Journal of Mathematics *Tạp chí Toán học* **21**(1 & 2)(1993), 20–37

AVERAGING METHOD FOR HYPERBOLIC INCLUSIONS

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Abstract. *We consider the averaging problem for hyperbolic inclusions of arbitrary order. A theorem is proved on the mutual approximation between the solution sets of the original and the averaging inclusions.*

1. INTRODUCTION

The averaging method is an asymptotic method that was widely used in the study of nonlinear mechanics problems. First elaborated by Bogoliubov and Krylov [2], it was subsequently developed in the works of many authors [1, 3, 7, 14, 15]. The basic idea of the method is the use of an averaging operator to associate with the given original system a simpler one which preserves the main features of the object, while being easier to study. By averaging one can reduce a nonstationary system to a stationary one, thus allowing time saving in the calculations.

In control theory, the foundation of averaging methods for differential inclusions was established first by Plotnikov [16] for ordinary differental inclusions and subsequently by Vitiuk and Klimenko [23] for second order hyperbolic differential inclusions with no derivative on the right hand side, by Tuan [18] and Khapaev, Filatov [9, 10] for ordinary differential inclusions with fast and slow variables. The proof of these authors is based on the construction of approximating functions to solutions of the given inclusion on subintervals and applying Filippov's lemma [6]. That proof is some what complicated and rather of a special character.

A simpler and more general approach to the foundation of averaging method for differential inclusions was introduced by the author in [19, 20, 21]. This approach has enabled us to established Bogoliubov's type averaging theorems for various classes of differential inclusions in Banach space, including differential inclusions in standard form, differential inclusions with fast and slow variables, differential inclusions with retarded arguments and differential inclusions in semi-implicit form.

In the present paper we are concerned with the averaging problem for hyperbolic inclusions in standard form of arbitrary order which may contain partial derivatives on the right hand side.

The paper consists of 4 sections. After the Introduction, in Section 2 we shall present some auxilinary results which may have by themselves an independent interest. In Section 3, using the same approach as in [19, 20, 21] we will establish a theorem on the continuous dependence on a parameter for the solution set of inclusions with a right hand side integrally continuously depending on that parameter. Then in Section 4 we shall prove results on the mutual approximation between the solution sets of the original and the averaging inclusions.

Throughout in the sequel Comp R^n (Conv R^n) will denote the collection of all nonempty compact (convex and compact, resp.) subsets of R^n, equipped with the Hausdorff metric $\alpha(.,.)$, $\|.\|$ the norm in R^n, $\rho(u, A)$ the distance from point u to set A, $|A|$ the modulus of the set A, i.e. the number $\alpha(A, \theta)$ where θ is the null element of R^n. Integrals of multivalued maps are understood in Aumann sense. For $G \in \operatorname{Comp} R^m$, $L(G)$ $(C(G), AC(G)$, resp.) will denote the space of Lebesque integrable (continuous, absolutely continuous, resp.) functions from G into R^n. For $D \in \operatorname{Comp} R^n$, $C(G, D)$ will denote the set of all functions $f(.) \in C(G)$ taking values in D. Finally, $I = [0, a]$ is a segment of the real positive axis $R^+ = [0, +\infty)$.

2. PRELIMINARY PROPOSITIONS

We begin with some auxiliary results

Lemma 1. *Let $D \in$ Comp R^n, $\Lambda \subset R$ and let $F(x, u, \lambda) : I^m \times R^n \times \Lambda \to$ Conv R^n be a multivalued map satisfying the following conditions:*

1) F is measurable in x for fixed (u, λ);

2) F is uniformly continuous in u on D for fixed (x, λ);

3) There exists an integrable function $\omega(x)$ on I^m such that

$$|F(x, u, \lambda)| \leq \omega(x) \quad \textit{for all} \quad (x, u, \lambda);$$

4) F is integrally continuous in λ at some given limit point $\lambda_0 \in \Lambda$, i.e.

$$\lim_{\lambda\to\lambda_0} \alpha\Big(\int_0^x F(y,u,\lambda)dy, \int_0^x F(y,u,\lambda_0)dy\Big) = 0$$

for every $x \in I^m$, $u \in D$.

Then for any family of continuous functions $\{u(x,\lambda),\ \lambda \in \Lambda\} \subset C(I^m, D)$ *that converges uniformly with respect to* $x \in I^m$ *to a function* $u(x,\lambda_0)$ *as* $\lambda \to \lambda_0$, *we have:*

$$\lim_{\lambda\to\lambda_0} \alpha\Big(\int_0^x F(y,u(y,\lambda),\lambda)dy, \int_0^x F(y,u(y,\lambda_0),\lambda_0)dy\Big) = 0.$$

In particular, if $E \subset C(I^m, D)$ *is a compact subset of the Banach space* $C(I^m)$, *then for any* $\eta > 0$ *there exists a neighbourhood* $U(\lambda_0)$ *of point* λ_0 *such that for all* $\lambda \in U(\lambda_0)$ *and* $u(.) \in E$

$$\alpha\Big(\int_0^x F(y,u(y),\lambda)dy, \int_0^x F(y,u(y),\lambda_0)dy\Big) \le \eta.$$

Proof. See [20].

The argument is similar to that used to establish the corresponding result in [11] or [12].

Lemma 2. *Assume that F satisfies all the conditions of Lemma 1 and, in addition,* $|F(x,u,\lambda)| \le M$, $M > 0$, *for every* (x,u,λ). *Let* $E \subset C(I^m, D)$ *be a compact subset of* $C(I^m)$. *Then for any* $\eta > 0$ *there exists a neighbourhood* $U(\lambda_0)$ *of point* λ_0 *such that for all* $\lambda \in U(\lambda_0)$ *and all* $u(.) \in E$ *the following properties hold:*

1) For each measurable selection $v(x) \in F(x,u(x),\lambda)$ *there exists a measurable selection* $\overline{v}(x) \in F(x,u(x),\lambda_0)$ *satisfying*

$$\Big\|\int_0^x (v(y) - \overline{v}(y))dy\Big\| \le \eta \quad \forall x \in I^m. \tag{1}$$

2) For each measurable selection $\overline{v}(x) \in F(x,u(x),\lambda_0)$ *there exists a measurable selection* $v(x) \in F(x,u(x),\lambda)$ *satisfying (1).*

Proof.

We need only prove the first assertion because the proof of the second assertion is similar.

Divide the interval I into ℓ equal subintervals by the points t_j, $j = 0, 1, \ldots, \ell$. Let $G_{j_1,\ldots,j_m} = \{(x_1, x_2, \ldots, x_m) \in I^m : t_{j_k} < x_k \le t_{j_k+1},\ k = 1, 2, \ldots, m\}$. Then I^m is the union of ℓ^m subdomains $G_{j_1,\ldots,j_m}$ corresponding to all possible indices $\sigma = (j_1, \ldots, j_m)$. To simplify the notation, whenever convenient we shall write G_σ for $G_{j_1,\ldots,j_m}$. According to Lemma 1, for any $\overline{\eta} > 0$ there exists a neighbourhood $\overline{U}(\lambda_0)$ of point λ_0 such that

$$\alpha\Big(\int_0^x F(y, u(y), \lambda)dy,\ \int_0^x F(y, u(y), \lambda_0)dy\Big) \le \overline{\eta} \quad \forall\lambda \in \overline{U}(\lambda_0),\ \forall x \in I^m.$$

Hence, upon simple computation,

$$\alpha\Big(\int_{G_\sigma} F(y, u(y), \lambda)dy,\ \int_{G_\sigma} F(y, u(y), \lambda_0)dy\Big) \le 2^m\overline{\eta}.$$

This implies the existence of a measurable selection $v_\sigma \in F(x, u(x), \lambda_0)$ $(x \in G_\sigma)$ such that

$$\Big\|\int_{G_\sigma} v(y)dy - \int_{G_\sigma} v_\sigma(y)dy\Big\| \le 2^m\overline{\eta}. \tag{2}$$

Define now a measurable selection $\overline{v}(x) \in F(x, u(x), \lambda_0)$ by setting

$$\overline{v}(x) = v_\sigma(x) \quad \text{for} \quad x \in G_\sigma.$$

Then for any $x \in I^m$ (i.e. $x \in G_\sigma$ for some $\sigma = (j_1 \ldots j_m)$) we derive from (2)

$$\Big\|\int_0^x v(y)dy - \int_0^x \overline{v}(y)dy\Big\| \le \sum_{i_1=0}^{j_1} \cdots \sum_{i_m=0}^{j_m} \Big\|\int_{G_{i_1\ldots i_m}} (v(y) - \overline{v}(y))dy\Big\|$$

$$+\Big\|\int_{x_{j_1}}^{x_1} \cdots \int_{x_{j_m}}^{x_m} (v(y) - \overline{v}(y))dy\Big\| \le \ell^m 2^m\overline{\eta} + 2Ma^m/\ell^m.$$

To complete the proof it remains to select ℓ and $\overline{\eta}$ so that $2Ma^m/\ell^m \leq \eta/2$, $\overline{\eta} \leq \eta/\ell^m 2^{m+1}$. □

Lemma 3. *Let $P : I^m \to \text{Comp } R^n$ be a measurable multivalued map and $\omega : I^m \to R^n$ a measurable function on I^m. Then there exists a measurable selection $g(x) \in P(x)$ such that*

$$\|\omega(x) - g(x)\| = \rho(\omega(x), P(x)) \quad \text{for almost every} \quad x \in I^m.$$

Proof.

The function $r(x) = \rho(\omega(x), P(x))$ is obviously measurable on I^m, so the map $R(x) = \omega(x) + r(x).S$ with $S = \{u \in R^n : \|u\| = 1\}$ is also measurable on I^m. By Lemma 1.7.5 in [24] the map $Q : I^m \to \text{Comp } R^n$ defined by $Q(x) = P(x) \cap R(x)$ is also measurable on I^m and by Lemma 1.7.7 in [24] there exists a measurable selection $g(x) \in Q(x)$ which obviously satisfies the required condition. □

Lemma 4. *Let $x_i \in I$, $i = 1, \ldots, m$. For natural number $\overline{n}$ we have*

$$\int_0^{x_1} \Big(\overline{x}_1 + \sum_{i=2}^m x_i\Big)^{\overline{n}} d\overline{x}_1 \leq \Big(\sum_{i=1}^m x_i\Big)^{\overline{n}+1} / (\overline{n}+1), \tag{3}$$

$$\int_0^{x_1}\int_0^{x_2} \Big(\overline{x}_1 + \overline{x}_2 + \sum_{i=3}^m x_i\Big)^{\overline{n}} d\overline{x}_2 d\overline{x}_1 \leq b\Big(\sum_{i=1}^m x_i\Big)^{\overline{n}+1} / (\overline{n}+1), \tag{4}$$

$$\int_0^{x_1}\int_0^{x_2}\cdots\int_0^{x_k} \Big(\sum_{i=1}^k \overline{x}_i + \sum_{i=k+1}^m x_i\Big)^{\overline{n}} d\overline{x}_k \ldots d\overline{x}_2 d\overline{x}_1 \leq b^{k-1}\Big(\sum_{i=1}^m x_i\Big)^{\overline{n}+1} / (\overline{n}+1), \tag{5}$$

$$\int_0^{x_1}\int_0^{x_2}\cdots\int_0^{x_m} \Big(\sum_{i=1}^m \overline{x}_i\Big) d\overline{x}_m \ldots d\overline{x}_2 d\overline{x}_1 \leq b^{m-1}\Big(\sum_{i=1}^m x_i\Big)^{\overline{n}+1} / (\overline{n}+1), \tag{6}$$

where $b = ma/3$.

Proof.

The inequality (3) is obivious. To prove (4) observe that

$$\int_0^{x_1}\int_0^{x_2}\Big(\overline{x}_1+\overline{x}_2+\sum_{i=3}^m x_i\Big)^{\overline{n}} d\overline{x}_2 d\overline{x}_1 = \int_0^{x_1}\frac{1}{\overline{n}+1}\Big[\Big(\overline{x}_1+\sum_{i=2}^m x_i\Big)^{\overline{n}+1}$$

$$-\Big(\overline{x}_1+\sum_{i=3}^m x_i\Big)^{\overline{n}+1}\Big]d\overline{x}_1 \le \int_0^{x_1}\frac{1}{\overline{n}+1}\Big(\overline{x}_1+\sum_{i=2}^m x_i\Big)^{\overline{n}+1} d\overline{x}_1$$

$$\le \Big(\sum_{i=1}^m x_i\Big)^{\overline{n}+2}/(\overline{n}+1)(\overline{n}+2) \le ma\Big(\sum_{i=1}^m x_i\Big)^{\overline{n}+1}/(\overline{n}+1)(\overline{n}+2).$$

Since $\overline{n} \ge 1$ the inequality (4) is proved. By induction the inequalities (5) and (6) can be easily proved in an analogous manner. □

Consider now a differential inclusion

$$\partial^m u(x)/\partial x \in F[u](x), \quad x \in I^m \tag{7}$$

with the boundary conditions

$$u^i(x) = \theta \quad (i = 1, 2, \ldots, m). \tag{8}$$

Here $F[u](x) = F(x, [u]_x, u(x))$; $[u]_x = \big(\partial u(x)/\partial x_1, \ldots, \partial u(x)/\partial x_m,$ $\partial^2 u(x)/\partial x_1\partial x_2, \ldots, \partial^2 u(x)/\partial x_{m-1}\partial x_m, \ldots, \partial^{m-1}u(x)/\partial x_2\partial x_3\ldots\partial x_m\big)$, i.e. $[u]_x$ is composed of (2^m-2) mixed derivatives of the function $u(x) : \partial^k u/\partial x_{i_1}\ldots\partial x_{i_k}$, $i \le k \le m-1$, $i_\ell < i_j$ for $\ell < j$; $u^i(x) = u(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_m)$, $x = (x_1, x_2, \ldots, x_m)$, $\partial x = \partial x_1 \partial x_2 \ldots \partial x_m$.

By a solution of the problem (7)-(8) we mean any absolutely continuous function on I^m satisfying the inclusion (7) almost everywhere on I^m and satisfying, moreover, the boundary conditions (8).

Denote $C_m^k = m!/k!(m-k)!$, $H_k = a^{k-1}/k$, $H = \sum_{k=1}^m C_m^{m-k} H_k$, $b = ma/3$, $d = [(b+1)^m - 1]/b$, $\overline{m} = (2^m-1)n$. Then the following proposition holds.

Theorem 1. *Assume the map* $F : I^m \times R^{\overline{m}} \times R^n \to \mathrm{Comp}\, R^n$ *satisfies the following conditions:*

1) F is measurable in x for each fixed (y, u) and there exists a function $\gamma(x)$ integrable on I^m such that

$$|F(x,y,u)| \le \gamma(x) \quad \textit{for all} \quad (x,y,u).$$

2) F satisfies a Lipschitz condition in (y,u) with constant χ, i.e.

$$\alpha(F(x,y^1,u^1),F(x,y^2,u^2)) \le \chi(\|y^1-y^2\|+\|u^1-u^2\|)$$
$$\forall x \in I^m,\ y^i \in R^{\overline{m}},\ u^i \in R^n \quad (i=1,2).$$

Let $\omega(x)$ be an absolutely continuous function on I^m such that

$$\omega^i(x) = \omega(x_1,\dots,x_{i-1},0,x_{i+1},\dots,x_m) = 0, \quad i=1,2,\dots,m, \tag{9}$$

$$\rho(\partial^m\omega(x)/\partial x,\ F[\omega](x)) \le \lambda, \quad \lambda > 0 \tag{10}$$

for almost every $x \in I^m$. Then there exists a solution $u(x)$ of problem (7)-(8) satisfying

$$\|u(x)-\omega(x)\| \le \lambda H b^{m-1}\exp\Big(\chi^d\Big(\sum_{i=1}^m x_i\Big)\Big)/\chi^{d^2}, \tag{11}$$

$$\|\partial^k u(x)/\partial x_{i_1}\dots\partial x_{i_k} - \partial^k\omega(x)/\partial x_{i_1}\dots\partial x_{i_k}\|$$
$$\le \lambda H b^{m-k-1}\exp\Big(\chi^d\Big(\sum_{i=1}^m x_i\Big)\Big)/\chi^{d^2} \quad (k=1,2,\dots,m-1) \tag{12}$$

for almost every $x \in I^m$.

Proof.

The method of proof is in many respects analogous to that used in [6].

For every $x=(x_1,x_2,\dots,x_m) \in I^m$ denote

$$G_x = \prod_{i=1}^m [0,x_i], \quad G_x^{i_1\dots i_k} = \prod_{j=1}^k\Big(\prod_{\ell=i_{j-1}+1}^{i_j-1}[0,x_\ell]\Big) \times \prod_{j=i_k+1}^m [0,x_j],$$

where $1 \le i_1 \le \dots \le i_k \le m$, $1 \le k \le m-1$, $i_0 = 0$, $\prod_{j=\ell_1}^{\ell_2}[0,x_j] = \emptyset$ if $\ell_2 < \ell_1$.

Define a bijection $L^{i_1\dots i_k} : \{1,2,\dots,m-k\} \to \{1,2,\dots,m\}/\{i_1,\dots,i_k\}$ such that $L^{i_1\dots i_k}(j) < L^{i_1\dots i_k}(\ell)$ for $j<\ell$. Then define $g_x^{i_1\dots i_k} : G_x^{i_1\dots i_k} \to R^m$ by setting $g_x^{i_1\dots i_k}(z^{m-k}) = (\overline{x}_1,\overline{x}_2,\dots,\overline{x}_m)$ for $z^{m-k}=(z_1,z_2,\dots,z_{m-k})$, where $\overline{x}_j = x_j$ if $j=i_\ell$ for some ℓ $(1<\ell<k)$, and $\overline{x}_j = z_\ell$ otherwise, with ℓ being the index such that $j = L^{i_1\dots i_k}(\ell)$. To each function $v(.)\in L(I^m)$ let us associate the function $v_{xi_1\dots i_k} : G_x^{i_1\dots i_k} \to R^n$ defined by

$$v_{xi_1\dots i_k}(z^{m-k}) = v\big(g_x^{i_1\dots i_k}(z^{m-k})\big).$$

Setting $u^0(x)=\omega(x)$, by Lemma 3 we can choose a measurable selection $v^0(x)\in F[u^0](x)$ satisfying, almost everywhere on I^m,

$$\|\partial^m u^0(x)/\partial x - v^0(x)\| = \rho\big(\partial^m u^0(x)/\partial x,\quad F[u^0](x)\big) \le \lambda. \tag{13}$$

Then, setting

$$u^1(x) = \int_{G_x} v^0(\overline{x})d\overline{x},$$

$$p^0_{i_1\dots i_k}(x) = \int_{G_x^{i_1\dots i_k}} \big(\partial^m u^0(x)/\partial x\big)_{xi_1\dots i_k}(z^{m-k})dz^{m-k},$$

$$p^1_{i_1\dots i_k}(x) = \int_{G_x^{i_1\dots i_k}} v^0_{xi_1\dots i_k}(z^{m-k})dz^{m-k},$$

and noting that $x_{i_1}\dots x_{i_k} \le H_k h(x)$, for $x\in I^m$, $h(x)=\sum_{i=1}^m x_i$, we have

$$\|u^1(x)-u^0(x)\| \le \lambda x_1x_2\dots x_m \le \lambda H_m h(x), \tag{14}$$

$$\|p^1_{i_1\dots i_k}(x) - p^0_{i_1\dots i_k}(x)\| \le \lambda x_{i_1}\dots x_{i_k} \le \lambda H_{m-k}h(x). \tag{15}$$

Now by Lemma 3 we can take a measurable selection $v^1(x)\in F[u^1](x)$ such that

$$\|v^0(x)-v^1(x)\| = \rho\big(v^0(x), F[u^1](x)\big) \quad \text{almost everywhere on } I^m \tag{16}$$

and since $[u]_x$ contains C_m^k mixed derivatives of k-th order, from (14) and (15) we have

$$\|v^0(x) - v^1(x)\| \le \alpha\big(F[u^0](x), F[u^1](x)\big) \le \lambda\chi H h(x). \tag{17}$$

Setting $u^2(x) = \int\limits_{G_x} v^1(\overline{x})d\overline{x}$, $p^2_{i_1\ldots i_k}(x) = \int\limits_{G_x^{i_1\cdots i_k}} v^1_{x i_1\ldots i_k}(z^{m-k})dz^{m-k}$, and applying Lemma 4 yields the following estimates from (17):

$$\|u^2(x) - u^1(x)\| \le \lambda\chi H \int\limits_{G_x} h(\overline{x})d\overline{x} \le \lambda\chi H b^{m-1} h^2(x)/2,$$

$$\|p^2_{i_1\ldots i_k}(x) - p^1_{i_1\ldots i_k}(x)\| \le \lambda\chi H \int\limits_{x^{i_1\cdots i_k}} h(z^{m-k})dz^{m-k} \le \lambda\chi H b^{m-k-1} h^2(x)/2.$$

Further, choosing a measurable selection $v^2(x) \in F[u^2](x)$ such that

$$\|v^1(x) - v^2(x)\| = \rho(v^1(x), F[u^2](x)) \quad \text{almost everywhere on } I^m,$$

we easily see that

$$\|v^1(x) - v^2(x)\| \le \alpha(F[u^1](x), F[u^2](x)) \le \lambda\chi^2 H d h^2(x)/2!.$$

Then, setting $u^3(x) = \int\limits_{G_x} v^2(\overline{x})d\overline{x}$, $p^3_{i_1\ldots i_k}(x) = \int\limits_{G_x^{i_1\cdots i_k}} v^2_{x^{i_1\cdots i_k}}(z^{m-k})dz^{m-k}$, and using the same argument as previously we obtain

$$\|u^3(x) - u^2(x)\| \le \lambda\chi^2 H d b^{m-1} h^3(x)/3!, \tag{18}$$

$$\|p^3_{i_1\ldots i_k}(x) - p^2_{i_1\ldots i_k}(x)\| \le \lambda\chi^2 H d b^{m-k-1} h^3(x)/3!. \tag{19}$$

In a general manner once $u^\ell(x)$, $p^\ell_{i_1\ldots i_k}(x)$ have been constructed, we can take a measurable selection $v^\ell(x) \in F[u^\ell](x)$ such that

$$\|v^{\ell-1}(x) - v^\ell(x)\| = \rho\big(v^{\ell-1}(x), F[u^\ell](x)\big) \quad \text{almost everywhere on } I^m,$$

and setting $u^{\ell+1} = \int\limits_{G_x} v^\ell(\overline{x})d\overline{x}$, $p^{\ell+1}_{i_1\ldots i_k}(x) = \int\limits_{G_x^{i_1\cdots i_k}} v^\ell_{x^{i_1\cdots i_k}}(z^{m-k})dz^{m-k}$, we can by induction verify that, similarly to (18), (19),

$$\|v^\ell(x) - v^{\ell-1}(x)\| \le \lambda H d^{-1}\big(\chi d h(x)\big)^\ell/\ell! \quad \text{for almost all } x, \tag{20}$$

$$\|u^\ell(x) - u^{\ell-1}(x)\| \le \lambda H b^{m-1}(\chi d^2)^{-1}\big(\chi d h(x)\big)^{\ell+1}/(\ell+1)!, \tag{21}$$

$$\|p^{\ell+1}_{i_1\ldots i_k}(x) - p^{\ell}_{i_1\ldots i_k}(x)\| \le \lambda H b^{m-k-1}(\chi d^2)^{-1}\big(\chi d h(x)\big)^{\ell+1}/(\ell+1)! \quad \text{for almost all } x_{i_j},\ j=1,2,\ldots,k, \tag{22}$$

$$\rho\big(v^{\ell+1}(x), F[u^\ell](x)\big) \le \lambda H d^{-1}\big(\chi d h(x)\big)^\ell/\ell! \quad \text{for almost all } x. \tag{23}$$

From (20) it is not hard to see that the sequence $\{v^\ell\}_{\ell=0}^\infty$ is a Cauchy sequence in $L(I^m)$ and hence converges to some function $v(.) \in L(I^m)$ almost everywhere. From (21) it follows that the sequence $\{u^\ell\}_{\ell=0}^\infty$ uniformly converges on I^m to some absolutely continuous function $u(x)$. From (22) it follows that the sequence $\{p^\ell_{i_1\ldots i_k}\}_{\ell=0}^\infty$ converges to some function $p_{i_1\ldots i_k}(x)$ for almost all x_{i_j}, $j=1,2,\ldots,k$ and for all values of the other variables. By virtue of Legesgue's Theorem (see [13], p. 302) we then obtain

$$u(x) = \int_{G_x} v(\bar{x})d\bar{x}, \quad p_{i_1\ldots i_k}(x) = \int_{G_x^{i_1\ldots i_k}} v_{x^{i_1\ldots i_k}}(z^{m-k})dz^{m-k}.$$

These relations show that $p_{i_1\ldots i_k}(x) = \partial^k u(x)/\partial x_{i_1}\ldots\partial x_{i_k}$ for almost all x_{i_j}, $j=1,2,\ldots,k$ and, taking account of (23) it is not hard to see that

$$\partial^m u(x)/\partial x \in F[u](x) \quad \text{for almost all} \quad x \in I^m,$$

i.e. $u(x)$ is a solution of problem (7)-(8) since the boundary conditions (8) automatically hold according to the definition of $u(x)$.

In view of (21) we have

$$\begin{aligned}\|u(x) - \omega(x)\| &\le \sum_{\ell=1}^{\infty} \|u^\ell(x) - u^{\ell-1}(x)\| \\ &\le \sum_{\ell=1}^{\infty} \lambda H b^{m-1}(\chi d^2)^{-1}\big(\chi d h(x)\big)^{\ell+1}/(\ell+1)! \\ &\le \lambda H b^{m-1}(\chi d^2)^{\ell-1}\exp\big(\chi d h(x)\big),\end{aligned}$$

proving (11). Since the relation (12) can be established in exactly the same way, the proof of Theorem 1 is complete. □

3. CONTINUOUS DEPENDENCE ON PARAMETER OF THE SOLUTION SET OF HYPERBOLIC DIFFERENTIAL INCLUSIONS

Consider the hyperbolic differential inclusion

$$\partial^m u(x)/\partial x \in F(x, u(x), \lambda) \tag{24}$$

with the boundary conditions

$$u^i(x) = \theta, \quad i = 1, 2, \ldots, m, \quad x \in I^m, \tag{25}$$

where $F : I^m \times R^n \times \Lambda \to \text{Conv } R^n$, $\Lambda \subset R$.

Theorem 2. *Assume that:*

1) For fixed (u, λ), F is measurable in x;

2) For fixed (x, λ), F satisfies a Lipschitz condition in u with constant γ;

3) $|F(x, u, \lambda)| \le M \ \ \forall (x, u, \lambda)$, where $M > 0$.

Assume, furthermore, that there exists a bounded domain $D \subset R^n$ such that:

4) For some limit point $\lambda_0 \in \Lambda$ we have

$$\lim_{\lambda \to \lambda_0} \alpha\Big(\int_0^x F(y, u, \lambda) dy, \ \int_0^x F(y, u, \lambda_0) dy \Big) = 0$$

uniformly with respect to $u \in D$, $x \in I^m$.

5) There exists a neighbourhood $\overline{U}(\lambda_0)$ of point λ_0 such that for any $\lambda \in \overline{U}(\lambda_0)$ any solution of problem (24)-(25) lies in the interior of D.

Then for any $\eta > 0$ there exists a neighbourhood $U(\lambda_0) \subset \overline{U}(\lambda_0)$ of point λ_0 such that for every solution $u(x)$ of problem (24)-(25) with $\lambda \in U(\lambda_0)$ there exists a solution $u_0(x)$ of problem

$$\partial^m u_0(x)/\partial x \in F(x, u_0(x), \lambda_0), \tag{26}$$

$$u_0^i(x) = \theta, \quad i = 1, 2, \ldots, m \tag{27}$$

satisfying

$$\|u(x) - u_0(x)\| \le \eta \quad \forall x \in I^m, \tag{28}$$

and conversely, for any solution $u_0(x)$ of problem (26)-(27) there exists a solution $u(x)$ of problem (24)-(25) satisfying (28).

Proof.

Denote by E the set of all absolutely continuous functions $u(x)$ on I^m that satisfy the boundary condition (25) and possess mixed derivatives $\partial^m u(x)/\partial x$ such that $\|\partial^m u(x)/\partial x\| \leq M$ for almost every $x \in I^m$. From the generalized Arzela's Theorem [13] it follows that E is a compact subset of $C(I^m)$. Let $u(x)$ be an arbitrary solution of problem (24)-(25), so that $u(.) \in E$. According to Lemma 2, for every $\eta_1 > 0$ there exists a neighbourhood $\overline{U}_1(\lambda_0)$ of point λ_0 such that, whenever $\lambda \in \overline{U}_1(\lambda_0)$, there exists a measurable selection $\overline{v}(x) \in F(x, u(x), \lambda_0)$ satisfying

$$\left\| \int_0^x \left(\partial^m u(y)/\partial y - \overline{v}(y)\right) dy \right\| \leq \eta_1 \quad \forall x \in I^m. \tag{29}$$

Define now a selection $f_0(x, u) \in F(x, u, \lambda_0)$ by the relation

$$\|\overline{v}(x) - f_0(x, u)\| = \rho\big(\overline{v}(x), F(x, u, \lambda_0)\big).$$

By virtue of the convexity and compactness of $F(x, u, \lambda_0)$ such a selection exists and is uniquely defined. Furthermore, by a theorem of Himmelberg [8] f_0 is measurable in x for fixed u and since F is continuous in u, it follows from a known result (see [6], Lemma 5) that f_0 is continuous in u for fixed x. Using the condition $\|f_0(x, u)\| \leq M \ \ \forall (x, u)$, we deduce from the just obtained result that the equation

$$\partial^m u_0(x)/\partial x = f_0(x, u_0(x)),$$
$$u_0^i(x) = \theta \quad (i = 1, 2, \ldots, m),$$

admits a solution $u_0(x)$ on I^m which is obviously also a solution of problem (26)-(27). Furthermore, we have

$$\begin{aligned} &\|\overline{v}(x) - \partial^m u_0(x)/\partial x\| = \rho\big(\overline{v}(x), F(x, u_0(x), \lambda_0)\big) \\ &\leq \alpha\big(F(x, u(x), \lambda_0), F(x, u_0(x), \lambda_0)\big) \leq \gamma \|u(x) - u_0(x)\|. \end{aligned}$$

Hence, taking account of the estimate (29) we obtain

$$\|u(x)-u_0(x)\| = \Big\| \int_0^x [\partial^m u(y)/\partial y - \partial^m u_0(y)/\partial y]\,dy \Big\|$$

$$\leq \Big\| \int_0^x [\partial^m u(y)/\partial y - \overline{v}(y)]\,dy \Big\| + \Big\| \int_0^x [\overline{v}(y) - \partial^m u_0(y)/\partial y]\,dy \Big\|$$

$$\leq \eta_1 + \int_0^x \|\overline{v}(y) - \partial^m u_0(y)/\partial y\|\,dy \leq \eta_1 + \gamma \int_0^x \|u(y) - u_0(y)\|\,dy.$$

By virtue of the generalized Gronwall's Lemma [5] we can write

$$\|u(x)-u_0(x)\| \leq \eta_1\Big[1 + \int_0^x \exp\Big(\gamma \prod_{j=1}^m (x_j - y_j)\Big)\,dy\Big] \leq \eta_1\big(1+\exp(\gamma a^m)\big).$$

By taking $\eta_1 \leq \eta\big(1+\exp(\gamma a^m)\big)^{-1}$ this proves the first assertion of the theorem. The second assertion is proved similarly. □

Note that the above theorem includes both Theorem 1 in [22] and Theorem 1 in [11] as special cases.

4. AVERAGING METHOD FOR HYPERBOLIC INCLUSIONS

From Theorem 1 we can now derive a Bogoliubov type averaging theorem for the problem

$$\partial^m u(x)/\partial x \in \varepsilon^m G(x,u), \tag{30}$$

$$u^i(x) = \theta \quad (i = 1,2,\ldots,m;\ x \in [0,+\infty)^m). \tag{31}$$

Theorem 3. *Assume that the right hand side of problem (30)-(31) and that of the problem*

$$\partial^m \overline{u}(x)/\partial x \in \varepsilon^m \overline{G}(\overline{u}(x)), \tag{32}$$

$$\overline{u}^i(x) = \theta \quad (i = 1, 2, \ldots, m;\ x \in [0, +\infty)^m) \tag{33}$$

satisfy the following conditions:

1) The map $G : [0, +\infty)^m \times R^n \to \operatorname{Conv} R^n$ is measurable in x for fixed u and satisfies a Lipschitz condition in u with constant γ; in addition $|G(x, u)| \leq M$ $\forall (x, u)$ with $M > 0$;

2) The map $\overline{G} : R^n \to \operatorname{Conv} R^n$ satisfies a Lipschitz condition in u with constant γ; in addition $|\overline{G}(u)| \leq M \quad \forall u$.

Assume also that a compact domain $D \subset R^n$ exists such that

3) $\lim_{T \to +\infty} \alpha\Big(\frac{1}{T^m} \int_{[0,T]^m} G(x, u)dx, \overline{G}(x)\Big) = 0$ uniformly with respect to $u \in D$;

4) Any solution to problems (30)-(31) or (32)-(33) lies in the interior of D.

Then for any $\eta > 0$, $L > 0$ there exists $\varepsilon_0 > 0$ such that on every domain $[0, L\varepsilon^{-1}]^m$ with $0 < \varepsilon \leq \varepsilon_0$, for any solution $\overline{u}(x)$ of problem (30)-(31) there exists a solution $u(x)$ of the inclusion (32)-(33) satisfying

$$\|u(x) - \overline{u}(x)\| \leq \eta, \tag{34}$$

and for any solution $\overline{u}(x)$ of problem (32)-(33) there exists a solution $u(x)$ of problem (30)-(31) satisfying (34).

Proof.

By the change of variable $z = x/\varepsilon$ the problems under consideration take the form

$$\partial^m u(z)/\partial z \in G(z/\varepsilon, u(z)),$$

$$u^i(z) = \theta \quad (i = 1, 2, \ldots, m;\ z \in [0, L]^m)$$

and

$$\partial^m \overline{u}(z)/\partial z \in \overline{G}(\overline{u}(z)),$$

$$\overline{u}^i(z) = \theta \quad (i = 1, 2, \ldots, m;\ z \in [0, L]^m).$$

Setting

$$F(z, u, \varepsilon) = G(z/\varepsilon, u) \quad \text{if } \varepsilon > 0,$$

$$F(z, u, \varepsilon) = \overline{G}(u) \quad \text{if } \varepsilon = 0,$$

we easily see that F satisfies all the conditions of Theorem 2. The conclusion follows. □

Note that Theorem 3 includes the results in [16] and [23] as special cases when $m = 1$ and $m = 2$, respectively.

Consider now the differential inclusion

$$\partial^m u(x)/\partial x \in \varepsilon^m F(x, [u]_x, u(x)), \quad 0 < \varepsilon \leq \bar{\varepsilon}_0 \tag{35}$$

with boundary conditions

$$u(x)\big|_{x_i=0} = g_i(x), \quad i = 1, 2, \ldots, m \tag{36}$$

where $u(x)\big|_{x_i=0} = u(x_1, \ldots, x_{i-1}, 0, x_i, \ldots, x_m)$. We assume the functions $g_i(x)$ to be differentiable in each variable and such that $g_i(x)\big|_{x_j=0} = g_j(x)\big|_{x_i=0}$ for all i, j.

Setting

$$g(x) = \sum_{i_1=1}^{m} g_{i_1}(x) - \frac{1}{2!}\sum_{i_1=1}^{m}\Big(\sum_{\substack{i_2=1\\ i_2\neq i_1}}^{m} g_{i_2}(x)\Big)\Big|_{x_{i_1}=0} + \cdots$$
$$+ \frac{(-1)^m}{m!}\sum_{i_1=1}^{m}\Big(\cdots\Big(\sum_{\substack{i_m=1\\ i_m\neq i_j,\ j=1,2,\ldots,m-1}}^{m} g_{i_m}(x)\Big)\Big|_{x_{i_{m-1}}=0}\cdots\Big)\Big|_{x_{i_1}=0},$$

we associate problem (35)-(36) with the following one:

$$\partial^m \bar{u}(x)/\partial x \in \varepsilon^m \overline{F}(\bar{u}(x)), \tag{37}$$

$$\bar{u}(x)\big|_{x_i=0} = \theta, \quad i = 1, 2, \ldots, m, \tag{38}$$

where

$$\overline{F}(u) = \lim_{T\to\infty}\frac{1}{T^m}\int_{[0,T]^m} F(x, [g]_x, u + g(x))dx. \tag{39}$$

From the results in Section 2 we can derive the following averaging theorem for problem (35)-(36):

Theorem 4. *Assume that the set-valued map $F : [0,\infty)^m \times R^m \times R^m \to \mathrm{Conv}\ R^n$ satisfies all the conditions of Theorem 1 and, furthermore, $|F(x, y, u)| \leq M$ $(M > 0)$ for all (x, y, u). Let $D \in \mathrm{Comp}\ R^n$, $D_1 \in \mathrm{Comp}\ R^{\overline{m}}$ be two domains such that:*

1) the limit (39) exists, uniformly with respect to $u \in D$;

2) any solution of problem (35)-(36) or of problem (37)-(38) lies, together with its derivatives, in the interior of $D \times D_1$.

Then for every $\eta > 0$, $L > 0$, there exists an $\varepsilon_0 > 0$ such that on any $[0, L\varepsilon^{-1}]^m$ with $0 < \varepsilon \le \varepsilon_0$: for every solution $u(x)$ of problem (35)-(36) there exists a solution $\overline{u}(x)$ of problem (37)-(38) satisfying

$$\|u(x) - g(x) - \overline{u}(x)\| \le \eta, \tag{40}$$

and conversely, for every solution $\overline{u}(x)$ of problem (37)-(38) there exists a solution $u(x)$ of problem (35)-(36) satisfying (40).

Proof.

If suffices to prove the first assertion of the theorem, since the second assertion can be proved in an analogous manner.

Let $u(x)$ be any solution of problem (35)-(36). It is easily seen that, by means of the substitution $u_1(x) = u(x) - g(x)$, problem (35)-(36) can be converted into the following form:

$$\partial^m u_1(x)/\partial^m x \in \varepsilon^m F(x, [u_1]_x + [g]_x, u_1(x) + g(x)), \tag{41}$$

$$u_1(x)\big|_{x_i=0} = \theta, \quad i = 1, 2, \dots, m. \tag{42}$$

Consider also the problem

$$\partial^m u_2(x)/\partial x \in \varepsilon^m F(x, [g]_x, u_2(x) + g(x)), \tag{43}$$

$$u_2(x)\big|_{x_i=0} = \theta, \quad i = 1, 2, \dots, m. \tag{44}$$

From the fact that the map F is bounded by the constant M we get the following estimate on $[0, L\varepsilon^{-1}]^m$:

$$\|\partial^k u_j(x)/\partial x_{i_1} \dots \partial x_{i_k}\| \le L^{m-k}\varepsilon^k M, \quad j = 1, 2.$$

Hence,

$$\|[u_j]_x\| \le \overline{c}\varepsilon, \quad j = 1, 2, \tag{45}$$

with $\overline{c} = M \sum\limits_{i=1}^{m-1} C_m^i L^{m-i} \overline{\varepsilon}_0^{i-1}$.

Let $u_1(x) = u(x) - g(x)$, i.e. $u_1(x)$ is a solution of problem (41), (42). Then in view of (45) we can write:

$$\begin{aligned}&\rho\big(\partial^m u_1(x)/\partial x,\ \varepsilon^m F(x,[g]_x,u_1(x)+g(x))\big)\\ &\le \varepsilon^m \alpha\big(F(x,[u_1]_x+[g]_x,u_1(x)+g(x)), F(x,[g]_x,u_1(x)+g(x))\big)\\ &\le \varepsilon^m \chi \|[u_1]_x\| \le \varepsilon^{m+1}\chi\bar{c}.\end{aligned}$$

Therefore, by Theorem 1 we can find a constant c, independent of ε, L, for which there exists a solution $u_2(x)$ of problem (43)-(44) such that on $[0, L\varepsilon^{-1}]^m$ we have

$$\|u_1(x) - u_2(x)\| \le c\varepsilon. \tag{46}$$

By Theorem 3, for $\eta/2$ there exists an $\varepsilon_1 > 0$ such that on any $[0, L\varepsilon^{-1}]^m$ with $0 < \varepsilon \le \varepsilon_1$ there exists a solution $\overline{u}(x)$ of problem (37)-(38) satisfying

$$\|u_2(x) - \overline{u}(x)\| \le \eta/2. \tag{47}$$

Combining then (46) and (47) yields

$$\begin{aligned}\|u(x) - g(x) - \overline{u}(x)\| = \|u_1(x) - \overline{u}(x)\| &\le \|u_1(x) - u_2(x)\|\\ &+ \|u_2(x) - \overline{u}(x)\| \le c\varepsilon + \eta/2.\end{aligned}$$

The proof of the Theorem is complete by choosing $\varepsilon_0 = \min\{\varepsilon_1, \frac{\eta}{2c}\}$.

REFERENCES

1. L. D. Akulenko, *Asymptotic methods of optimal control*, Moscow, 1987 (Russian).
2. N. N. Bogoliubov, *On some statistical methods in mathematical physics*, Kiev, Ukrainian Acad. of Science Publishing House, 1945 (Russian).
3. N. N. Bogoliubov and Yu. A. Mitropolski, *Asymptotic method in the theory of nonlinear oscillation*, Moscow, 1974 (Russian).
4. C. Castaing and M. Valadier, *Convex analysis and measurable multifunctions*, Lecture Notes in Math., 580, Springer-Verlag, 1977.
5. J. Chandra and P. W. Davis, *Linear generalizations of Gronwall's inequalities*, Proc. Amer. Math. Soc. **60** (1976), 157-160.
6. A. F. Filippov, *Classical solutions of differential equations with multivalued right hand side*, SIAM J. on Control and Optimization **5** (1967), 609-621.

7. E. A. Grebelnikov and Yu. A. Ryabov, *Constructive method for the analysis of nonlinear systems*, Moscow, 1979 (Russian).

8. C. J. Himmelberg, *Measurable relations*, Fundamenta Mathematicae **87** (1975), 53-72.

9. M. M. Khapaev and O. P. Filatov, *On the mutual ε-approximation for solution to a differential inclusion and the averaging inclusion*, Matematicheskie Zametki **47** (1990), 127-134 (Russian).

10. ________ , *The averaging of differential inclusions with fast and slow variables*, Matematicheskie Zametki **47** (1990), 102-108 (Russian).

11. M. Kisielewicz, *Bogoliubov's type theorem for hyperbolic equations*, Ukrainian Math. J. **22** (1970), 374-379 (Russian).

12. ________ , *Method of averaging for differential equations with compact convex valued solutions*, Rend. Math. **9** (1976), 397-408.

13. A. N. Kolmogorov and S. V. Fomin, *Elements of theory of functions and functional analysis*, Moscow, 1981 (Russian).

14. Yu. A. Mitropolski and G. P. Khoma, *Mathematical foundation of asymptotic methods in nonlinear mechanics*, Kiev, 1983 (Russian).

15. Yu. A. Mitropolski and B. N. Moiseenkov, *Asymptotic solutions of partial differential equations*, Kiev, 1976 (Russian).

16. V. A. Plotnikov, *The averaging method for differential inclusions and its application to optimal control problems*, Differentialnye Uravnenya **15** (1979), 1427-1443 (Russian).

17. ________ , *Partial averaging for differential inclusions*, Matematicheskie Zametki **27** (1980), 947-952 (Russian).

18. H. D. Tuan, *Some questions on the foundation of the averaging method for differential inclusions*, Preprint VIINTI, Odessa Univ., 1990 (Russian).

19. ________ , *A theorem on continuous dependence upon parameter of solutions of differential inclusions in Banach space with closed right hand side*, Ukrainian Math. J. **43** (1991), 562-565 (Russian).

20. ________ , *Asymptotic construction of solutions to differential systems with set-valued right hand side*. Ph. D. Thesis, Odessa Univ., Odessa, 1990 (Russian).

21. ________ , *Theorem of averaging for differential inclusions in Banach spac with fast and slow variables*. Differentialnye Uravnenya (to appear, Russian).

22. A. B. Vasiliev, *On continuous dependence on parameter of solutions to differential inclusions*, Ukrainian Math. J. **35** (1983), 607-611 (Russian).

23. A. M. Vitiuk and S. S. Klimenko, *A Bogoliubov's theorem for hyperbolic differential inclusions*, Ukrainian Math. J. **39** (1987), 641-645 (Russian).

24. J. Warga, *Optimal control of differential and functional equations*, Academic Press, 1972.

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Received February 21, 1992