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ON THE MAXIMAL INEQUALITY

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Abstract. Let (Φ, Ψ) be a pair of Young's functions and (φ, ψ) their density functions such that

 $\sup_{x\in\mathbb{R}_+}\varphi(x)=K<\infty\quad and\quad \xi(u)=u\varphi(u)-\Phi(u)=\Psi[\varphi(u)]$

Let $\{X_n\}$ be an integrable martingale. Then, the following maximal inequality holds:

$$E\left[\xi\left(\sup_{1\leq k\leq n}|X_k|/\rho||X_n||_{\Phi}\right)\right]\leq \frac{1}{\rho-1}$$

where $\rho > 1$ is a constant and $\|\cdot\|_{\Phi}$ denotes the Luxemburg's norm.

1. INTRODUCTION

Throughout this paper we shall work with a fixed probability space (Ω, \mathcal{A}, P) and a sequence of σ -fields $\mathcal{B}_0 \subset \mathcal{B}_1 \subset \cdots \subset \mathcal{B}_n \subset \cdots \subset \mathcal{A}$, such that $\bigvee_{n=1}^{\infty} \mathcal{B}_n = \mathcal{B}_{\infty} \subset \mathcal{A}$.

The martingales or submartingales are always supposed to be adapted to the sequence of σ -fields $\{\mathcal{B}_n\}_{n\in\mathbb{N}}$.

For a pair of Young's conjugate functions (Φ, Ψ) , we shall denote by φ (resp. ψ) the density function of Φ (resp. Ψ), and we shall set by ξ the increasing function from \mathbf{R}_+ to \mathbf{R}_+ defined by

$$\xi(u) = u\varphi(u) - \Phi(u) = \Psi[\varphi(u)]. \tag{1}$$

For a sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$ we shall set

$$S_n = \sup_{1 \le k \le n} |X_k|.$$

Let $||X||_{\Phi}$ be the Luxemburg's norm of the random variable X. Our aim is to prove that if Φ is a generalized Young's function, that is some a weak to prove that if Φ is a generalized Young's function, that is some as well as the following that the following the following that the following that the following that the following the following the following that the following the following that the following the following that the following the following

(8)
$$\sup_{x \in \mathbf{R}_+} \varphi(x) = K < +\infty, \quad \text{que} = (u)$$

and $\{X_n\}_{n\in\mathbb{N}}$ is an integrable martingale such that $0<\|X_n\|_{\Phi}<+\infty$, then

$$E\left[\xi\left(\frac{S_n}{\rho\|X_n\|_{\Phi}}\right)\right] \leq \frac{1}{\rho-1} \text{ and evisiting of a specific support } (3)$$

where $\rho > 1$ is a constant.

 $\Psi(v) = / \psi(u)du, v \in \mathbb{R}_+$ It is well known that the inequality (3) was proved by J. Neveu in the case where Φ is a Young's function, that is

$$onumber \sum_{x = -\infty}^{\infty} \frac{1}{\varphi(x)} = -\infty$$
 $onumber \sum_{x = -\infty}^{\infty} \frac{1}{\varphi(x)} = -\infty$
 $onumber \sum_{x = -\infty}^{\infty} \frac{1}{\varphi(x)} = -\infty$

see [3]. In the case $\Phi(t) = t^p$, 1 , from (3) we obtain (vilsupeni s'anuo)

$$||S_n||_p \le \frac{p}{p-1} ||X_n||_p \tag{4}$$

with $\rho = \frac{[\varphi(t), \psi(t+0)] \Leftrightarrow t \in [\psi(v), \psi(v+q)]}{[\psi(v), \psi(v+q)]} \Rightarrow \sigma \Leftrightarrow (11)$

For $\Phi(t) = t \log^+(t)$, we get

$$E\left[\sup_{n\in\mathbb{N}}|X_n|\right] \leq \frac{e}{e-1}\left(1 + \sup_{n\in\mathbb{N}}E\left[|X_n|\log^+|X_n|\right]\right). \tag{5}$$

2. YOUNG'S FUNCTIONS

Let $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be an increasing function, continuous from the left and $\varphi(0) = 0$. Its primitive function (0 geor) W to not some engines and

of
$$\Phi$$
 (resp. Ψ).

(6) A Young's function Φ is $\operatorname{cax} b(x) \varphi^{\dagger} = (t) \Phi$ Young's function if its density function φ satisfies the condition

is a continuous increasing convex function on R and $\Phi(0) = 0$. Suppose that

and we define a function $\psi: \mathbf{R}_+ \to \mathbf{R}_+$ being the inverse of φ by

$$\psi(u) = \sup\{x : \varphi(x) < u\} \quad \text{for every } u > 0. \tag{8}$$

It is clear that ψ is increasing continuous from the left, $\psi(0)=0$ and $\lim_{u\to+\infty}\psi(u)=+\infty$.

Denote the primitive function of ψ by Ψ , i.e.

where
$$\rho > 1$$
 is a constant. $+\mathbf{R} = \mathbf{v}$, $ub(u)\psi$ $= (v)\psi$. It is well known that the inequality (3) was proved by J. Neveu in the case where Φ is a Young's function, that is

It is easy to verify that

$$tv \le \Phi(t) + \Psi(v), \quad t, v \in \mathbb{R}_+$$
 (10)

see [3]. In the case $\Phi(t) = t^p$, 1 , from (3) we obtain (Young's inequality)

$$tv = \Phi(t) + \Psi(v) \Leftrightarrow v \in [\varphi(t), \varphi(t+0)] \Leftrightarrow t \in [\psi(v), \psi(v+0)] \tag{11}$$

 $\lim \varphi(x) = +\infty$

for every $t, v \in \mathbb{R}_+$. Furthermore,

(5)
$$\Psi(v) = \sup_{t \in \mathbb{R}_{+}} [tv - \Phi(t)], \qquad (12)$$

$$\Phi(t) = \sup_{v \in \mathbb{R}_+} [tv - \Psi(v)]. \tag{13}$$

The pair (Φ, Ψ) is called a pair of Young's functions, Φ (resp. Ψ) is called the conjugate function of Ψ (resp. Φ). The function φ (resp. ψ) is called density of Φ (resp. Ψ).

A Young's function Φ is called generalized Young's function if its density function φ satisfies the condition

defines a norm on L[®] being the Luxemburg's norm of X, and there exist positive

$$\sup_{x \in \mathbb{R}_+} \varphi(x) = K < +\infty^{1/4} \text{ down so } 0 \text{ bas } 10 \text{ statistically}$$

In this case the density function ψ , inverse of φ , is obtained by

for all random variables
$$X \in L^0$$
.

(15) The space L^0 is complete and is called Orlicz spaces and $K = 0$ is called Orlicz spaces.

and the conjugate function of Φ is

$$\Psi(x) = \begin{cases} \int_{0}^{x} \psi(t)dt & \text{if } x < K \\ \int_{0}^{x} \psi(t)dt & \text{if } x < K \end{cases}$$
evods beautiful (16)

Assertion 2. Let Φ be a generalized Young's function annoinisonmos entringale satisfying $0 \le ||X_n||_{\Phi} < \infty$, $n \in \mathbb{N}$. Then, for every constant $\rho > 1$

$$\Phi[\psi(x)] = x\psi(x) - \Psi(x) \tag{17}$$

can take the value $+\infty$. But the composition

$$\Psi[\varphi(x)] = x\varphi(x) - \Phi(x) \tag{18}$$

is finite for all $x \in \mathbb{R}_+$.

(t)
$$\phi bt = tb(t) \phi - (u) \phi u = (u) \Phi - (u) \phi u = (u) \beta$$

3. ORLICZ SPACES

Let (Ω, \mathcal{A}, P) be a probability space and (Φ, Ψ) a pair of Young's functions.

Assertion 1. The set $L^{\Phi}(\Omega, A, P)$ of the equivalence classes of reals, random variables X defined on (Ω, A, P) for which exists, at least one real number a > 0 such that

$$E\left[\Phi\left(\frac{|X|}{a}\right)\right] \leq \frac{1}{2} \qquad (19)$$

is a subspace of $L^1(\Omega, A, P)$. Furthermore, the expression

(82)
$$\|X\|_{\Phi} = \inf\left\{a : a > 0, E\left[\Phi\left(\frac{|X|}{a}\right)\right] \le 1\right\}$$
 (20)

defines a norm on L^{Φ} being the Luxemburg's norm of X, and there exist positive constants C_1 and C_{∞} such that $A_1 = A_2 = A_3 = A_4$

vd benisto
$$C_1 \|X\|_1 \le \|X\|_{\Phi} \le C_{\infty} \|X\|_{\infty_{\text{neb ent seas sint nI}}}$$
 (21)

and the conjugate function of Φ is

for all random variables $X \in L^{\Phi}$.

The space L^{Φ} is complete and is called Orlicz space. For more details about the Orlicz spaces and the proof of this assertion see [2], [3].

4. MAXIMAL INEQUALITY

In this section we will use the notations as introduced above.

Assertion 2. Let Φ be a generalized Young's function and $\{X_n\}_{n\in\mathbb{N}}$ be a martingale satisfying $0 \le \|X_n\|_{\Phi} < \infty$, $n \in \mathbb{N}$. Then, for every constant $\rho > 1$

(17)
$$\frac{\Phi\left[\psi(x)\right]}{E\left[\xi\left(\frac{S_n}{N_n\|\Phi}\right)\right]} = x\psi(x) - \Psi(x)$$

$$\frac{1}{E\left[\xi\left(\frac{S_n}{\rho\|X_n\|\Phi}\right)\right]} \leq \frac{1}{E\left[\frac{S_n}{\rho\|X_n\|\Phi}\right]}$$
(22)
$$\frac{1}{E\left[\frac{S_n}{\rho\|X_n\|\Phi}\right]} = \frac{1}{E\left[\frac{S_n}{\rho\|X_n\|\Phi}\right]}$$

Proof.

$$\Psi[\varphi(x)] = x\varphi(x) - \Phi(x)$$

It is clear that

$$\xi(u) = u\varphi(u) - \Phi(u) = u\varphi(u) - \int_{0}^{u} \varphi(t)dt = \int_{0}^{u} t d\varphi(t)$$

Let (Ω, A, P) be a probability space and (Φ, Ψ) a pair of young s functions.

$$E\left[\xi\left(rac{S_n}{a}
ight)
ight]=E\left[\int\limits_0^{S_n/a}tdarphi(t)
ight]$$

and

(e1)
$$E\left[\xi\left(\frac{S_n}{a}\right)\right] = \sum_{k=1}^n E\left[\int_{a}^{S_k/a} t d\varphi(t)\right]$$

$$\leq \sum_{k=1}^{n} E\left\{\frac{S_k}{a} \left[\varphi\left(\frac{S_k}{a}\right) - \varphi\left(\frac{S_{k-1}}{a}\right)\right]\right\},$$
 (23)

$$E\left[\xi\left(\frac{S_{n}}{a}\right)\right] \leq \sum_{k=1}^{n} E\left\{\frac{|X_{k}|}{a}\left[\varphi\left(\frac{S_{k}}{a}\right) - \varphi\left(\frac{S_{k-1}}{a}\right)\right]\right\}$$

$$\leq \sum_{k=1}^{n} E\left\{\frac{|X_{n}|}{a}\left[\varphi\left(\frac{S_{k}}{a}\right) - \varphi\left(\frac{S_{k-1}}{a}\right)\right]\right\} = E\left[\frac{|X_{n}|}{a}\varphi\left(\frac{S_{n}}{a}\right)\right]$$
(24)

By virtue of Young's inequality we get

$$E\left[\frac{|X_{n}|}{a}\varphi\left(\frac{S_{n}}{a}\right)\right] \leq bE\left[\Phi\left(\frac{|X_{n}|}{ab}\right)\right] + bE\left\{\Psi\left[\varphi\left(\frac{S_{n}}{a}\right)\right]\right\}$$

$$= bE\left[\Phi\left(\frac{|X_{n}|}{ab}\right)\right] + bE\left[\xi\left(\frac{S_{n}}{a}\right)\right], \tag{25}$$

where 0 < b < 1. From (24) and (25) we have given been $\infty \uparrow 0$ guitted wolf

$$E\left[\xi\left(\frac{S_n}{a}\right)\right] \le bE\left[\Phi\left(\frac{|X|}{ab}\right)\right] + bE\left[\xi\left(\frac{S_n}{a}\right)\right]. \tag{26}$$

If $E\left[\xi\left(\frac{S_n}{a}\right)\right] < +\infty$ then

$$(1-b)E\left[\xi\left(\frac{S_n}{a}\right)\right] \le bE\left[\Phi\left(\frac{|X_n|}{a}\right)\right]. \tag{27}$$

The inequality (27) was proved under the condition (26). We shall show that it always holds under the condition of the assertion. In fact, we have

$$\max_{1 \le k \le n} \min (|X_k|, C) = \min (S_n, C) = T_n$$

$$1 \le k \le n$$

$$\min_{1 \le k \le n} \min_{1 \le k \le n} (28)$$

J. Neveu, Martingales à temps discret, Masson, Paris, 1972.

and by the same reasoning as above we obtain new Market State of the same reasoning as above we obtain and Year and Year

$$E\left[\xi\left(\frac{\min(S_n,C)}{a}\right)\right] = E\left[\xi\left(\frac{T_n}{a}\right)\right] \le \sum_{k=1}^n E\left\{\frac{T_k}{a}\left[\varphi\left(\frac{T_k}{a}\right) - \varphi\left(\frac{T_{k-1}}{a}\right)\right]\right\}$$

$$= \sum_{k=1}^n E\left\{\frac{T_k}{a}\left[\varphi\left(\frac{T_k}{a}\right) - \varphi\left(\frac{T_{k-1}}{a}\right)\right]\right\} \le \sum_{k=1}^n E\left\{\frac{|X_n|}{a}\left[\varphi\left(\frac{T_k}{a}\right) - \varphi\left(\frac{T_{k-1}}{a}\right)\right]\right\}$$

$$= E\left\{\frac{|X_n|}{a}\varphi(T_n)\right\} = E\left\{\frac{|X_n|}{a}\varphi\left[\frac{\min(S_n,C)}{a}\right]\right\}.$$

It follows from the Young's inequality that

$$E\Big[\xi\Big(\frac{\min(S_n,C)}{a}\Big)\Big] \le bE\Big[\Phi\Big(\frac{|X_n|}{ab}\Big)\Big] + bE\Big[\xi\Big(\frac{\min(S_n,c)}{a}\Big)\Big]$$
 where $0 < b < 1$.

It is clear that

By virtue of Young's inequality we get
$$E\left[\xi\left(\frac{\min(S_n,C)}{a}\right)\right] \leq E\left[\xi\left(\frac{C}{a}\right)\right] < +\infty.$$

So, for every constant 0 < b < 1, $0 \ge \left[\left(\frac{n^2}{a} \right) \circ \left[\frac{n^2}{a} \right] \right]$

$$(1-b)E\left[\xi\left(\frac{\min(S_n,C)}{a}\right)\right] \le bE\left[\Phi\left(\frac{|X_n|}{ab}\right)\right]. \tag{29}$$

Now, letting $C \uparrow \infty$ and using the Beppo-Levi theorem we have

$$\left((1-b)E\left[\xi\left(\frac{S_n}{a}\right)\right] \le bE\left[\Phi\left(\frac{|X_n|}{ab}\right)\right],\tag{30}$$

which implies that

 $E\left[\xi\left(\frac{S_n}{\rho\|X_n\|_{\Phi}}\right)\right] \leq \frac{1}{\rho - 1}.$

The proof is complete.

The inequality (27) was proved under the condition (26). We shall show that it always holds under the condition of the assertion. In fact, we have

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