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ON THE CONVERGENCE OF TWO-PARAMETER MARTINGALES AND GEOMETRIC CHARACTERIZATIONS OF BANACH SPACES

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Abstract. *Let* $(M) = (M_{m,n}, (m,n) \in N^2)$ *be a two-parameter Banach - valued martingale. In this note some interrelations between the almost sure convergence of* (M) *and the finiteness of its r-order conditional variation (and its r - order variation) are investigated. Also the three-series theorem is given in the presence of geometric characterizations of Banach spaces.*

1. INTRODUCTION

Let (Ω, F, F_n, P), $n = 1, 2, \ldots$, be a filtration probability, $(B, \|\cdot\|)$ a Banach space. Let $(M) = (M_m, F_n)$ be a B-valued martingale. We define for $1 < r \leq 2$, $n = 1, 2, \ldots$

$$B_n(r) = \sum_{k=1}^{n} E\big(\|\Delta M_k\|^r \big| F_k\big), \tag{1.1}$$

(r-order conditional variation of (M)) and

$$V_n(r) = \sum_{k=1}^{n} \|\Delta M_k\|^r \tag{1.2}$$

(r-order variation of (M)), where $\Delta M_n = M_{n+1} - M_n, \quad M_0 = 0$.

D. H. Thang and N. D. Tien [10] have studied the interrelation between the finiteness of $B_n(r)$ ($V_n(r)$) and the almost sure convergence of (M) in the present of geometric charaterization of B. Some inequalities of Doob's type concerning $B(r)$ and $V(r)$ are also investigated. Note that this result appeared earlier in the real case [1], [4], [6].

The aim of this note is to extend some of these results to the case of two-parameter B-valued martingales. Our methods are based on results of the one-parameter case and an idea of Fefforman which was used to obtain such results in the real case by Brossord [1]. Note that, as far as we know, this approach does not allow us to obtain every result which has been done for one-parameter Banach-valued martingales because of the lack of stopping times. This note can be regarded as a continuation of [9].

2. NOTATIONS AND PRELIMINARIES

Let $N^2 = N * N$ be a set of parameters with the order $(i,j) \leq (m,n)$ if $i \leq m$, $j \leq n$. Let (Ω, F, P) be a probability space and let $(F_{m,n})$ be an increasing family of sub-σ-fields of F. Throughout $(F_{m,n})$ are assumed to satisfy the usual condition (F_4) (c.f. [2] for definition).

A sequence $(X_{m,n})$ is said to be adapted if $X_{m,n}$ is $F_{m,n}$ - measurable for every $(m,n) \in N^2$.

Given a sequence $(X_{m,n})$ we set for $(m_1, n_1) \leq (m_2, n_2)$,

$$X[(m_1,n_1),(m_2,n_2)] = X_{m_2,n_2} - X_{m_2,n_1} - X_{m_1,n_2} + X_{m_1,n_1} = \Delta X_{(m_1,n_1),(m_2,n_2)}.$$

This quantity is called the increment of (X) on the rectangle $\{(m,n) \in N^2 : (m_1,n_1) < (m,n) \leq (m_2,n_2)\}$.

Unless otherwiser stated, a sequence $(X_{m,n})$ is assumed to be Bochner integrable, adapted to σ-fields $(F_{m,n})$ and it takes its values in a Banach space $(B, \|\cdot\|)$.

A L^1-integrable adapted sequence $(M_{m,n})$ is said to be a martingale if

$$E(M_{m,n} | F_{i,j}) = M_{i,j} \quad \text{whenever} \quad (i,j) \leq (m,n) \in N^2.$$

A Banach space B is said to be p-smoothable $(1 < p \leq 2)$ if (possible after equivalent renorming)

$$q_B(t) = \sup\left\{ \frac{(\|x+ty\| - \|x-ty\|)}{2} - 1 \ :\cdot \|x\| = \|y\| = 1 \right\}$$
$$= 0(t^p) \qquad \text{as } t \longrightarrow 0.$$

B is superreflexive if it is p-smoothable for a $p > 1$. With a B-valued martingale $(M_{m,n}, F_{m,n})$, we denote $\Delta M_{m,n}$, $(m,n) \in N^2$, for its increments.

In what follows, we shall assume $M_{m,n} = 0$ if m or $n = 0$, i.e. $M_{m,n}$ is null on the axes, in which condition $(M_{m,n})$ can be written as

$$M_{m,n} = \sum_{i=1}^{m}\sum_{j=1}^{n} \Delta M_{i,j} \overset{\text{df}}{=} \sum_{i=1}^{m}\sum_{j=1}^{n} X_{i,j}.$$

We now present some results which are used very often later on. Their proofs can be found in [9].

Proposition 2.1. *Let B be a separable Banach space and $1 < p \leq 2$. The following properties are equivalent:*

(i) Every two-parameter B-valued martingale $(M_{m,n})$ satisfying the $(L\log^+ L)$-condition converges a.s. and in L^1.

(ii) Every two-parameter B-valued martingale $(M_{m,n})$ satisfying the L^p-condition converges a.s. and in L^p.

(iii) B has the Random - Nikodym property (RNP).

Recall that a martingale $(M_{m,n})$ is said to be satisfiable the $(\text{L}\log^+\text{L})$-condition if

$$\sup_{(m,n)} E\big(\|M_{m,n}\|\log^+\|M_{m,n}\|\big) < \infty.$$

It satisfies the L^p-condition if

$$\sup_{(m,n)} E\|M_{m,n}\|^p < \infty.$$

Proposition 2.2. *Let B be a separable Banach space and $1 < p \leq 2$. The following three assertions are equivalent:*

(i) B is isomorphic to a p-smoothable Banach space,

(ii) There exists a constant C_p (only depending on p) such that for every two-parameter B-valued martingale $(M_{m,n})$ we have

$$\sup_{(m,n)} E\|M_{m,n}\|^p \leq C_p \sum_{i=1}^{m}\sum_{j=1}^{n} E\|\Delta M_{i,j}\|^p$$

(Assauad - Pisier's inequality)

(iii) For every two-parameter B-valued martingale $(M_{m,n})$ satisfying the condition

$$\sum_{i=1}^{m}\sum_{j=1}^{n} E\|\Delta M_{i,j}\|^p/(ij)^p < \infty,$$

we have that the martingale $M_{m,n} = \sum_{i=1}^{m}\sum_{j=1}^{n} \Delta M_{i,j}/ij$ *converges a.s. in the norm of B.*

Note that from (iii) and Kronecker's lemma (cf.[10]) we get

$(1/mn)\sum_{i=1}^{m}\sum_{j=1}^{n} \Delta M_{i,j} \longrightarrow 0$ a.s.

The following lemma (due to Brossord [1]) is used to get Theorem 3.1.

Lemma 2.1. *Let A be an even and* $a_{m,n} = E\big[1_A \big| F_{m,n}\big]$. *Set* $B = \inf_{m,n} a_{m,n} \geq 1-\alpha$, $\alpha \in (0,1]$. *Then we have* $B \subset A$ *a.s. and* $P\ (B^c) \leq cP(A^c)$, *where* 1_A *denote the indicator function of the set A, C is a constant not depending on* α *and* $B^c = \Omega \setminus B$, $A^c = \Omega \setminus A$.

3. THE MAIN RESULTS

Let us define, for $(m,n) \in N^2$,

$$M^* = \sup_{(m,n)} \|M_{m,n}\|,$$

$$B_{m,n}(p) = \sum_{i=1}^{m}\sum_{j=1}^{n} E\big(\|\Delta M_{i,j}\|^p \big| F_{i,j}\big),$$

(p-order conditional variation of (M)),

$$V_{m,n}(p) = \sum_{i=1}^{m}\sum_{j=1}^{n} \|\Delta M_{i,j}\|^p,$$

(p-order variation of (M)).

Denote $B(p)$, $V(p)$ for $B_\infty(p)$, $V_\infty(p)$. Note that if $p = 2$, we have the Hilbert space case and $B_{m,n}$, $V_{m,n}$ have been used in [1] to pursue the same purpose here.

Theorem 3.1. *Let B be a p-smoothable Banach space, $1 < p \le 2$ and $(M_{m,n})$ be a B-valued martingale. For every $\lambda > 0$ we have the folllowing inequality:*

$$P(M^* > \lambda) \le C_p\{P(B^{1/p}(p) > \lambda) + (1/\lambda^p)E[B(p); B^{1/p}(p) \le \lambda]\}. \quad (3.1)$$

Proof.

Set $A = B^{1/p}(p) \le \lambda$, B and $a_{m,n}$ being defined as in Lemma 2.1 with $\alpha = 1/2$, i.e. $B = \inf\limits_{(m,n)} a_{m,n} \ge 1/2$.

We have for an arbitrary fixed $\lambda > 0$

$$P(M^* > \lambda) \le P(B^c) + P(M^* > \lambda; B).$$

In view of Lemma 2.1, to obtain (3.10) we only have to dominate $P(M^* > \lambda; B)$.

Now we observe that $M_{m,n}$ coincides with the martingale $M_{m,n} = \sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n} 1_{\alpha_{i,j}} \Delta M_{i,j}$ on B. Thus

$$\begin{aligned}
P(M^* > \lambda; B) &\le P(M^* > \lambda) \\
&\le (1/\lambda^p)E(M^{*p}) \\
&\text{(Chebyshev's inequality)} \\
&\le (C_p/\lambda^p) \sup_{(m,n)} E\|M_{m,n}\|^p \\
&\text{(Doob's inequality)} \\
&\le (C_p^2/\lambda^p)E\{\sum_{(m,n)\in N^2} 1_{\{a_{m,n}\ge 1/2\}}\|\Delta M_{m,n}\|^p\} \\
&\text{(Assauad-Pisier's inequality)} \\
&\le (C_p/\lambda^p)E\{\sum_{(m,n)\in N^2} E(\|\Delta M_{m,n}\|^p|F_{m,n})\cdot \\
&\qquad 1_{\{a_{m,n}\ge 1/2\}}\|\Delta M_{m,n}\|^p\} \\
&\le (2C_p/\lambda^p)E\{\sum_{(m,n)\in N^2} E(\|\Delta M_{m,n}\|^p|F_{m,n})a_{m,n}\} \\
&= (2C_p/\lambda^p)E[B(p); a_\infty] \\
&= (2C_p/\lambda^p)E[B(p); B(p) \le \lambda] \quad \text{Q.E.D.}
\end{aligned}$$

Note that by putting $p = 2$ in the above theorem we obtain Theorem 2.1 in [1].

Theorem 3.2. *Suppose that B is a p-smoothable Banach space, $1 < p \leq 2$. Then we have the following facts*

(i) $\{B(p) < \infty\} \subset \{M_{m,n}$ *converges a.s.*$\}$ *a.s.*

(ii) $M_{i,\infty}$, $M_{\infty,j}$ *and* $M_{\infty\infty}$ *exist a.s. on the set* $\{B(p) < \infty\}$

(iii) For each $0 \leq r < 1$

$$E(M^*)^{r+1} \leq C_{p,r} E[B(p)]^{(r+1)/p}. \tag{3.2}$$

Proof.

The proof of (i) and (ii) are similar to that of Theorem 2 in [1], and so we omit them.

Now we prove (iii). Note at first that

$$E(M^*)^{r+1} = \int_0 \lambda^r P(M^* > \lambda) d\lambda.$$

Multiplying two sides of (3.1) with λ^r and taking integration in λ from 0 to ∞, we get

$$\begin{aligned}
E(M^*)^{r+1} &\leq C_p \Big\{ \int_0 \lambda^r P(M^* > \lambda) d\lambda \\
&\quad + \int_0 \lambda^{r-p} E\big[B(p); B(p) \leq \lambda\big] d\lambda \Big\} \\
&= C_p \Big\{ E\big(B^{(r+1)/p}(p)\big) + \int_0 \lambda^{r-p} d\lambda \int_0 x^{p-1} P\big(B^{1/p}(p) > x\big) dx \Big\} \\
&= C_{p,r} \Big\{ E\big(B^{(r+1)/p}(p)\big) \\
&\quad + \int_0 \lambda^{r-p} d\lambda \int_0 x^{p-1} P\big(B^{1/p}(p) > x\big) dx \int_0 \lambda^{r-p} d\lambda \Big\} \\
&= C_{p,r} \Big\{ E\big(B^{(r+1)/p}(p)\big) + \int_0 x^r P\big(B^{1/p}(p) > x\big) dx \Big\} \\
&= C_{p,r} E\big(B^{(r+1)/p}(p)\big).
\end{aligned}$$

Q.E.D.

Remarks. 1) Taking $r = 0$ in (3.2) we get

$$E(M^*) \le C_p E\big(B^{1/p}(p)\big). \tag{3.3}$$

Inequality (3.3) is useful in the one-parameter case (cf. [1],[2]).

2) Taking $r + p - 1$ in (3.2) we obtain

$$E(M^*)^p \le C_p E\big(B(p)\big). \tag{3.4}$$

Inequality (3.4) was found by Brossard ([1], Theorem 1) in the real case.

We now deal with the case in the presence of $V_\infty(p)$.

Theorem 3.3. *Suppose that B is a separable Banach space and $1 < p \le 2$. The following assertions are equivalent:*

(1) B is isomorphic to a p-smoothable Banach space,

(2) For every two-parameter B-valued martingale $(M_{m,n}, F_{m,n})$ we have

$$\{B_\infty(p) < \infty\} \subset \{M_{m,n} \text{ converges a.s.}\} \quad \text{a.s.}$$

(3) for every two-parameter B-valued martingale $(M_{m,n}, F_{m,n})$ satisfying the condition $E(M^*)^p < \infty$ we have

$$\{V_\infty(p) < \infty\} \subset \{M_{m,n} \text{ converges a.s.}\} \quad \text{a.s.}$$

To prove this theorem we need the following lemma which is an extension of Theorem 2.1 in [7].

Lemma 3.1. *Let $(u_{i,j})$ be a non-negative real-valued sequence of two-parameter random variables such that $u_{i,j}$ is $F_{i,j}$ - measurable.*

Set, for all $(i,j) \in N^2$ and $(m,n) \in N^2$,

$$\begin{aligned} b_{i,j} &= E(u_{i,j} \mid F_{i-1,j-1}), \\ b_{i,j} &= 0 \text{ if } i \text{ or } j = 0, \\ V_{m,n} &= \sum_{i=1}^{m} \sum_{j=1}^{n} u_{i,j}, \\ B_{m,n} &= \sum_{i=1}^{m} \sum_{j=1}^{n} b_{i,j}. \end{aligned}$$

If $E(\sup_{(m,n)} u_{m,n}) < \infty$, *then we have*

$$\{V(p) < \infty\} = \{B(p) < \infty\}.$$

Proof.

We have at first the following observations:

(1) $V_{m,n}(p)$ and $B_{m,n}(p)$ are non-negative and $B_{m,n}(p)$ are $F_{m,n}$ - preditable.

(2) We always have for all $(m,n) \in N^2$ that

$$V_{k,k}(p) \leq V_{m,n}(p) \leq V_{r,r}(p),$$
$$B_{k,k}(p) \leq B_{m,n}(p) \leq B_{r,r}(p),$$

where $k = \min(m,n)$, $r = \max(m,n)$.

The two above facts let us see that it suffices to prove the conclusion in the one-parameter case, precisely in the diagonal. Hence the rest of the proof can be proceeded as Theorem 2.1 in [7]. Q.E.D.

Proof of Theorem 3.3. We shall prove Theorem 3.3 as the following implications: (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2) is a consequence of Theorem 3.2.

(2) $\Rightarrow$ (3) follows from Theorem 3.2 and Lemma 3.1.

(3) $\Rightarrow$ (1). Suppose that $(M_{m,n}, F_{m,n})$ is B-valued martingale satisfying the condition

$$\sum_{i=1}^{m}\sum_{j=1}^{n} E\|\Delta M_{i,j}\|^p/(ij)^p < \infty. \tag{3.5}$$

Consider the following B-valued sequence

$$Y_{m,n} = (1/mn)\Delta M_{m,n}, \quad (m,n) \in N^2,$$
$$Y_{m,n} = 0 \text{ if } m \text{ or } n = 0.$$

Clearly, $\left(N_{m,n} = \sum_{i=1}^{m}\sum_{j=1}^{n} Y_{i,j}, F_{m,n}\right)$ is a B-valued martingale and

$$V(p,N) = \sum_{i=1}\sum_{j=1} \|\Delta N_{i,j}\|^p/(ij)^p$$
$$= \sum_{i=1}\sum_{j=1} \|\Delta M_{i,j}\|^p/(ij)^p,$$

$$E \sup_{(m,n)} \|\Delta M_{m,n}\|^p = E \sup \|\Delta M_{m,n}\|^p$$
$$\leq \sum_{i=1}^{m}\sum_{j=1}^{n} E\|\Delta M_{i,j}\|^p/(ij)^p < \infty,$$

by (3.5).

Thus, $E[V(p,N)] < \infty$ and

$$E \sup_{(m,n)} \|\Delta M_{m,n}\|^p < \infty.$$

From (3) we get that the martingale $(M_{m,n}, F_{m,n})$ converges a.s. Hence the conclusion follows from an application of Proposition 2.2. Q.E.D.

Now the above results can be applied to obtain the conditional version of the three series theorem in Banach space for two-parameter martingales.

Recall that the three-series theorem holds on a Banach space B if for every B-valued sequence $\left(M_{m,n} = \sum_{i=1}^{m}\sum_{j=1}^{n} \Delta M_{i,j}, F_{m,n}\right)$ we have that $M_{m,n}$ converges a.s. on B provided the following conditions hold:

(i) $\sum_{i=1}^{m}\sum_{j=1}^{n} E\left(\Delta Y_{i,j}^c \middle| F_{i,j}\right)$ converges a.s.,

(ii) $\sum_{i=1}^{m}\sum_{j=1}^{n} E\left(\|Y_{i,j}^c - E(Y_{i,j}^c|F_{i,j})\|^p \middle| F_{i,j}\right)$ is finite a.s.,

(iii) $\sum_{i=1}\sum_{j=1} P\left(\|Y_{i,j}\| > c \middle| F_{i,j}\right) < \infty$,

where $Y_{i,j} = \Delta M_{i,j}$ and

$$Y_{i,j}^c = \begin{cases} \Delta M_{i,j} & \text{if } \|\Delta M_{i,j}\| \leq c, \\ 0 & \text{otherwise,} \end{cases}$$

c being an arbitrary constant.

Theorem 3.4. *(The three-series theorem)*

The following three assertions are equivalent:

(1) B is isomorphic to a p-smoothable Banach space,

(2) For every B-valued martingale $M_{m,n}$, if the condition $B_\infty(p) < \infty$ a.s. holds then we have that $M_{m,n}$ converges a.s.,

(3) The three-series theorem holds on B.

Proof.

(1) ⇒ (2). This implication is a conclusion of Theorem 3.2.

(2) ⇒ (3). Suppose that we have (2) and

$$\left(M_{m,n} = \sum_{i=1}^{m}\sum_{j=1}^{n} \Delta M_{i,j} = \sum_{i=1}^{n}\sum_{j=1}^{n} Y_{i,j}\right)$$

is an arbitrary B-valued martingale.

Now the condition $B_\infty(p) < \infty$ a.s. and the second condition (ii) of three-series theorem imply that

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\left(Y_{i,j}^c - E\left(Y_{i,j}^c \middle| F_{i,j} \|\right)\right)$$

converges a.s. and, by the first condition (i) of the three-series theorem,

$$\sum_{i=1}^{m}\sum_{j=1}^{n} Y_{i,j}^c$$

converges a.s.

The condition (iii) of the three-series theorem and Borel-Cantell's lemma end the proof of (2) ⇒ (3).

(3) ⇒ (1). We again use Proposition 2.2. Suppose that $(M_{m,n})$ is a B-valued martingale satisfying the condition (3.5). Put

$$Y_{m,n} = (1/mn)\Delta M_{m,n}, \quad (m,n) \in N^2,$$
$$Y_{m,n} = 0 \text{ if } m \text{ or } n = 0.$$

It is easy to see that $(Y_{m,n})$ are adapted to σ-fields $(F_{m,n})$. Further we observe that

$$P(\|Y_{m,n}\| > c|F_{m,n}) \leq CE\|\Delta M_{(m,n)}\|^p/(mn)^p, \tag{3.6}$$

(Chebyshev's inequality).

$$\begin{aligned} E\|E(Y^c_{(m,n)}|F_{m,n})\| &= \frac{1}{mn}E\|E(\Delta M_{m,n}1_{\{\|\Delta M_{m,n}\|>C_{mn}\}}|F_{m,n})\| \\ &\leq \frac{1}{mn}E\|E(\Delta M_{m,n}1_{\{\|\Delta M_{m,n}\|>C_{mn}\}})\| \\ &= C^{1-p}\frac{1}{(mn)^p}E\|\Delta M_{m,n}\|\Big(\frac{1}{mn}\Big)^{p-1}1_{\{\|\Delta M_{m,n}\|>C_{mn}\}} \\ &\leq C^{1-p}\frac{1}{(mn)^p}E\| \ \|\Delta M_{m,n}\|, \end{aligned} \tag{3.7}$$

$$\begin{aligned} E(\|Y^c_{m,n} - E(Y^c_{m,n}|F_{m,n})\|^p|F_{m,n}) &\leq 2^p E(\|Y^c_{m,n}\|^p|F_{m,n}) \\ &\leq 2^p\frac{1}{(mn)^p}E(\|\Delta M_{m,n}\|^p|F_{m,n}). \end{aligned} \tag{3.8}$$

From (3.6), (3.7), (3.8), and (3.5) we conclude that the conditions (i), (ii) and (iii) of the three-series theorem hold for the sequence $(Y_{m,n})$. Hence, according to (3), we get that the martingale $M_{m,n} = \sum_{i=1}^{m}\sum_{j=1}^{n}\Delta M_{i,j}$ converges a.s.. Finally Proposition 2.2 ends the proof Q.E.D.

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