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THE HURWITH THEOREM FOR BANACH-VALUED MEROMORPHIC FUNCTIONS IN INFINITE DIMENSION

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Abstract. *The aim of the note is to prove the Hurwith theorem for Banach-valued meromorphic functions in infinite dimension. The main results of the note is following:*

Theorem. *Let G be an open subset in a Banach space F and S be an analytic set in G with* codim $S \geq 2$. *If f is a meromorphic function on $G \setminus S$ with values in Banach space F then f can be meromorphically extended on G.*

Theorem. *Let $\mathbf{CP}(E)$ denote a Banach projective space associated with a Banach space E. If f is meromorphic function defined $\mathbf{CP}(E)$ with values in a Banach space F, then f is rational i. e. $f = P/Q$, where P and Q are homogeneous polynomials with values in F and in $\mathbf{C}$ respectively.*

In 1950 Hurwith has proved that every meromorphic function of $\mathbf{C}P^n$ is rational, i. e. $f = P/Q$, where P and Q are homogeneous polynomials. In this paper we prove the Hurwith theorem for Banach-valued meromophic functions on Banachprojective space.

Let X be a Banach complex manifold and G be a dense subset in X. A holomorphic function f on G with values in a Banach space F is called meromorphic on X if for each $z \in X$, there exists a neighbourhood U of z and holomorphic functions g, σ on U with values in F and in $\mathbf{C}$ respectively such that

$$f|_{U \cap G} = g/\sigma|_{U \cap G}$$

Put

$$P(f) = \{z \in X \ : \ f \text{ is not holomorphic at } z\}.$$

By the paracompactness of X, it follows that $P(f)$ is either empty or contained in a hypersurface of X. For a Banach space E, let $\mathbf{C}P(E)$ denote the Banach projective space associated with E. It is known in [1] that every analytic set S of finite codimension is algebraic, i. e. S can be written of the form

$$S = Z(P_1, \ldots, P_s);$$

where $P_1, \ldots, P_s$ are homogeneous polynomials on E.

Using the results in [1], we shall prove the ratiolization of meromorphic functions on $\mathbf{C}P(E)$ with values in a Banach space.

1. EXTENSION OF BANACH-VALUED MEROMORPHIC FUNCTIONS THROUGH AN ANALYTIC SET

First, we study the extension of meromorphic functions through an analytic set of codimension ≥ 2

Theorem 1.1. *Let G be an open subset in a Banach space E and S be an analytic set in G with* codim $S \geq 2$. *If f is a meromorphic function on $G \setminus S$ with values in a Banach space F, then f can be meromorphically extended on G.*

In order to prove Theorem 1.1 we need the following lemma.

Lemma 1.2. *Let $\beta \in O_{E,0}$, where O_E is the sheaf of germs of holomorphic functions on E, be irreducible then there exists a neighbourhood U of* 0 *such that*

$$\text{Codim } Z(\beta, D\beta)_y \geq 2 \qquad (y \in U),$$

where $Z(\beta, D\beta)$ is the zero-set of β and $D\beta$.

Proof.

By [1], it is sufficient to show that $Z(\beta, D\beta)$ does not contain a pricipal germ. Assume for the contrary that $Z(\beta, D\beta)$ contains a pricipal germ. Then there exists $\sigma \in O_{E,0}$ such that $Z(\sigma)_0 \subset Z(\beta, D\beta)_0$. From the irreducibility of σ and β, we have

$$\sigma = \gamma\beta \quad \text{for some element } \gamma \in O_{E,0} \quad [1]$$

Hence

$$Z(\beta)_0 \subset Z(\sigma)_0 \subset Z(D\beta)_0$$

Take $y \in E$ such that $D(\beta|_L)_0(y) \neq 0$, where $L = \mathbf{C}y$. We can write $D(\beta|_L)(z)(y) = \beta^k(z)g(z)$ for all z in a neighbourhood of zero in L with $g(0) \neq 0$. Define a holomorphic function h on a neighbourhood of zezo in $\mathbf{C}$ by

$$h(\lambda) = \beta(\lambda y).$$

Then

$$\begin{aligned} h'(\lambda) &= (D\beta)(\lambda y)(y) = \beta^k(\lambda y)g(y) \\ &= h^k(\lambda)\varphi(\lambda), \end{aligned}$$

where $\varphi(0) \neq 0$. Obviously, this is impossible.

Proof of Theorem 1.1.

Let f be a Banach-valued meromorphic function on $G \setminus S$. Choose a decreasing basis of convex neighbourhood $\{U_k\}$ of zero in E. For each $x^* \in F^*$ the dual space of F, we denote $\widehat{x^*f}$ the meromorphic extension of x^*f on G. By the Remmert - Stein theorem [1], $P(f)$ can be extended to an analytic set V with Codim $V = 1$.

Take $z \in S$, without loss of generality, we can assume that $z = 0 \in S$. Since the ring $O_{E,0}$ is factorial [1], by Lemma 1.2. we can find a neighbourhood W of zero and $\sigma \in O(W)$ such that

$$V \cap W = Z(\sigma)_0 = \sigma^{-1}(0) \text{ and}$$

$$\text{Codim } Z(\sigma, D\sigma)_y \geq 2 \quad \text{for all } y \in W.$$

For each $(k,n) \in N \times N$, put

$$A_{k,n}\{x^* \in F^* \ : \ \sigma^n \ \widehat{x^*f} \in O(U_k)\}.$$

Then $F^* = UA_{k,n}$ and $A_{k,n}$ are closed in F^*. Indeed, for each $x \in F^*$, $\widehat{x^*f}$ can be written of the form g/β where $g, \beta \in O(U_k)$ for some $k \geq 0$ such that g_y is prime respect to β_y for $y \in U_k$. This implies that

$$P(\widehat{x^*f}) \cap U_k = Z(\beta)_0 \subset V \cap U_k = Z(\sigma)_0 \cap U_k.$$

Hence

$$\sigma^n|_{U_i} = \gamma\beta|_{U_i} \text{ for some } i \text{ and } \gamma \in O(U_i).$$

Thus $\sigma^n \widehat{x^* f} \in O(U_i)$ and the equality $F^* = \bigcup_{k,n} A_{k,n}$ is proved.

Let $\{x^*_\alpha\} \subset A_{k,n}$ and $\{x^*_\alpha\}$ converge to x^* in F^*. Then $\{\sigma^n \widehat{x^* f} \subset O(U_k)$ and converges to $\sigma^n \widehat{x^* f}$ in $O(U_k)$ i. e. $x^* \in A_{k,n}$. Hence $A_{k,n}$ is closed in F^*.

Using the Baire property of F^*, we can find (k,n) such that $\sigma^n \widehat{x^* f}$ holomorphic on U_k for all $x^* \in F^*$. Now assume that $x_0 \in z(\sigma) \setminus \{Z(\sigma) \cap Z\ (D\sigma)\}$. Then there exists a biholomorphic map u of a neighbourhood of $0 \in \mathbb{C}a + \ker D\sigma(x_0)$, where $a \in E$ and $D\sigma(x_0)(a) = 1$ onto a neighbourhood of x_0 such that $f.\ u$ can be written of the form

$$f \,.\, u(\lambda, x') = \sum_{k=-\infty}^{\infty} a_k(x')\lambda^k$$

Since $\sigma u(0,0) = 0$, we can write $\sigma^n u(\lambda, x') = \lambda^n v(\lambda, x')$. Where v is a holomorphic function in a neighbourhood of zero. From the holomorphicity of $\sigma^n \widehat{x^* f}$ on U_k and from the relation

$$x^* \sigma^n(u(\lambda, x')) f(u(\lambda, x')) = v(\lambda, x') \sum_{k=-\infty}^{\infty} \lambda^n x^* a_k(x')\lambda^k$$

it follows that $x^* a_k = 0$ for $k < -n$. This yields the holomorphicity of $x^* \sigma^n f$ on $U_k \setminus \{Z(\sigma) \cap Z(D\sigma)\}$ for all $x \in F^*$. Thus f is extended meromorphically at 0.

Theorem 1.1 is proved.

2. THE HURWITH THEOREM FOR BANACH-VALUED MEROMORPHIC FUNCTIONS ON BANACH PROJECTIVE SPACE

Theorem 2.1. *Let $\mathbb{C}P(E)$ be a Banach projective space associated with a Banach space E. If f is meromorphic function defined on $\mathbb{C}P(E)$ with values in a Banach space F, then f is rational, i. e. $f = P/Q$, where P and Q are homogeneous polynomials with values in F and in $\mathbb{C}$ respectively.*

Proof.

Take a hypersurface H in $\mathbb{C}P(E)$ containing $P(f)$. By [1], $\theta^{-1}(H)U\{0\}$ is algebraic hypersurface in E, where $\theta\ :\ E \setminus \{0\} \longrightarrow \mathbb{C}P(E)$ is the canonical projection.

Then

$$\theta^{-1}(H)U\{0\} = Z(Q)$$

for some homogeneous continous polynomial Q of degree m. Using Theorem 1.1 and Lemma 1.2, we can find $\varepsilon > 0$ and a holomorphic function on

$$B(\varepsilon) = \{x \in E \ : \ \|x\| < \varepsilon\} \quad \text{such that}$$

$$\begin{cases} \sigma.f.\theta \quad \text{is holomorphic on } B(\varepsilon) \\ \text{Codim } Z(\sigma, D\sigma)_y \geq 2 \quad (\forall y \in B(\varepsilon)) \\ Z(\sigma) \subset Z(Q) \end{cases}$$

By the implicite function Theorem, we have $Q = \beta\sigma$ for some holomorphic function β on $B(\varepsilon)$. From this, $P = Q.f.\theta$ is also holomorphic on $B(\varepsilon)$. From the relation

$$Q(x)f(\theta(x)) = \frac{\|x\|^m}{\varepsilon^m} Q\Big(\varepsilon \frac{x}{\|x\|}\Big) f\theta\Big(\varepsilon \frac{x}{\|x\|}\Big) = \frac{\|x\|^m}{\varepsilon^m} P\Big(\varepsilon \frac{x}{\|x\|}\Big) ,$$

it follows that

$$P(x) = \frac{\|x\|^m}{\varepsilon^m} P\Big(\frac{\varepsilon}{\|x\|} \cdot x\Big).$$

This shows that P is a homogeneous polynomial of degree m on E. Thus, Theorem 2.1 is completely proved.

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