Journal of Mathematics *Tạp chí Toán học* **21**(1 & 2)(1993), 1–12

ON THE STRUCTURE OF INVOLUTION RINGS WITH CHAIN CONDITION

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Abstract. *Recent results due to U. A. Aburawash, K. I. Beĭdar, M. Domokos, P. H. Lee, N. V. Loi and the author ([1], [2], [3], [7], [8], [10], [14], [15] , [16]) are surveyed on the structure of rings with involution. Working within the category of all rings with involution and of all homomorphisms preserving also involution, one has the slight advantage of having more calculation rules, but one faces the disadvantage of having less homomorphisms. Thus, to be consistent in describing the structure of involution rings, one cannot work with one-sided ideals. An appropriate and efficient left - and right - symmetric notion is that of *- biideals. Imposing chain conditions on *-biideals the involutive versions of the Wedderburn - Artin, Goldie, Litoff - Ánh, Ayoub - Dinh Van Huynh theorems can be proved, and also stronger statements than for associative rings, can be achieved (for instance, if an involution ring has d.c.c. on *-biideals, then its Jacobson radical has d.c.c. on additive subgroups). An involution ring A is semiprime and finite if and only if the polynomial ring over A has d.c.c. on *-biideals. This latter result can be regarded as a counterpart of the Hilbert Basis Theorem. Also some open problems are posed.*

1. INTRODUCTION

The purpose of this note is to survey the most recent developments in describing the structure of involution rings satisfying chain conditions. These results have been published or are being published in the papers [1], [2], [3], [8], [10], [14], [15] and [16].

An *involution ring* A is a ring with an additional unary operation $*$, called *involution*, subjected to the familiar identities

$$(x+y)^* = x^* + y^*, \quad (xy)^* = y^* x^*, \quad x^{**} = x, \quad \forall x, y \in A.$$

The most common and natural examples for involution rings are the real and complex matrix rings with the usual involution.

Though involution rings have been intensively investigated in the context of ring theory with applications to Jordan algebras, Banach algebras and operator algebras, less attention was paid till recently to the study of involution rings as such, that is, to their description in terms within the category of involution rings. While working in the category of involution rings one has the slight advantage of having more calculation rules, but on the other hand one has less homomorphic mappings, than in the category of rings: all homomorphisms considered (and so also embeddings) have to preserve involution too. Thus only subrings are admitted which are closed under involution. Since every one-sided ideal which is closed under involution must be a two-sided ideal, the term "one-sided ideal with involution" cannot play any role. As a consequence there is no "module theory" over involution rings. In the investigation of rings it is quite efficient to impose chain conditions on one-sided ideals (chain conditions on two-sided ideals are usually too weak for describing the structure of rings). Since the notion of one-sided ideals is meaningless for involution rings, an appropriate notion has had to be found which has generalized that of one-sided ideals and which has worked in describing the structure of involution rings. It has turned out that such a desired notion is that of *-biideals. A **-biideal* B of an involution ring A is a subring such that

$$BAB \subseteq B \quad \text{and} \quad B^{(*)} \subseteq B$$

where

$$B^{(*)} = \{b^* \in A : b \in B\}.$$

For rings without involution biideals have been introduced by Lajos and Szász [13] in 1971, and the first significant application of that notion in describing the structure of rings appeared in the Habilitationschrift of Widiger [19] in 1978. It was Loi who first used successfully *-biideals in proving structure theorems for involution rings in his paper [15].

The *-biideal of an involution ring generated by a subset S will be denoted by $(S)^*$. A *principal *-biideal* is a *-biideal generated by a single element. Obviously, the principal *-biideal $\langle a\rangle^*$ of an involution ring A is given by

$$\langle a\rangle^* = \mathbf{Z}a + \mathbf{Z}a^* + aAa + aAa^* + a^*Aa + a^*Aa^*$$

where $\mathbf{Z}$ denotes the ring of integers. In the ring $M_n(\mathbf{R})$ of all $n \times n$ real matrices with the usual involution, let us consider the matrix e_{ij} with a single 1 at the (i,j) position and 0 everywhere else. As one can easily verify, the principal *-biideal $\langle e_{ij}\rangle^*$ consists of all matrices which have arbitrary real numbers at the (i,i), (i,j), (j,i) and (j,j) positions and 0 everywhere else.

In the sequel, a ring will mean, if not specified, an associative ring not necessarily with 1. If I is an ideal of an involution ring A which is closed under involution, then we shall say that I is a **-ideal* of A and denote this fact by $I \triangleleft^* A$. For descending chain condition and ascending chain condition we shall briefly write d.c.c. and a.c.c., respectively.

Let A be any ring with involution and A^{op} its oposite ring. On the direct sum

$$C = A \oplus A^{op}$$

we may define an involution *, called the *exchange involution*, by the rule

$$(x,y)^* = (y,x) \qquad \forall (x,y) \in C.$$

If the ring A happens to be simple, then the only *-ideals of C are 0 and C itself. We say that an involution ring C is *-simple, if $C^2 \neq 0$ and 0 and C are the only *-ideals of C. By the above example, a *-simple involution ring need not be simple as a ring, althoulgh the converse is trivially true.

Throughout this paper $M_n(D)$ will denote the $n \times n$ matrix ring over a division ring D. Notice that if $M_n(D)$ is an involution ring, then so is also D. Further , $K_m(D)$ will stand for the direct sum $K_m(D) = M_m(D) \oplus M_m^{op}(D)$ endowed with the exchange involution.

2. SEMIPRIME INVOLUTION RINGS

As is well-known, a ring A is said to be *semiprime*, if $I \triangleleft A$ and $I^2 = 0$ force $I = 0$. Correspondingly, an involution ring A may be called **-semiprime*, if $I \triangleleft^* A$ and $I^2 = 0$ imply $I = 0$. As one can easily prove, an involution ring A is *-semiprime if and only if A is semiprime, and therefore there is no need to distinguish these notions.

In [1] Aburawash described the structure of semiprime involution rings with d.c.c. on *-biideals. Let us recall that a ring is said to be *artinian*, if it satisfies d.c.c. on right ideals.

Theorem 1. *For a semiprime ring A with involution * the following conditions are equivalent:*

*(i) A has d.c.c. on *-biideals,*

(ii) A is artinian as a ring without involution,

(iii) A is a finite direct sum

$$A \cong \sum_{i=1}^{k}{}^{\oplus} M_{n_i}(D_i) \oplus \sum_{j=1}^{\ell}{}^{\oplus} K_{m_j}(D_j).$$

Corollary 1. *Every semiprime involution ring with d.c.c. on *-biideals satisfies also a.c.c. on *-biideals.*

For *-simple rings with d.c.c. on *-biideals Theorem 1 reads as follows.

Corollary 2 ([2]). *An involution ring A is *-simple and has d.c.c. on *-biideals if and only if either A is a semisimple ring and $A \cong M_n(D)$ or it is not a simple ring and $A \cong K_m(D)$.*

Theorem 1 and Corollary 2 are, in fact, the involutive versions of the Wedderburn Artin Structure Theorems.

Imposing d.c.c. on principal *-biideals is certainly a weaker requirement, than d.c.c. on *-biideals inasmuch as any infinite direct sum of involution rings with d.c.c. on *-biideals satisfies d.c.c. on principal *-biideals. As we shall see, also d.c.c. on principal *-biideals yields fairly good structure theorems.

Chronologically the following theorem of Loi [15] was the first result on classifying the structure of involution rings in terms of *-biideals and it has become the starting point of the later investigations.

Theorem 2. *A semiprime involution ring A satisfies d.c.c. on principal *-biideals if and only if A as a ring without involution satisfies d.c.c. on principal right ideals.*

Theorem 2 describes completely the structure of A inasmuch as A as a ring is a discrete direct sum of simple rings of linear transformations of finite rank of a vector space over a division ring. Moreover, a direct summand of that decomposition is either closed under involution or not, and correspondingly a similar, but may be infinite decomposition can be achieved as in Theorem 1 (iii).

3. INVOLUTION RINGS WITH RADICAL

Given a Kurosh-Amitsur radical ρ of rings the question arises as whether the radical $\rho(A)$ of an arbitrary involution ring A is closed under involution. To solve this problem, a very easily testable criterion was given in [14].

Theorem 3. *For a Kurosh-Amitsur radical ρ of rings the following two conditions are equivalent:*

*(i) the radical $\rho(A)$ is a *-ideal in every involution ring A,*

(ii) $\rho(R) = R$ implies $\rho(R^{op})$ for every ring R.

*In particular, the Baer (that is, prime) radical $\beta(A)$, the Jacobson radical $J(A)$ as well as the other classical radicals (the Levitzki, Köthe's nil, the Brown-McCoy radical) are *-ideals in every involution ring A.*

Let us recall that the maximal torsion subgroup of an involution ring A forms a *-ideal T, called the *maximal torsion ideal* of A.

Theorem 4 ([16]). *If A is an involution ring with d.c.c. on principal *-biideals, then*

$$\beta(A) = J(A) \subseteq T.$$

*In particular, if an involution ring with d.c.c. on *-biideals is torsionfree, then it is Jacobson semisimple.*

Applying Theorems 2 and 3 also the following *splitting theorem* was proved in [16].

Theorem 5. *Let A be an involution ring and T be its maximal torsion *-ideal. If A satisfies d.c.c. on pricipal *-biideals, then A decomposes into a direct sum*

$$A = T \oplus F$$

*where F is a uniquely determined torsionfree *-ideal. Moreover, F is semiprime and has d.c.c. on principal *-biideals.*

The statement of Theorem 5 can be considered as the involutive version of the splitting theorem of Christine Ayoub [6] and Dinh Van Huynh [9].

Imposing d.c.c. on *-biideals, also the structure of non-semiprime involution rings can be described.

Theorem 6 ([8]). *If A is an involution ring with d.c.c. on *-biideals, then its Jacobson radical $J(A)$ satisfies d.c.c. on additive subgroups and $J(A)$ is nilpotent. If, in addition, $J(A)$ is reduced (that is, it has no divisible additive subgroup), then $J(A)$ is finite.*

A relatively easy consequence of the not so easy Theorem 6 is

Corollary 3 ([8]). *An involution ring A has d.c.c. on *-biideals if and only if A is an artinian ring and so is $J(A)$.*

Let us remind the reader that the structure of artinian rings with artinian Jacobson radical has been fully described by Kertész and Widiger [12] (cf. also [11] Theorems 65.2 and 67.4): an artinian ring A with artinian Jacobson radical is the direct sum of finitely many matrix rings over infinite division rings and of a ring C which satisfies d.c.c. on additive subgroups (also the structure of C is fully determined, $J(A) \subseteq C$ and $C/J(A)$ is a finite direct sum of matrix rings over finite fields).

4. INVOLUTION RINGS WITH D.C.C. ON PRINCIPAL *-BIIDEALS

In Theorems 2, 3 and 5 we have already seen results on the structure of involution rings with d.c.c. on principal *-biideals. Here we shall present more results on that area.

Let us recall that the structure of a simple ring possessing a minimal right ideal is described by the Litoff-Ánh Theorem [4]. We say that A is a *strongly locally matrix ring* if there exists a division ring D such that every finite subset F of A can be embedded into a biideal B of A such that B is isomorphic to a matrix ring $M_n(D)$ over D for some $n \geq 1$. The *Litoff-Ánh Theorem* asserts that A is a *simple ring with minimal right ideal if and only if A is a strongly locally matrix ring.* (In [4] Ánh used Steinfeld's quasi-ideal (cf. [11] or [18]) for biideal but in our context these notions coincide.) Moreover, a simple ring has a minimal right ideal if and only if it has d.c.c. on principal right ideals, and hence it is a ring of linear transformations of finite rank of a vector space over a division ring.

The involutive version of the Litoff-Ánh Theorem has been proved in [2]. Following Aburawash [2] we say that an involution ring A is a **-strongly locally matrix ring* over a division ring D, if every finite subset F of A can be embedded into a *-biideal B of A such that $B \cong M_n(D)$ whenever A is a simple ring, and $B \cong K_m(D)$ whenever A is not simple as a ring (the number n or m depends on F).

Theorem 7 ([2]). *An involution ring A is *-simple and possesses a minimal *-biideal if and only if A is a *-strongly locally matrix ring.*

In Theorem 7 the *-simple involution ring A is, of course, semiprime, and has d.c.c. on principal *-biideals. Thus, in view of Theorem 2, the structure of semiprime involution rings with d.c.c. on principal *-biideals has been completely determined also in terms of *-strongly locally matrix rings.

In the description of involution rings having nonzero Baer radical (or Jacobson radical), demanding only d.c.c. on principal *-biideals seems not to be sufficient, also d.c.c. on *-biideals on the Jacobson radical is needed.

Theorem 8 ([3]). *For an involution ring A the following conditions are equivalent:*

*(i) A has d.c.c. on principal *-biideals and its Jacobson radical $J(A)$ has d.c.c. on *-biideals,*

(ii) A as a ring without involution has d.c.c. on principal right ideals and $J(A)$ is an artinian ring,

(iii) A is a discrete direct sum

$$A = \sum_{\lambda\in\Lambda}{}^{\oplus} B_\lambda \oplus C,$$

*where each B_λ is a *-strongly locally matrix ring over an infinite division ring D with involution, such that $J(A) \subseteq C$ and $C/J(A)$ is a discrete direct sum of *-strongly locally matrix rings over finite fields with involution.*

Let us mention that the non-involutive version of Theorem 8 concerning rings with d.c.c. on principal right ideals and with artinian Jacobson radical has been given by Ánh in [5].

5. A.C.C. ON *-BIIDEALS

The results of this section have been proved in [8].

Before going to present results on involution rings with a.c.c. on *-biideals, we illustrate the connection between a.c.c. on right ideals (that is, being *right noetherian*) and a.c.c. on biideals on rings without involution.

Even for commutative rings, a.c.c. on biideals is a stronger requirement than being noetherian. Namely, for a commutative ring A the following conditions are equivalent:

(i) A has a.c.c. on biideals,

(ii) A is noetherian and for every ideal I of A the additive group of I/I^2 is finitely generated,

(iii) A is noetherian and for every ideal I of A the additive group of the Baer radical $\beta(A/I)$ of A/I is finitely generated.

A not necessarily commutative but associative ring A has a.c.c. on biideals if and only if A is left and right noetherian and for every subring G of A the additive group of the factor ring $G/(\langle G\rangle \cap G)$ is finitely generated where $\langle G\rangle$ stands for the biideal generated by G.

In the case of involution rings there exist an involution ring A such that A is left and right noetherian as a ring but A does not satisfy a.c.c. on *-biideals. We can state also

Theorem 9 ([8]). *If an involution ring A has a.c.c. on *-biideals, then for every *-subring G of A the additive group of the factor ring $G/(\langle G\rangle^* \cap G)$ if finitely generated. If an involution ring A is left anf right noetherian as a ring and for every *-subring G of A the additive group of $G/(\langle G\rangle^* \cap G)$ is finitely generated, then A has a.c.c. on *-biideals.*

The connection between a.c.c. on biideals (a.c.c. on *-biideals) and other chain conditions seems to be a delicate question and far from being settled. In this context we repeat the problems posed in [8].

Problem 1. *Does a semiprime involution ring with a.c.c. on *-biideals satisfy a.c.c. on biideals ?*

Problem 2. *Does there exist a prime non-artinian ring with a.c.c. on biideals ?*

Concerning the Baer radical we have

Theorem 10. *If an involution ring A has a.c.c. on *-biideals, then its Baer radical $\beta(A)$ is nilpotent and the additive group of $\beta(A)$ is finitely generated.*

The strong effect of a.c.c. on *-biideals is exhibited by the following surprising result which has been recently proved by Beĭdar [7]: if an involution ring A has a.c.c. on *-biideals and A is an algebra over an infinite field, then A is a semiprime artinian ring. Also the corresponding assertion holds true for rings without involution. In view of Theorem 1 and Corollary 1 Beĭdar's result [7] can be formulated as

Theorem 11. *Let A be an involution ring as well as an algebra over an infinite field (with the same addition and multiplication). A satisfies a.c.c. on *-biideals if and only if A is semiprime and satisfies d.c.c. on *-biideals.*

This result can be regarded as a partial converse of Corollary 1 and a partial sharpening of Theorem 10.

Considering the polynomial ring $A[x]$ over an involution ring A, the involution $*$ of A may be extended to $A[x]$ by defining either $x^* = x$ or $x^* = -x$. Speaking of a polynomial ring $A[x]$ over an involution ring A, we assume in the sequel that the involution of A is extended to $A[x]$ in one of these two possibilities.

Imposing a.c.c. on *-biideal on the polynomial ring $A[x]$ has a very strong effect on the involution ring A, as we see it in the following

Theorem 12. *An involution ring A is semiprime and finite (hence a finite direct sum of matrix rings over finite fields) if and only if the polynomial ring $A[x]$ has a.c.c. on *-biideals.*

Theorem 12 can be considered as a counterpart of the *Hilbert Basis Theorem* which states that a ring A with identity is right noetherian if and only if the polynomial ring $A[x]$ is right noetherian.

Let us mention that in [8] Theorem 9 has been proved for not necessarily associative rings (with necessary modification of the definition of a *-biideal) and Theorem 12 is valid also for a not necessarily associative ring A with an extra condition imposed on A in terms of fields of rational functions.

6. GOLDIE'S THEOREMS

In the classical structure theory of rings a very important major branch is Goldie's theory of rings of quotients (cf. for instance [11] or [17]). In this theory again chain conditions on certain one-sided ideals play an important role. So it is natural to ask for involutive versions of Goldie's Theorems formulated in terms of left and right symmetric conditions. This job was done recently by M. Domokos [10].

As is well-known, a ring Q is said to be a *right (classical) ring of quotients* of its subring R, or in other words, the subring R is a *right order* in the ring Q if the following three conditions hold:

(i) Q has an identity 1,

(ii) each elements $s \in S(R)$ is a *unit* in Q (that is, s has a multiplicative inverse in Q) where $S(R)$ denotes the set of all cancellative elements (that is, non-zero devisors) of R,

(iii) to every element $x \in Q$ there are elements $a \in R$ and $s \in S(R)$ such that $x = as^{-1}$ holds.

Left ring of quotients and *left order* are defined correspondingly. We say that Q is a *ring of quotients* of R (or R is and *order* in Q), if Q is both a right and a left ring of quotients of R. The involution makes the situation symmetric as seen from the next assertion.

Proposition 1. *An involution ring has a right ring of quotients if and only if it has a left ring of quotients, and they are isomorphic.*

Thus we may speak only of rings of quotients of involution rings.

Proposition 2. *If R is an involution ring and Q is a ring of quotient of R, then there is a unique involution on Q which extends that of R to Q.*

In accordance with our terminology, a **-subring* will mean a subring closed under involution. We say that an involution ring Q is a **-ring of quotients* of its *-subring R (or R is a **-order* in Q), if Q is a ring of quotients of the *-subring R. Proposition 2 shows that an involution ring R has a *-ring of quotients if and only if R has a ring of quotients, further, the *-ring of quotients of R is uniquely determined up to isomorphism.

The right annihilator $r_R(G)$ of any subset G of a ring R is a right ideal, and the left annihilator $\ell_R(G)$ is a left ideal. One has clearly the relation $r_R(G)^{(*)} = \ell_R(G^{(*)})$, furthermore, the intersection $r_R(G) \cap r_R(G)^{(*)}$ is a *-biideal of the involution ring R. A *-biideal B of the involution ring R is called an *annihilator *-biideal*, if there exists a subset G of R such that $B = r_R(G) \cap r_R(G)^{(*)}$.

An involution ring R will be called a *Goldie involution ring*, if the following two conditions are fulfilled:

(i) there is no infinite sequence $B_1, \ldots, B_n, \ldots$ of nonzero *-biideals of R such that $(B_1 + \cdots + B_n)^* \cap B_{n+1} = 0$ for all $n = 1, 2, \ldots$, that is, the *maximum condition on *-biideal direct sums* is satisfied,

(ii) there is no infinite strictly ascending chain

$$r_R(G_1) \cap r_R(G_1)^{(*)} \subset \cdots \subset r_R(G_n) \cap r_R(G_n)^{(*)} \subset \ldots$$

where $G_1, \ldots, G_n, \ldots$ are subsets of R, that is, R satisfies the *a.c.c. on annihilator *-biideals.*

After these preparations we are ready to formulate the involutive versions of Goldie's Theorems as given by Domokos in [10].

Theorem 13. *For an involution ring R the following conditions are equivalent:*

(i) R is a semiprime Goldie involution ring

*(ii) R has a *-ring of quotients Q and*

$$Q \cong \sum_{i=1}^{k}{}^{\oplus} M_{n_i}(D_i) \oplus \sum_{j=1}^{\ell} K_{m_j}(D_j),$$

*that is, R is a *-order in the semiprime involution ring Q with d.c.c. on *-biideals.*

Theorem 13 corresponds to Goldie's Second Theorem on rings of quotients.

An involution ring R is said to be **-prime*, if R does not contain nonzero *-ideals K and L such that $KL = 0$. In contrast to *-simplicity, *-primeness does not imply primeness only semiprimeness. The involutive version of Goldie's First Theorem reads as follows.

Theorem 14 ([10]). *The following two conditions are equivalent for an involution ring R:*

*(i) R is a *-prime Goldie involution ring,*

*(ii) R has a *-ring of quotients Q and*

$$Q \cong M_n(D) \quad \text{or} \quad Q \cong K_m(D),$$

*that is, R is a *-order in the *-simple involution ring Q with d.c.c. on *-biideals.*

An immediate consequence of Theorems 13 and 14 is

Corollary 4. *If R is a semiprime (*-prime) involution ring with a.c.c on *-biideals, then R is a *-order in a semiprime (*-simple) involution ring Q with d.c.c. on *-biideals.*

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Received February, 1993