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ON THE MAXIMAL INEQUALITY

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Abstract. *Let (Φ, Ψ) be a pair of Young's functions and (φ, ψ) their density functions such that*

$$\sup_{x \in \mathbf{R}_+} \varphi(x) = K < \infty \quad \text{and} \quad \xi(u) = u\varphi(u) - \Phi(u) = \Psi[\varphi(u)]$$

Let $\{X_n\}$ be an integrable martingale. Then, the following maximal inequality holds:

$$E\Big[\xi\Big(\sup_{1 \le k \le n} |X_k|/\rho \|X_n\|_\Phi\Big)\Big] \le \frac{1}{\rho - 1}$$

where $\rho > 1$ is a constant and $\|\cdot\|_\Phi$ denotes the Luxemburg's norm.

1. INTRODUCTION

Throughout this paper we shall work with a fixed probability space $(\Omega, \mathcal{A}, P)$ and a sequence of σ-fields $\mathcal{B}_0 \subset \mathcal{B}_1 \subset \cdots \subset \mathcal{B}_n \subset \cdots \subset \mathcal{A}$, such that $\bigvee_{n=1}^{\infty} \mathcal{B}_n = \mathcal{B}_\infty \subset \mathcal{A}$.

The martingales or submartingales are always supposed to be adapted to the sequence of σ-fields $\{\mathcal{B}_n\}_{n \in \mathbf{N}}$.

For a pair of Young's conjugate functions (Φ, Ψ), we shall denote by φ (resp. ψ) the density function of Φ (resp. Ψ), and we shall set by ξ the increasing function from $\mathbf{R}_+$ to $\mathbf{R}_+$ defined by

$$\xi(u) = u\varphi(u) - \Phi(u) = \Psi\big[\varphi(u)\big]. \tag{1}$$

For a sequence of random variables $\{X_n\}_{n \in \mathbf{N}}$ we shall set

$$S_n = \sup_{1\le k\le n} |X_k|.$$

Let $\|X\|_\Phi$ be the Luxemburg's norm of the random variable X. Our aim is to prove that if Φ is a generalized Young's function, that is

$$\sup_{x\in\mathbf{R}_+} \varphi(x) = K < +\infty, \tag{2}$$

and $\{X_n\}_{n\in\mathbf{N}}$ is an integrable martingale such that $0 < \|X_n\|_\Phi < +\infty$, then

$$E\Big[\xi\Big(\frac{S_n}{\rho\|X_n\|_\Phi}\Big)\Big] \le \frac{1}{\rho - 1} \tag{3}$$

where $\rho > 1$ is a constant.

It is well known that the inequality (3) was proved by J. Neveu in the case where Φ is a Young's function, that is

$$\lim_{x\to+\infty} \varphi(x) = +\infty$$

see [3]. In the case $\Phi(t) = t^p$, $1 < p < \infty$, from (3) we obtain

$$\|S_n\|_p \le \frac{p}{p-1}\|X_n\|_p \tag{4}$$

with $\rho = \dfrac{p}{p-1}$.

For $\Phi(t) = t\,\log^+(t)$, we get

$$E\Big[\sup_{n\in\mathbf{N}} |X_n|\Big] \le \frac{e}{e-1}\Big(1 + \sup_{n\in\mathbf{N}} E\big[|X_n|\log^+|X_n|\big]\Big). \tag{5}$$

2. YOUNG'S FUNCTIONS

Let $\varphi : \mathbf{R}_+ \longrightarrow \mathbf{R}_+$ be an increasing function, continuous from the left and $\varphi(0) = 0$. Its primitive function

$$\Phi(t) = \int_0^t \varphi(x)dx \tag{6}$$

is a continuous increasing convex function on $\mathbf{R}$ and $\Phi(0) = 0$. Suppose that

$$\lim_{x \to \infty} \uparrow \varphi(x) = +\infty \tag{7}$$

and we define a function $\psi : \mathbf{R}_+ \to \mathbf{R}_+$ being the inverse of φ by

$$\psi(u) = \sup\{x \,:\, \varphi(x) < u\} \quad \text{for every } u > 0. \tag{8}$$

It is clear that ψ is increasing continuous from the left, $\psi(0) = 0$ and $\lim\limits_{u \to +\infty} \psi(u) = +\infty$.

Denote the primitive function of ψ by Ψ, i.e.

$$\Psi(v) = \int_0^v \psi(u)du, \quad v \in \mathbf{R}_+. \tag{9}$$

It is easy to verify that

$$tv \leq \Phi(t) + \Psi(v), \quad t, v \in \mathbf{R}_+ \tag{10}$$

(Young's inequality) and

$$tv = \Phi(t) + \Psi(v) \Leftrightarrow v \in \big[\varphi(t), \varphi(t+0)\big] \Leftrightarrow t \in \big[\psi(v), \psi(v+0)\big] \tag{11}$$

for every $t, v \in \mathbf{R}_+$. Furthermore,

$$\Psi(v) = \sup_{t \in \mathbf{R}_+} [tv - \Phi(t)], \tag{12}$$

$$\Phi(t) = \sup_{v \in \mathbf{R}_+} [tv - \Psi(v)]. \tag{13}$$

The pair (Φ, Ψ) is called a pair of Young's functions, Φ (resp. Ψ) is called the conjugate function of Ψ (resp. Φ). The function φ (resp. ψ) is called density of Φ (resp. Ψ).

A Young's function Φ is called generalized Young's function if its density function φ satisfies the condition

$$\sup_{x\in\mathbf{R}_+} \varphi(x) = K < +\infty. \tag{14}$$

In this case the density function ψ, inverse of φ, is obtained by

$$\psi(t) = \begin{cases} \sup\{s \ : \ \varphi(s) < t\} & \text{if } t < K \\ +\infty & \text{if } t \geq K \end{cases} \tag{15}$$

and the conjugate function of Φ is

$$\Psi(x) = \begin{cases} \int_0^x \psi(t)dt & \text{if } x < K \\ +\infty & \text{if } x \geq K. \end{cases} \tag{16}$$

The composition

$$\Phi\big[\psi(x)\big] = x\psi(x) - \Psi(x) \tag{17}$$

can take the value $+\infty$. But the composition

$$\Psi\big[\varphi(x)\big] = x\varphi(x) - \Phi(x) \tag{18}$$

is finite for all $x \in \mathbf{R}_+$.

3. ORLICZ SPACES

Let $(\Omega, \mathcal{A}, P)$ be a probability space and (Φ, Ψ) a pair of Young's functions.

Assertion 1. *The set $L^\Phi(\Omega, \mathcal{A}, P)$ of the equivalence classes of reals, random variables X defined on $(\Omega, \mathcal{A}, P)$ for which exists, at least one real number $a > 0$ such that*

$$E\Big[\Phi\Big(\frac{|X|}{a}\Big)\Big] \leq 1 \tag{19}$$

is a subspace of $L^1(\Omega, \mathcal{A}, P)$. Furthermore, the expression

$$\|X\|_\Phi = \inf\Big\{a \ : \ a > 0, \ E\Big[\Phi\Big(\frac{|X|}{a}\Big)\Big] \leq 1\Big\} \tag{20}$$

defines a norm on L^Φ being the Luxemburg's norm of X, and there exist positive constants C_1 and C_∞ such that

$$C_1\|X\|_1 \le \|X\|_\Phi \le C_\infty \|X\|_\infty \tag{21}$$

for all random variables $X \in L^\Phi$.

The space L^Φ is complete and is called Orlicz space. For more details about the Orlicz spaces and the proof of this assertion see [2], [3].

4. MAXIMAL INEQUALITY

In this section we will use the notations as introduced above.

Assertion 2. *Let Φ be a generalized Young's function and $\{X_n\}_{n\in\mathbf{N}}$ be a martingale satisfying $0 \le \|X_n\|_\Phi < \infty$, $n \in \mathbf{N}$. Then, for every constant $\rho > 1$*

$$E\Big[\xi\Big(\frac{S_n}{\rho\|X_n\|_\Phi}\Big)\Big] \le \frac{1}{\rho-1}. \tag{22}$$

Proof.

It is clear that

$$\xi(u) = u\varphi(u) - \Phi(u) = u\varphi(u) - \int_0^u \varphi(t)dt = \int_0^u t\,d\varphi(t)$$

for all $u \in \mathbf{R}_+$. Furthermore, we have

$$E\Big[\xi\Big(\frac{S_n}{a}\Big)\Big] = E\Big[\int_0^{S_n/a} t\,d\varphi(t)\Big] \quad (a > 0)$$

and

$$E\Big[\xi\Big(\frac{S_n}{a}\Big)\Big] = \sum_{k=1}^n E\Big[\int_{S_{k-1}/a}^{S_k/a} t\,d\varphi(t)\Big]$$

$$\le \sum_{k=1}^n E\Big\{\frac{S_k}{a}\Big[\varphi\Big(\frac{S_k}{a}\Big) - \varphi\Big(\frac{S_{k-1}}{a}\Big)\Big]\Big\}, \tag{23}$$

$$E\Big[\xi\Big(\frac{S_n}{a}\Big)\Big] \le \sum_{k=1}^{n} E\Big\{\frac{|X_k|}{a}\Big[\varphi\Big(\frac{S_k}{a}\Big)-\varphi\Big(\frac{S_{k-1}}{a}\Big)\Big]\Big\}$$
$$\le \sum_{k=1}^{n} E\Big\{\frac{|X_n|}{a}\Big[\varphi\Big(\frac{S_k}{a}\Big)-\varphi\Big(\frac{S_{k-1}}{a}\Big)\Big]\Big\} = E\Big[\frac{|X_n|}{a}\varphi\Big(\frac{S_n}{a}\Big)\Big] \tag{24}$$

By virtue of Young's inequality we get

$$E\Big[\frac{|X_n|}{a}\varphi\Big(\frac{S_n}{a}\Big)\Big] \le bE\Big[\Phi\Big(\frac{|X_n|}{ab}\Big)\Big] + bE\Big\{\Psi\Big[\varphi\Big(\frac{S_n}{a}\Big)\Big]\Big\}$$
$$= bE\Big[\Phi\Big(\frac{|X_n|}{ab}\Big)\Big] + bE\Big[\xi\Big(\frac{S_n}{a}\Big)\Big], \tag{25}$$

where $0 < b < 1$. From (24) and (25) we have

$$E\Big[\xi\Big(\frac{S_n}{a}\Big)\Big] \le bE\Big[\Phi\Big(\frac{|X|}{ab}\Big)\Big] + bE\Big[\xi\Big(\frac{S_n}{a}\Big)\Big]. \tag{26}$$

If $E\Big[\xi\Big(\frac{S_n}{a}\Big)\Big] < +\infty$ then

$$(1-b)E\Big[\xi\Big(\frac{S_n}{a}\Big)\Big] \le bE\Big[\Phi\Big(\frac{|X_n|}{a}\Big)\Big]. \tag{27}$$

The inequality (27) was proved under the condition (26). We shall show that it always holds under the condition of the assertion. In fact, we have

$$\max_{1\le k\le n} \min\big(|X_k|, C\big) = \min\big(S_n, C\big) = T_n \tag{28}$$

and by the same reasoning as above we obtain

$$E\Big[\xi\Big(\frac{\min(S_n,C)}{a}\Big)\Big] = E\Big[\xi\Big(\frac{T_n}{a}\Big)\Big] \le \sum_{k=1}^{n} E\Big\{\frac{T_k}{a}\Big[\varphi\Big(\frac{T_k}{a}\Big)-\varphi\Big(\frac{T_{k-1}}{a}\Big)\Big]\Big\}$$
$$= \sum_{k=1}^{n} E\Big\{\frac{T_k}{a}\Big[\varphi\Big(\frac{T_k}{a}\Big)-\varphi\Big(\frac{T_{k-1}}{a}\Big)\Big]\Big\} \le \sum_{k=1}^{n} E\Big\{\frac{|X_n|}{a}\Big[\varphi\Big(\frac{T_k}{a}\Big)-\varphi\Big(\frac{T_{k-1}}{a}\Big)\Big]\Big\}$$
$$= E\Big\{\frac{|X_n|}{a}\varphi(T_n)\Big\} = E\Big\{\frac{|X_n|}{a}\varphi\Big[\frac{\min(S_n,C)}{a}\Big]\Big\}.$$

It follows from the Young's inequality that

$$E\Big[\xi\Big(\frac{\min(S_n, C)}{a}\Big)\Big] \leq bE\Big[\Phi\Big(\frac{|X_n|}{ab}\Big)\Big] + bE\Big[\xi\Big(\frac{\min(S_n, c)}{a}\Big)\Big]$$

where $0 < b < 1$.

It is clear that

$$E\Big[\xi\Big(\frac{\min(S_n, C)}{a}\Big)\Big] \leq E\Big[\xi\Big(\frac{C}{a}\Big)\Big] < +\infty.$$

So, for every constant $0 < b < 1$,

$$(1-b)E\Big[\xi\Big(\frac{\min(S_n, C)}{a}\Big)\Big] \leq bE\Big[\Phi\Big(\frac{|X_n|}{ab}\Big)\Big]. \tag{29}$$

Now, letting $C \uparrow \infty$ and using the Beppo-Levi theorem we have

$$(1-b)E\Big[\xi\Big(\frac{S_n}{a}\Big)\Big] \leq bE\Big[\Phi\Big(\frac{|X_n|}{ab}\Big)\Big], \tag{30}$$

which implies that

$$E\Big[\xi\Big(\frac{S_n}{\rho\|X_n\|_\Phi}\Big)\Big] \leq \frac{1}{\rho - 1}.$$

The proof is complete.

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