions of Kamke type was investigated. Some interesting results on the differential inclusions of Kamke type in Banach spaces can be found in the book by Tolstonogov [16].

Let $x_0 \in R^n$ be a fixed point, B a closed ball of radius b centred at x_0, X a bounded subset of R^n and comp X stand for the family of nonempty compact subsets of X. Let F be a map defined on an open subset $Y \subset R^{n+1}$, $(0, x_0) \in Y$, with compact but not necessarily convex values. For simplicity, we can assume that F is defined on $[0,a] \times B$. Set $M = \sup\{|x| \,:\, x \in X\}$, $T = \min\{a, \frac{b}{M}\}$, $I = [0,T]$ and $J = [0, 2b]$.

In this paper we consider the Cauchy problem (1.1) where $F \,:\, I \times B \to$ comp X is continuous and such that a nonlinear condition of the form

$$D(F(t,x), F(t,y)) \leq \omega(t, |x-y|) \quad \text{a.e. on } I \tag{1.5}$$

is satisfied for every $x, y \in B$ with a Kamke function $\omega \,:\, I \times R_+ \to R_+ \,:= [0, \infty)$, i.e. ω is such that:

(K_1) $\omega(t, 0) \equiv 0$ a.e. on I,

(K_2) $\omega(t, u)$ is measurable in t for each fixed u,

(K_3) $\omega(t, u)$ is continuous in u for each fixed t.

(K_4) $\omega(t, u) \leq m(t)$ with $m \in L^1(I)$.

If F is Lipschitzian in x, then ω is linear in u, i.e. $\omega(t, u) = k(t)u$ with $k \in L^1(I)$ and, of course, all the conditions $(K_1 - K_4)$ are satisfied.

2. THE RESULT

Theorem. *Suppose that $F : I \times B \to compX$ is continuous and the conditions (1.5) and $(K_1 - K_4)$ are satisfied. Let $y : I \to B$ be an absolutely continuous function such that $\rho(t) := d(\dot{y}(t), F(t, y(t)))$ is integrable $(\rho \in L^1(I))$. Then there exists a solution $x = x(t)$ to (1.1) verifying*

$$|x(t) - y(t)| \leq r(t) \tag{2.1}$$

where $r(t)$ is the maximal solution of

$$\dot{u} = \omega(t, u) + \rho(t), \quad u(0) = \delta \,:= |x(0) - y(0)|. \tag{2.2}$$

If, moreover, the following condition is also satisfied:

(K_5) $\omega(t, u)$ is nondecreasing in u for each fixed t, then we have

$$|\dot{x}(t) - \dot{y}(t)| \leq \dot{r}(t) \quad \text{a.e. on } I. \tag{2.3}$$

Remark. It is well known that under the conditions $(K_2 - K_4)$ and the assumption that $\rho \in L^1(I)$ the maximal solution of the Carathéodory equation (2.2) exists (see, e.g., [12, p. 41]). If F is Lipschitzian, i.e. $\omega(t,u) = k(t)u$, then (2.2) becomes the linear equation, $\dot{u} = k(t)u + \rho(t)$, $u(0) = \delta$, hence $r(t)$ is now just $\eta(t)$ given in (1.4). Thus, we get the Filippov-Gronwall inequalities (1.2) and (1.3) as a special case of our inequalities of Filippov-Peano type (2.1) and (2.3), respectively. Moreover, unlike the theorem of Filippov our theorem can be also applied to the non-Lipschitzian differential inclusions, e.g. to the differential equation $\dot{x} = \sqrt{x}$, $x(0) = 0$, $t, x \in [0,1]$, with the Kamke function $\omega(t,u) = \sqrt{u}$, or more general to differential inclusions of the form (1.1) with a Kamke function like $\omega(t,u) = k(t)u^\alpha$ for any $0 < \alpha < 1$ $(k \in L^1(I))$, i.e. F satisfies Hölder's condition.

P r o o f of the Theorem. We shall prove the theorem by modifying the techniques of Antosiewicz-Cellina [1] and Pianigiani [14] and applying a Carathéodory version of the comparison theorem of Peano [8, 12]. Define

$$K = \{u \in C(I) \ : u(0) = x_0, |\dot{u}(t)| \leq M, \ t \in I\}.$$

It is clear that K is a compact and convex subset of $C(I)$. Let $w(t)$ be a measurable selection such that $w(t) \in F(t, y(t))$ and $|w(t) - \dot{y}(t)| = d(\dot{y}(t), F(t, y(t))) =: \rho(t)$ [9, 10]. We claim that there exists a continuous map $g : K \to L^1(I)$ such that for a.e. $t \in I$ and every $u \in K$, $g(u)(t) \in F(t, u(t))$ and

$$|g(u)(t) - w(t)| \leq \omega(t, |u(t) - y(t)|). \tag{2.4}$$

Let $\{\varepsilon_n\}_{n=0}^{\infty}$ be a sequence of positive numbers such that $\varepsilon_n \downarrow 0$ as $n \uparrow +\infty$ e.g. $\varepsilon_n = 2^{-n}$. By (K_2, K_3) and Scorza-Dragoni's theorem [15], for $\varepsilon_0 > 0$ there exists a closed subset I_0 of I such that $\mu(I \setminus I_0) < \varepsilon_0$, where μ is the Lebesgue measure in I, and $\omega(t,u)$ is (uniformly) continuous on $I_0 \times J$. Let $\delta_0 > 0$ be such that $|t - s| < \delta_0$, $|x - y| < \delta_0$ imply $|\omega(t, |x|) - \omega(s, |y|)| < \varepsilon_0$ $(s, t \in I_0)$ and $D(F(t,x), F(t,y)) < \varepsilon_0$ $(s, t \in I)$. By the compactness of K there exists an open cover $(U_i)_{i=1}^N$ of K with $\operatorname{diam}(U_i) < \delta_0$. Let (p_i) be a continuous partition of unity relative to (U_i). For each u, put $t_0(u) = 0, \ldots, t_i(u) = t_{i-1}(u) + Tp_i(u)$, $i = 1, \ldots, N$. For each $i = 1, \ldots, N$, fix $u_i \in U_i$ and let $v_i(t)$ be a measurable selection such that $v_i(t) \in F(t, u_i(t))$ and $|w(t) - v_i(t)| = d(w(t), F(t, u_i(t)))$.

Define $g^0 \ : K \to L^1(I)$ by $g^0(u)(t) = \sum_{i=1}^{N} \chi_i^u(t) v_i(t)$ where χ_i^u is the characteristic function of $I^i(u) \ := [t_{i-1}(u), t_i(u))$. For each $t \in [0, T)$ there exists $i \in \{1, \ldots, N\}$ such that $t \in I^i(u)$. Then, for every $u \in K$,

$$d(g^0(u)(t), F(t, u(t))) \leq d(v_i(t), F(t, u_i(t))) + D(F(t, u_i(t)), F(t, u(t))) < \varepsilon_0$$

for $t \in I$. and

$$|g^0(u)(t) - w(t)| = d(w(t), F(t, u_i(t))) \leq d(w(t), F(t, y(t)))$$
$$+ D(F(t, y(t)), F(t, u_i(t))) \leq \omega(t, |y(t) - u_i(t)|) \leq \omega(t, |u(t) - y(t)|) + \varepsilon_0$$

for $t \in I_0$.

Note that if $t \in I^i(u) \cap I^i(v)$ then $g^0(u)(t) = g^0(v)(t) = v_i(t)$. Since $\{t_i(\cdot)\}_{i=1}^N$ is a finite family of continuous real functions, it is an equicontinuous family, hence for every $\gamma > 0$ there exists $\beta > 0$ such that if $|u - v|_C < \beta$ then $\mu\{t : |g^0(u)(t) - g^0(v)(t)| > 0\} < \gamma$. This implies that $g^0 : K \to L^1(I)$ is continuous.

For $\varepsilon_1 > 0$ there exists a closed subset I_1 of I such that $\mu(I \setminus I_1) < \varepsilon_1$ and $\omega(t, u)$ is continuous on $I_1 \times J$. Let $\delta_1 > 0$ ($\delta_1 < \delta_0$) be such that $|t - s| < \delta_1$, $|x - y| < \delta_1$ and $|u - v|_C < \delta_1$ imply $|\omega(t, |x|) - \omega(t, |y|)| < \varepsilon_1$ ($s, t \in I_1$), $D(F(t, x), F(t, y)) < \varepsilon_1$ ($s, t \in I$) and $\mu\{t : |g^0(u)(t) - g^0(v)(t)| > 0\} < \varepsilon_0$. Let $(U_i^1)_{i=1}^{N_1}$ be an open cover of K with diam $(U_i^1) < \delta_1$ and let $p_i^1(u)$, $t_i^1(u)$ and χ_i^{1u} be defined as in the previous step. Fix $u_i^1 \in U_i^1$ and choose $v_i^1(t) \in F(t, u_i^1(t))$ such that $|v_i^1(t) - g^0(u_i^1)(t)| = d(g^0(u_i^1)(t), F(t, u_i^1(t)))$.

Define $g^1(u)(t) = \sum_{i=1}^{N_1} \chi_i^{1u}(t) v_i^1(t)$. Similarly as before, we see that $g^1 : K \to L^1(I)$ is continuous, $d(g^1(u)(t), F(t, u(t))) < \varepsilon_1$ ($t \in I$), $|g^1(u)(t) - w(t)| \leq \omega(t, |u(t) - y(t)|) + \varepsilon_1$, ($t \in I_1$) and

$$|g^1(u)(t) - g^0(u)(t)| \leq |v_i^1(t) - g^0(u_i^1)(t)| + |g^0(u_i^1)(t) - g^0(u)(t)|$$
$$\leq \varepsilon_0 + |g^0(u_i^1)(t) - g^0(u)(t)|.$$

Hence

$$\mu\{t : |g^1(u)(t) - g^0(u)(t) > \varepsilon_0\} \leq \mu\{t : |g^0(u_i^1)(t) - g^0(u)(t)| > 0\} < \varepsilon_0.$$

By induction, for evey $n = 0, 1, 2, \ldots$ we can construct a continuous map $g^n : K \to L^1(I)$ such that $d(g^n(u)(t), F(t, u(t))) < \varepsilon_n$ ($t \in I$), $|g^n(u)(t) - w(t)| \leq \omega(t, |u(t) - y(t)|) + \varepsilon_n$ ($t \in I_n$) with $\mu(I \setminus I_n) < \varepsilon_n$ and $\mu\{t : |g^{n+1}(u)(t) - g^n(u)(t)| > \varepsilon_n\} < \varepsilon_n$. Thus, for each $u \in K$, $\{g^n(u)\}$ converges in measure to some limit $g(u)$, hence there exists a subsequence $\{g^{n_k}(u)\}$ converges to $g(u)$ a.e. in I. It is clear that the map $g : K \to L^1(I)$ is continuous, $g(u)(t) \in F(t, u(t))$ and $|g(u)(t) - w(t)| \leq \omega(t, |u(t) - y(t)|)$ for every $u \in K$ and a.e. $t \in I$. Our claim (2.4) is proved.

Define $h : K \to K$ by

$$h(u)(t) = x_0 + \int_0^t g(u)(s)ds,$$

for every $u \in K$. Since h is continuous, by Schauder's theorem there exists a fixed point $x \in K$ such that

$$x(t) = h(x)(t) = x_0 + \int_0^1 g(x)(s)ds,$$

hence $\dot{x}(t) = g(x)(t) \in F(t, x(t))$ a.e. Moreover, by (2.4)

$$\begin{aligned} |\dot{x}(t) - \dot{y}(t)| &\le |g(x)(t) - w(t)| + |w(t) - \dot{y}(t)| \\ &\le \omega(t, |x(t) - y(t)|) + \rho(t). \end{aligned} \tag{2.5}$$

Denote the right derivative of a function f at t by $D_R f(t)$. Since $D_R|x(t) - y(t)| \le |\dot{x}(t) - \dot{y}(t)|$, (2.5) implies

$$D_R|x(t) - y(t)| \le \omega(t, |x(t) - y(t)|) + \rho(t) \quad \text{a.e.} \tag{2.6}$$

Applying a Carathéodory version of the comparison theorem of Peano [8, 12] we get (2.1) where $r(t)$ is the maximal solution of (2.2). Now if (K_5) is also satisfied, then by (2.5), (2.1) and (2.2) we have

$$|\dot{x}(t) - \dot{y}(t)| \le \omega(t, r(t)) + \rho(t) = \dot{r}(t).$$

Our theorem is proved.

The following result on the Hausdorff-continuous dependence of the set of solutions to the non-Lipschitzian differential inclusion on the initial data is obtained by an application of our theorem.

Corollary. *Let D be a closed subset of B. Suppose that the conditions of the theorem above and the following condition are satisfied:*
(K_6) the unique absolutely continuous solution of the Cauchy problem $\dot{u} = \omega(t, u)$, $u(0) = 0$, is $u(t) \equiv 0$.
Then the map $S : D \to AC(I)$ which associates to every $\xi \in D$ the set of solutions to $\dot{x} \in F(t, x)$, $x(0) = \xi$, is Hausdorff-continuous.

P r o o f. Let $\xi_1 \in D$ and $x_1 \in S(\xi_1)$. By our theorem, for every $\xi_2 \in D$ there exists a solution $x_2 \in S(\xi_2)$ such that

$$\int_0^T |\dot{x}_1(t) - \dot{x}_2(t)|dt \leq r(T) - |\xi_1 - \xi_2| \leq r(T) \tag{2.7}$$

where r is the maximal solution of

$$\dot{u} = \omega(t, u), \quad u(0) = |\xi_1 - \xi_2|. \tag{2.8}$$

By (K_6) and the upper simicontinuous dependence of the set of solutions to (2.8) on the initial data, given $\varepsilon > 0$ there exists $\delta > 0$ such that $|\xi_1 - \xi_2| < \delta$ implies $r(T) < \varepsilon$. Hence, by (2.7) $S(\xi_1) \subseteq B_{AC}(S(\xi_2), \varepsilon)$.

A symmetric argument shows also that $S(\xi_2) \subseteq B_{AC}(S(\xi_1), \varepsilon)$ if $|\xi_1 - \xi_2| < \delta$, hence concluding the proof.

The dependence of the set of solutions to Kamke differential inclusions on the initial data was also studied by Pianigiani [14, Th. 4] whose result is different from that stated in the corollary in this paper.

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