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DYNAMICAL SYSTEMS WITH STATE AND CONTROL CONSTRAINTS: CONTROLLABILITY AND RELATED TOPICS

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Abstract. *In this article we survey some results on controllability of dynamical systems in Banach spaces with constrained states and controls. Relationship between various notions of controllability is studied and their characterizations expressed in terms of the system data are given for both continuous and discrete-time systems. Applications of the results obtained for abstract system models to some concrete classes of systems such as those whose dynamics are described by periodic differential equations or retarded differential equations are presented. Some related questions on the properties of the reachable sets and on the continuity of the Bellman function for an optimal control problem are also considered.*

1. INTRODUCTION

The aim of this paper is to survey some results concerning the control theory of dynamical systems with state and control constraints. Consider the control system governed by the ordinary differential equation

$$\dot{x}(t) = f(t, x(t), u(t)), \quad t \geq 0, \tag{1.1}$$

$$x(t) \in N(t) \subset X, \quad u(t) \in \Omega(t) \subset U, \tag{1.2}$$

where $f : \boldsymbol{R}_+ \times X \times U \to X$ is a given function, X and U are linear spaces of states and controls, respectively; $N(t)$ and $\Omega(t)$ are non-empty constraint sets.

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Our attention is mainly focused on the study of various notions of controllability and some related topics (such as observability, properties of the reachable set, optimality conditions and continuity of the marginal function of the associated optimal control problem, etc.) for systems of the above type and applications of the results obtained for this general model to concrete classes of dynamical systems.

The notion of controllability is certainly one of the most fundamental concepts in the systems theory. They also play an important role in many problems in optimal control and optimal estimation theory (such as the existence of optimal solutions, the convergence of numerical algorithms, etc.). The first steps to formalize these matters as a separate area of research were undertaken by Kalman [34], [35], who proved some algebraic tests of controllabilty and observability, known in the literature as the Kalman rank conditions, for finite-dimensional linear autonomous systems with no constraints on states and controls. During the past three decades questions of controllability have been extensively examined for various types of dynamical systems with more complicated structures and under more general assumptions: linear systems with time-varying coefficients, nonlinear systems, systems described by differential equations with delayed arguments, distributed-parameters systems, etc. Recently, controllability problems have been studied also for implicit systems [26], [63] and systems described by multi-valued maps [25], [65].

The results to be surveyed in the present paper have been motivated by the following three developments in this area.

The first is the study of controllability for abstract control systems within the framework of the theory of one-parameter semigroups of bounded linear operators. The common object of research in this direction is the system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad t \geq 0, \ x(t) \in X, \ u(t) \in U, \tag{1.3}$$

where the operator A is usually assumed to be an infinitesimal generator of a strongly continuous semigroups $S(t), t \geq 0$, the state space X and the control space U are infinite dimensional Banach spaces. Initiated by the fundamental work due to Fattorini [23], [24], where some necessary and sufficient conditions of approximate controllability for linear systems in Hilbert spaces were presented, this direction has obtained further developments in a number of works by Delfour, Mitter, Triggiani, Korobov, Dolecki, and many others. Since it would be hard to list all interesting works in this area, we indicate instead the surveys of Russel [72], Curtain and Pritchard [13] and Conti [12], which contain along with systematic presentations also extensive bibliographies on this subject. One good reason motivating the study of such abstract models is that they make it possible to treat, in a mathematically unified manner, a variety of different dynamical systems, including those governed by integro-differential equations, partial differential equations of both parabolic and hyperbolic type, as well as difference differential equations. This fact is extremely well demonstrated in the context of recent development of the theory of functional differential equations (FDE) within the semigroups framework. Among the publications on this topic we quote Halle [30], Bernier and Manitius [6], Delfour and Manitius [14] and Salamon [73]. In particular, this approach allowed to carry out, for quite general classes of FDE, an analysis of controllability properties, leading from general abstract criteria to practical verifiable conditions, based on the matrices defining the orginal systems; see, for example, Manitius and Triggiani [44], Bartosiewicz [5], Salamon [74].

Second development motivating our investigations concerns with infinite-dimensional discrete-time systems of the form

$$x_{k+1} = Ax_k + Bu_k, \quad x_k \in X,\ u_k \in \Omega \subset U. \tag{1.4}$$

Discrete-time models are known to be more suitable for representing economic, biological and sociological systems. They arise also, for example, when one need to discretize continuous-time systems for the purpose of numerical implementation. Kalman [35], Weiss [84], Gabasov [29] were among the earlier investigators of the controllability of discrete-time systems. A detailed survey of works carried out in this direction is contained in a paper due to Faradjev, Shapiro and Phat [22]. For infinite-dimensional discrete-time systems, basic problems of controllability, observability and stability has been first treated in Fuhrmann [27], [28] and, more recently, in a series of works due to Przyluski [66], [67], and Przyluski and Rolewicz [68]. In Olbrot and Sosnowski [59], some duality theorems on control and observation are presented for linear discrete-time systems in Banach spaces. The growing interest in infinite-dimensional discrete-time models was motivated in part by the fact that this kind of systems provide quite a useful tool for studying some qualitative properties of systems governed by FDE. Although many control problems regarding the systems of FDE, as noted above, can be treated within the framework of the theory of semigroups, however, one of main disadvantages of the mentioned approach is that when applied to FDE it leads to the abstract model (1.3) with a unbounded infinitesimal generator A. This renderes the translation from abstract results into verifiable criteria expressed in terms of the original matrices quite a complicated problem, even for a simple class of retarded systems with one delay

$$\dot{x}(t) = A_0x(t) + A_1x(t-h) + Bu(t), \quad x(t) \in \mathbf{R}^n,\ u(t) \in \mathbf{R}^m, \tag{1.5}$$

see, for example, [44]. In the meantime, as has been shown in [59], it is always possible to transform linear differential equations with delays into discrete-time systems of the form (1.4) in a suitable function space X, where, in contrast to the abstract continuous-time model, the operators A and B are bounded.

The third important development is the study of controllability problems for dynamical systems with restrained controls. These problems are well known in the literature on optimal control. In most of works, the control constraint set Ω is usually assumed to be a bounded set containing 0 in its interior (for example, Ω is a unit ball or a unit cube centered at the origin in R^m). This classical assumption restricts, however, the scope of application of results, since it does not hold for many systems arising in practical fields such as electrical engineering, economics, ... where the controls usually represent input factors like energy, finance, materials, etc. which take only positive values. Problems of constrained controllability without the above mentioned assumption were first considered in Saperstone and Yorke [75], where a criterion of global controllability for finite-dimensional linear systems with a single positive input was established. Further developments in this direction have been done by Brammer [8], Heymann and Stern [32], Korobov, Marinich and Podol'skii [37] for linear autonomous systems in R^n; Evans and Murthy [21], [19], [20], Sontag [80] for finite-dimensional discrete-time systems. For the case of infinite dimension, the analogous problem has been examined in Korobov and Son [38], where some criteria of local controllability were proved for the linear autonomous system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(t) \in X, \quad u(t) \in \Omega \subset U, \tag{1.6}$$

with A,B being bounded linear operators.

In spite of all the research efforts mentioned above, the controllability theory of dynamical systems is still far from complete. In particular, the controllability conditions for the system (1.6) with unbounded operators are still unknown. Questions of controllability for dynamical systems with both state and control constraints have been almost not yet considered in the literature, although some results were obtained for discrete-time case, see e.g. Phat and Dieu [64]. We note that most of the works on controllability of FDE has been sofar concentrated on the case of unconstrained controls and, although this theory has been presently well developed, only few and scattered results are known for the systems described by FDE with constraints on controls, see [10], [11], [78].

Our investigations are called to solve some open problems in this area. In [46] we extended the results of [38] to the case of the system (1.6) with a unbounded operator A and obtained, in particular, criteria of local controllability and local approximate controllability for this class of systems. In [48] we studied a more general class of nonautonomous abstract systems of the form (1.1) in Banach spaces, with both state and control constraints. Some fundamental properties of the reachable set and the trajectory set of the system such as convexity and continuity with respect to the initial conditions have been examined. In particular, we obtained a generalized version of the well-known "bang-bang principle" for this class of systems. The mentioned results are summarized in the next Section 2.

In Section 3 we present the controllability theory of linear discrete-time systems in Banach spaces with constrained controls, developed in [47], [50], [51], [53], [54], [55]. Some notions of controllability are introduced and their relationship are examined. The main results, criteria of local and global controllability expressed in terms of the system data $\{A, B, \Omega\}$, have been established for several of the introduced controllability notions. In particular, the discrete-time systems with compact operators (which, in fact, are proved to be a suitable discrete-time model associated with systems of FDE) are studied in details, using the method of spectral decomposition of the state space.

Section 4 is devoted to discuss some applications of the above development. The results of Section 2 are used to prove the necessary and sufficient condition of optimality in a terminal optimal problem for the abstract evolutionary system with state and control constraints. The continuity of the Bellman function is also established. Further, as an application of the controllability theory of infinite-dimensional discrete-time systems presented in Section 3, we prove, for the nonautonomous system $\dot{x}(t) = A(t)x(t) + B(t)u(t)$, $x(t) \in X$, $u(t) \in \Omega \subset U$ with $A(t)$ and $B(t)$ being periodic functions, some creteria of local and global controllability. Recall now that our interest in controllability of infinite-dimensional discrete-time systems stems primarily from the possibility of their applications to study FDE. Although such a possibility was repeatedly emphasized in different papers, to our knowledge, there has been not yet significant attempts to apply this approach to examine controllability for any other classes of FDE more general than the retarded system (1.5). The results of [54], [55] are intended to fill this gap. Using a natural method of discretization and applying thereafter the results obtained previously for discrete-time systems we are able to derive a criterion of approximate controllability in the state space $\boldsymbol{R}^n \times L_p$ for a general class of retarded systems

$$\dot{z}(t) = \int_{-h}^{0} d\eta(\theta) z(\theta + h) + B_0 u(t), \quad z(t) \in \boldsymbol{R}^n, \; u(t) \in \Omega \subset \boldsymbol{R}^m, \tag{1.7}$$

where Ω is assumed to be a cone with the vertex at the origin. What is more important, these abstract criteria are then transformed, by means of the technical apparatus of the so-called structural operators (due to Bernier, Delfour and Manitius) into verifiable tests expressed in terms of the matrices of the original systems. A summary of these results is given also in Section 4.

To simplify the presentation, we list some notations and terminology used in this paper. Let $\boldsymbol{R}$ and $\boldsymbol{C}$ be the fields of all real and complex numbers, respectively, and N be the set of all non-negative integers. The symbol R^n will denote the n-dimensional Euclidean space. Given two Banach spaces X and Y, we denote by $L(X,Y)$ the Banach space of all bounded linear operators from X to Y and by $L(X)$ the space $L(X,X)$. The identity operator in $L(X)$ is denoted by I. The null space, the range and the spectrum of an operator A are denoted by Ker A, Im A, $\sigma(A)$, respectively. Let X^* be the topological dual of X and X_w be the space X endowed with the weak topology $\sigma(X,X^*)$. Then $\langle f,x\rangle$ denotes the value of $f\in X^*$ at $x\in X$ and A^* denotes the adjoint operator of a linear operator A; if $A\in L(X,Y)$ then $A^*\in L(Y^*,X^*)$. When we are dealing with Euclidean spaces, $\langle,\rangle$ is employed to denote the inner product and A^* denotes the transpose matrix of a matrix A. $\|.\|$ will denote the norm of whatever space under consideration. Given $a<b$, $C(a,b,X)$ will denote the topological vector space of all continuous functions $[a,b]\to X$ endowed with the topology of uniform convergence, $L_\infty(a,b,X)$ denotes the Banach space of all equivalent classes of strongly measurable functions $x(t):[a,b]\to X$ such that $\|x(t)\|$ is essentially bounded on $[a,b]$ and $L_1(a,b,X)$ is the Banach space of all equivalent classes of integrable functions $[a,b]\to X$. Throughout the paper, the Lebesgue measure is used and, for the functions with values in a Banach space, the integral is understood in the sense of Bochner. Further, a multivalued function $\Omega:[a,b]\to X$ is said to measurable if the set

$$\{t\in[a,b]:\Omega(t)\cap S\neq\emptyset\}.$$

is measurable for every open set $S\subset X$. A multivalued function F from a Banach space X into a topological vector space Y is called upper semi-continuous (resp., lower semi-continuous) at a point $x_0\in X$ if for every open neighbourhood $U(0)$ of the origin 0 in Y the exists $\delta>0$ s.t. $F(x)\subset F(x_0)+U(0)$ (resp., $F(x_0)\subset F(x)+U(0)$), whenever $\|x-x_0\|<\delta$. F is called continuous at x_0 if it is simultaneously uper and lower semi-continuous at x_0. Finally, let M be a subset of a Banach space X, then span M is the linear hull of M; the convex hull, the interior and the closure of M are denoted respectively by co M, int M and $\overline{M}$. The negative polar cone of M is defined by

$$M^0=\{f\in X^*:\langle f,x\rangle\leq 0,\quad \forall x\in M\}.$$

2. EVOLUTIONARY SYSTEMS WITH STATE AND CONTROL CONSTRAINTS

In this section we study some properties of the abstract system defined by an evolutionary process. By the term evolutionary process in a Banach space X we mean a strongly continuous operator function $E(t,s)$ from $\{(t,s) : 0 \leq s \leq t < \infty\}$ to $L(X)$ which has the transitive property $E(t,r)E(r,s) = E(t,s)$ and $E(t,t) = I$, $t \geq 0$. Thus $E(t,s)$ is the infinite dimensional analogue of a fundamental matrix associated with the solutions of a linear differential equation in $\boldsymbol{R}^n$.

2.1. Properties of the reachable set and the solutions set

Suppose that X, U are real separable Banach spaces. We consider the evolutionary system defined explicitly by

$$x(t) = x(t, x_0, u) = E(t,0)x_0 + \int_0^t E(t,s)B(s)u(s)ds, \quad t \geq 0. \tag{2.1}$$

where $E(.,.)$ is an evolutionary process, $B(.) : [0,\infty) \to L(U,X)$ is a given locally integrable function. A function $u(.) : [0,\infty) \to U$ is said to be admissible control on a subinterval (t',t'') of $[0,\infty)$ if $u(.) \in L_\infty(t',t'',U)$ and

$$u(t) \in \Omega(t) \quad \text{a.e.} \quad \text{on } (t',t''), \tag{2.2}$$

where $\Omega(.) : [0,\infty) \to U$ is a given measurable multi-valued function.

It is well known that for every admissible control $u(.)$, the function $x(t)$ defined by the formula (2.1) is strongly continuous and can be considered as a mild solution of a system of the form

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad t \geq 0, \tag{2.1'}$$

with the initial condition $x(0) = x_0 \in D$, where D is a dense set in X and $A(t)$ is a linear, in general, unbounded operator with the domain $D(A(t)) = D$. In the sequel, we shall call $x(t)$ the solution or the trajectory of the system. Controllability, stabilizability and other properties of evolutionary systems with unconstrained controls have been extensively investigated (see, for examle, [12], [61], [86]). Their properties in the case of constrained controls, however, have not been systematically treated. In [48] we study some properties of the set of all solutions of the system (2.1) with constrained controls (2.2) initiated at a given point $x_0 \in X$ and defined on $[0,T] \subset [0,\infty)$. We denote this set by $G(x_0,T)$. Clearly, $G(x_0,T) \subset C(0,T,X)$. The reachable set of the system at time $T > 0$ is defined by

$$R(x_0,T) = \{x(T) : x(.) \in G(x_0,T)\}.$$

Theorem 2.1 [48]. *The closure of the solutions set $G(x_0,T)$ of the system (2.1), (2.2) is convex in the space $C(0,T,X)$ and remains unchanged when the constraint (2.2) is replaced by the convex constraint*

$$u(t) \in \operatorname{co}\Omega(t) \quad a.e. \text{ on } \quad [0,T]. \tag{2.3}$$

Since the evaluation operator $x(.) \to x(T)$ from $C(0,T,X)$ to X is continuous, from the above theorem we obtain

Corollary 2.2. *The closure of the reachable set $R(x_0,T)$ of the system (2.1), (2.2) is convex in the state space X and remains unchanged when the constraint (2.2) is replaced by the convex constraint (2.3).*

This result was first proved in [38], [12] for the autonomous case, based essentially on the Ulh's generalization of the well known Lyapunov theorem on the convexity of the range of a vector-valued measure (see, e.g. [71]). The general case was treated in [48] by the method of discretization. Now, let denote by $\ddot{\Omega}(t)$ the set of all extreme points of $\Omega(t)$. Then by the Krein-Milman theorem, $\operatorname{co}\ddot{\Omega}(t) = \Omega(t)$, provided that $\Omega(t)$ is convex and compact. Therefore, as another consequence of Theorem 2.1, we obtain

Corollary 2.3. *Assume that the constraint set $\Omega(t)$ is compact and convex. Then the set of all trajectories of the system (2.1) generated by extreme controls $u(t) \in \ddot{\Omega}(t)$, a.e. $t \in [0,T]$, is dense in $G(x_0,T)$ (regarded as subsets of $C(0,T,X)$).*

The above result can be considered as a generalization of the well known classical "bang-bang principle", which, roughly speaking, claims that if a point x in the state space R^n is reachable along a trajectory of the linear dynamical system by some control $u(t) \in R^m$ with $\|u(t)\| \leq 1$ then it can also be reached by a suitable "bang-bang control" $u(t)$ which takes only the extreme values $+1$ and -1.

We consider now a more general situation. Suppose that the trajectory of the system (2.1), (2.2) must satisfy the state constraint

$$x(t) \in N(t), \quad \forall t \in [0,T], \tag{2.4}$$

where $N : [0,T] \to X$ is a given lower semi-continuous multi-valued function such that for each t, $N(t)$ is closed convex set with nonempty interior in X. Such a trajectory will be called admissible trajectory. Denote by

$$L(x_0,T) = \{x(.) \in G(x_0,T) : x(t) \in N(t), \quad \forall t \in [0,T]\}$$

the set of *admissible trajectories* and by

$$L_0(x_0,T) = \{x(.) \in G(x_0,T) : x(t) \in \text{int}\, N(t), \quad \forall t \in [0,T]\}$$

the set of *admissible interior trajectories* initiated at x_0. The following theorem extends the previous result to the case of systems with both state and control constraints.

Theorem 2.3 [48]. *The closure of $L_0(x_0,T)$ is convex in the space $C(0,T,X)$. If, in addition, $L_0(x_0,T)$ is nonempty, then the closure of $L(x_0,T)$ is also convex and remains unchanged when the constraint (2.2) is replaced by the convex constraint (2.3).*

As a consequence, the analogous result holds also for the reachable set, but we omit the formulation. It is important to notice that the condition that $L_0(x_0,T)$ is nonempty (i.e. the system possesses at least one admissible interior trajectory from x_0) is essential, as was shown by a simple example in [41]. We remark also that in all the above results we can establish only the convexity of the closure of the trajectory set or the reachable set. This is, in fact, a specific feature for infinite-dimensional systems. The closure convexity is however quite a convenient property, particularly in cases, where the separation theorems of convex sets play a crucial role.

Another property of the reachable set which is basic in some optimization problems is its continuity with respect to parameters perturbations. Taking the initial condition as a perturbed parameter for the evolutionary system (2.1) with the control constraint (2.2) and the state constraint (2.4) we can prove the following result.

Theorem 2.4 [48]. *Let X and U be reflexive separable Banach spaces. For every $T > 0$, the set of admissible trajectories $L(x_0,T)$ is lower semi-continuous when regarded as a multi-valued function from X to $C(0,T,X)$. If, in addition, the set of admissible interior trajectories $L_0(x_0,T)$ is nonempty at $x_0 \in X$, then $L(x_0,T)$ is continuous at x_0 when regarded as a multi-valued function from X to $C(0,T,X_w)$.*

As a corollary we can formulate the analogous result for the reachable set. In particular, we have

Corollary 2.5. *The reachable set $R(x_0,T)$ of the system (2.1)-(2.2)-(2.4) with $X = R^n$ and $U = R^m$ is a continuous multi-valued function of x_0 whenever the system possesses an admissible interior trajectory from x_0.*

2.2. Local controllability of evolutionary systems with constrained controls

As noted in Introduction, problems of controllability have sofar been systematically examined only for the evolutionary systems with unconstrained controls. The case of the classical constraint $\Omega = \{u \in U : \|u\| \leq \alpha\}$ is treated in some recent papers [1], [9]. We present now our results concerning with controllability of the evolutionary autonomous system with a more general type of control constraints. Remark first that in the autonomous case, the formula (2.1) is reduced to

$$x(t) = E(t)x_0 + \int_0^t E(t-s)Bu(s)ds, \quad t \geq 0, \tag{2.5}$$

where $\{E(t),\ t \geq 0\}$ is a strongly continuous semigroup of bounded linear operators in the Banach space X, $B \in L(U, X)$. Formula (2.5) gives a mild solution of the differential equation

$$\dot{x}(t) = Ax(t) + Bu(t), \quad t \geq 0,\ x(0) = x_0, \tag{2.6}$$

where A is the infinitesimal generator of the semigroup $E(t)$:

$$Ax = \lim_{t \to 0} \frac{E(t)x - x}{t},$$

for $x \in D$, D being a dense set in X. The set of admissible controls is defined by

$$\hat{\Omega}_T = \{u(.) \in L_\infty(0, T, U) : u(t) \in \Omega \quad \text{a.e. on } [0, T]\}, \tag{2.7}$$

where Ω is a given subset of a Banach space U. The only requirement imposed on Ω is

$$\exists u_0 \in \Omega : Bu_0 = 0. \tag{2.8}$$

The reachable set of the system (from the origin) is defined by

$$R^\Omega = \cup\{R_T^\Omega : T \geq 0\},$$

where R_T^Ω is the reachable set in time T : $R_T^\Omega = \left\{ \int_0^T E(T-s)Bu(s)ds : u(.) \in \hat{\Omega}_T \right\}$.

Definition 2.6. The evolutionary autonomous system (2.5) with constrained controls (2.7) is said to be locally controllable if $0 \in \operatorname{int} R^\Omega$ and locally approximately controllable if $0 \in \operatorname{int} \overline{R^\Omega}$

The following theorem gives the characterization of local controllability.

Theorem 2.7 [46]. *Suppose Ω is a convex set with nonempty interior and satisfies the condition (2.8). The system (2.5), (2.7) is locally controllable if and only if*

a) the associated system with unconstrained control is globally controllable, i.e. $R^U = X$, and

b) there exists no eigenvector of the dual operator A^* corresponding to a real eigenvalue and belonging to the negative polar cone of the set $B\Omega$, i.e.

$$\operatorname{Ker}(A^* - \lambda I^*) \cap (B\Omega)^0 = \{0\}, \quad \forall \lambda \in \mathbf{R}. \tag{2.9}$$

This result has been proved in [46] for the case of differentiable semigroup $E(t)$ and was extended to any strongly continuous semigroup in [76]. The proof is based on some properties of C_0-semigroups and the properties of the reachable set presented in 2.1, with the use, as a crucial tool, of the well known Krein-Rutman theorem, which we reformulate here for the purpose of reference in the sequel.

Theorem K - R [40]. *Let C be a convex cone with nonempty interior in a Banach space X and let $\{E(\gamma),\ \gamma \in \Gamma\} \subset L(X)$ be a family of commutative bounded linear operators mapping C into itself, i.e. $E(\gamma)C \subset C$ for all $\gamma \in \Gamma$. Then there exists a functional $f \in C^0 \subset X^*$ which is a common eigenvector of the dual operators $E(\gamma)^*$:*

$$E(\gamma)^* f = \lambda(\gamma) f,$$

where $\lambda(\gamma) \geq 0$, $\forall \gamma \in \Gamma$.

When the convexity of Ω is not assumed we can prove that a) b) is the sufficient condition for local approximate controllability of the system. We note that global (exact) controllability of infinite-dimensional linear systems with unconstrained controls is studied in Slemrod [79], Korobov and Rabah [39] and others. For example, in [39] it was shown that if $E(t)$ is a group then (2.5) is globally controllable iff

$$\exists m \in \mathbf{N}: \ \operatorname{span}\{BU, ABU, \ldots, A^{m-1}BU\} = X, \tag{2.10}$$

where A is the generator of $E(t)$. It is worth noticing that the above mentioned Krein-Rutman theorem has been extended for multi-valued operators and applied to obtain criteria of global controllability of discrete-time systems described by convex processes in a recent paper due to Phat and Dieu [65].

3. LINEAR DISCRETE-TIME SYSTEMS WITH CONSTRAINED CONTROLS

This section is devoted to present a systematic study of controllability of linear discrete-time systems in infinite-dimensional spaces.

3.1. Definition and the relationship between controllability concepts

Consider the control system (A, B, Ω) described by the difference equation

$$x_{k+1} = Ax_k + Bu_k, \quad x_k \in X, \; u_k \in \Omega \subset U, \tag{3.1}$$

where X and U are real Banach spaces of states and controls, respectively; $A \in L(X)$, $B \in L(U, X)$ and Ω is a nonempty *convex* subset, satisfying the following additional requirement

$$\exists u_0 \in \Omega : Bu_0 = 0. \tag{3.2}$$

Let us define, for each integer k, the following convex sets : $R_0 = \{0\}$,

$$R_k = \sum_{i=1}^{k} A^{k-i} B\Omega \quad \text{for} \quad k \geq 1 \quad \text{and} \quad S_k = \{x \in X : -A^k x \in R_k\} \tag{3.3}$$

and we set $R = \cup\{R_k : k \in \boldsymbol{N}\}$, $S = \cup\{S_k : k \in \boldsymbol{N}\}$. We observe that, by virtue of (3.2), the sets R_k and S_k are increasing, i.e.

$$R_k \subset R_{k+1}, \qquad S_k \subset S_{k+1}, \tag{3.4}$$

therefore the *reachable set* R and the *controllable set* S are also convex.

Definition 3.1. The system (3.1) is said to be

(i) locally controllable in time k, or $(LC)_k$, if $0 \in \text{int}\, R_k$;

(ii) locally null-controllable in time k, or $(LNC)_k$ if $0 \in \text{int}\, S_k$;

(iii) locally controllable, or (LC), if $0 \in \text{int}\, R$;

(iv) locally null-controllable, or (LNC), if $0 \in \text{int}\, S$.

The meaning of the above controllability properties is clear: for example, $(LC)_k$ means that each state in some neighbourhood of the origin can be reached from the origin by the system (3.1) in time k.

Definition 3.2. The system (3.1) is said to be

(i) locally approximately controllable in time k, or $(LAC)_k$, if $0 \in \text{int}\, \overline{R}_k$;

(ii) locally approximately null-controllable in time k, or $(LANC)_k$, if $0 \in \text{int}\, \overline{S}_k$;

(iii) locally approximately controllable, or (LAC), if $0 \in \text{int}\, \overline{R}$;

(iv) locally approximately null-controllable, or $(LANC)$, if $0 \in \operatorname{int} \overline{S}$.

For instance, (LAC) means that there exists a neighbourhood $U(0)$ of the origin such that for each $x \in U(0)$ and $\varepsilon > 0$ there exists a control sequence $u_i \in \Omega$, $i = \overline{0, k-1}$ such that

$$\left\| x - \sum_{i=1}^{k} A^{k-i} B u_{i-1} \right\| < \varepsilon.$$

In other words, this means that we can reach from the origin, in a finite time, arbitrarily close to every state in some neighbourhood of the origin by the system (3.1) with a suitable control sequence.

We can also consider the global controllability properties, corresponding to the local ones above. For instance, global controllability, or (GC), means that $R = X$ and global approximate controllability, or (GAC), means that $\overline{R} = X$. We observe that the above different notions of controllability may arise quite naturally in various control problems regarding the discrete-time system (A, B, Ω) and we pose a task to examine the relationship between them. The similar problem was considered in Dolecki [16] and Conti [12] for continuous-time systems without constraints on controls. Note that if the state space X is finite-dimensional then the notions of approximate controllability given in Definition 3.2 are abandoned and the relation between some other notions is easily seen. For example, in view of (3.4) and the convexity of the sets S_k and R_k, it is clear that (LC) implies $(LC)_k$ and (LNC) implies $(LNC)_k$ for some finite $k \in \boldsymbol{N}$. The situation is more delicate in the case of infinite-dimensional systems, as was shown by examples in [47]. The following theorem clarifies some relations, which are most specific for this case.

Theorem 3.4 [47]. *Suppose that the state space X is infinite-dimensional. For the linear discrete-time system (3.1) with a convex control set Ω satisfying (3.2) the following implications hold*

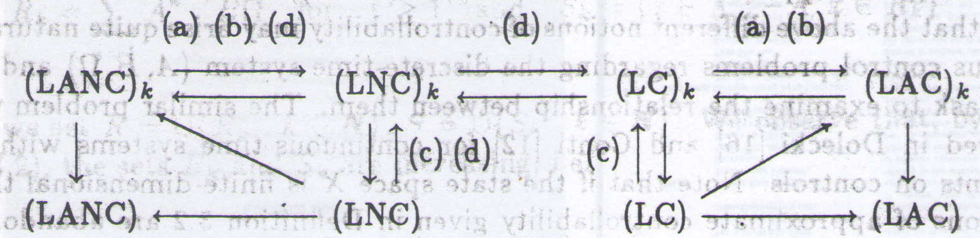
(i)	(LC)	$\Rightarrow$	$(\mathrm{LC})_k$, *if* $\operatorname{int} \Omega \neq \emptyset$;
(ii)	(LNC)	$\Rightarrow$	$(\mathrm{LNC})_k$, *if* $\operatorname{int} \Omega \neq \emptyset$ *and A is onto* : $AX = X$;
(iii)	(LC)	$\Rightarrow$	$(\mathrm{LAC})_k$;
(iv)	(LNC)	$\Rightarrow$	$(\mathrm{LANC})_k$;
(v)	$(\mathrm{LAC})_k$	$\Rightarrow$	$(\mathrm{LC})_k$, *if Ω is bounded and* $0 \in \operatorname{int} \Omega$;
(vi)	$(\mathrm{LANC})_k$	$\Rightarrow$	$(\mathrm{LNC})_k$, *if Ω is bounded*, $0 \in \operatorname{int} \Omega$ *and A is onto* : $AX = X$;
(vii)	$(\mathrm{LNC})_k$	$\Rightarrow$	$(\mathrm{LC})_k$, *if A is onto* : $AX = X$.

(For briefness, we omit the phrase "$\exists k \in \boldsymbol{N}$ such that" on the right hand side of the implications (i) - (iv))

The proof of the above theorem is based mainly on the Banach open mapping theorem and the Baire cathegory theorem, and on the use of the following lemma, which is of interest in itself.

Lemma 3.5. *Let M_j $(j = 0, 1, \dots)$ be a sequence of convex sets with nonempty interior in a Banach space X such that $0 \in M_j \subset M_{j+1}$ and $0 \in \operatorname{int} \cup \{M_j : j = 0, 1, \dots\}$. Then there exists an integer j_0 such that $0 \in \operatorname{int} M_{j_0}$.*

We gather some relations between various controllability notions in the following diagram, where implications are denoted by arrows $\rightarrow$; some arrows are drawn together with certain additional conditions, under which they hold. Namely, these conditions are briefly denoted as follows (a): Ω is bounded; (b): $0 \in \operatorname{int} \Omega$; (c): $\operatorname{int} \Omega \neq \emptyset$; (d): A is onto.



One may ask under which condition the implication $(\text{LAC}) \rightarrow (\text{LAC})_k$ holds (for some $k \in \boldsymbol{N}$). It will be shown later that for linear discrete-time systems associated with the retarded FDE (1.7) with no control constraint, the reachable set R_k increases with $k \leq n$ and remains constant thereafter. Consequently, for this class of systems, the above mentioned implication holds; in fact, we have (GAC) $\longrightarrow (\text{GAC})_n$. In general, this question is not solved yet. The same question can be posed also for the implication (LANC) $\longrightarrow (\text{LANC})_k$. Moreover, to investigate the relationship between the corresponding global controllability properties is still an open problem.

3.2. Criteria of local controllability

We note first that for the system with no control constraints (A, B, U), the notions of local controllability and global controllability coincide. In this case, the criteria of controllability and null-controllability have been established in Fuhrmann [27]. We reformulate them here for the sake of convenience.

Theorem F. *The linear discrete-time system (A, B, U) is (GC) iff*

$$\exists k \in \boldsymbol{N} : \ X = \operatorname{span}\{BU, ABU, A^2BU, \dots, A^{k-1}BU\} \tag{3.5}$$

and (GNC) *iff*

$$\exists k \in \boldsymbol{N}: \ \operatorname{Im} A^k \subset \operatorname{span}\{BU, ABU, \ldots, A^{k-1}BU\}. \tag{3.6}$$

For the property (GAC), a useful test is proved in our recent paper:

Theorem 3.6 [54]. *The linear discrete-time system* (A, B, U) *is* (GAC) *iff*

$$\exists k \in \boldsymbol{N}: \ \operatorname{Im} A^K \subset \overline{\operatorname{span}}\{BU, ABU, A^2BU, \ldots\} \tag{3.7}$$

and

$$\operatorname{Ker} A^* \cap \operatorname{Ker} B^* = \{0\}. \tag{3.8}$$

For the system with constrained controls, the following theorem gives the necessary and sufficient condition for (LC).

Theorem 3.7 [47]. *Suppose that the control set* Ω *is convex, has nonempty interior in* U *and satisfies the additional condition (3.2). Then for the linear discrete-time system* (A, B, Ω) *to be* (LC) *it is necessary and sufficient that*

a) the corresponding system with unconstrained controls (A, B, U) *is* (GC) *or, equivalently, (3.5) holds, and*

b) the adjoint operator A^* *has no eigenvectors, with non-negative eigenvalues, belonging to the negative polar cone of* $B\Omega$*, i.e.*

$$\operatorname{Ker}(A^* - \lambda I^*) \cap (B\Omega)^0 = \{0\}, \quad \forall \lambda \geq 0. \tag{3.7}$$

A sketchy explanation follows: the necessity of a) is obvious, the necessity of b) is easily derived from the contrary. Suppose a) and b) hold. From a) it follows that the reachable set R is a convex set with nonempty interior in X. Moreover, it is clearly invariant under A. Then, (LC) follows from b) via Theorem K-R.

For the finite-dimensional case, the condition a) is obviously reduced to the familiar Kalman rank condition

$$\operatorname{rank}[B, AB, \ldots, A^{n-1}B] = n. \tag{3.8}$$

Moreover, in this case, the assumption that Ω has nonempty interior can be removed and we obtain the following result.

Corollary 3.8. *Let Ω be a convex set of R^m satisfying (3.2). The linear discrete-time system*

$$x_{k+1} = Ax_k + Bu_k, \quad x_k \in R^n, \quad u_k \in \Omega \subset R^m \tag{3.9}$$

is (LC) if and only if

a) the transpose matrix A^* has no complex eigenvectors orthogonal to $B\Omega$, and

b) the transpose matrix A^* has no real eigenvectors, with non-negative eigenvalues, belonging to $(B\Omega)^0$.

Theorem 3.7 and Corollary 3.8 can be considered as the extentions of the results due to Brammer [8] and Korobov [37] to discrete-time systems. Note, however, that our methods are quite different from those used by these authors, which were based on the "oscillatory" character of the matrix A and the properties of almost periodic functions and, clearly, do not apply to infinite-dimensional systems.

Next we proceed to deal with the property of null-controllability. Since (LC) implies (LNC), the conditions for (LNC) are expected to be weaker than those for (LC). In fact, for finite-dimensional systems, we have the following result.

Theorem 3.9 [51]. *Let Ω be a convex set of R^m satisfying (3.2). The linear discrete-time system (3.9) is (LNC) if and only if*

a) the transpose matrix A^* has no complex eigenvectors, with nonzero eigenvalues, orthogonal to $B\Omega$, and

b) the transpose matrix A^* has no real eigenvectors, with positive eigenvalues, belonging to $(B\Omega)^0$.

Corollary 3.10. *Let Ω be a convex set of R^m satisfying (3.2) and having nonempty interior. The linear discrete-time system (3.9) is (LNC) if and only if*

a) the corresponding system with unconstrained controls is (GNC) or, equivalently,

$$\operatorname{rank}[B, AB, \dots, A^{n-1}B] = \operatorname{rank}[B, AB, \dots, A^{n-1}B, A^n], \tag{3.10}$$

b) the transpose matrix A^* has no real eigenvectors, with positive eigenvalues, belonging to $(B\Omega)^0$.

Remark that the condition (3.10) is simply a reformulation of (3.6) for the case where $X = R^n$ and $U = R^m$. However, a direct extension of the above result to infinite-dimensional systems (with (3.10) replaced by (3.6)) is not true, as shown by the Example 2.1 in [50]. In fact, the problem of (LNC) for infinite-dimensional systems is more difficult and we have sofar obtained only some particular results.

Theorem 3.11. *Let Ω be a convex set with nonempty interior in U and satisfying (3.2). Suppose, in addition, that there exists $m \in \mathbf{N}$ such that the quotient space $X/\mathrm{Ker}\, A^m$ is finite-dimensional. Then the system (A, B, Ω) is (LNC) if and only if (3.6) holds and*

$$\mathrm{Ker}\,(A^* - \lambda I^*) \cap (B\Omega)^0 = \{0\}, \quad \forall \lambda > 0. \tag{3.11}$$

The proof of the above result is based on the fact that (A, B, Ω) is (LNC) if and only if the *quotien system* $(\hat{A}, \hat{B}, \Omega)$ on the quotien space $X/\mathrm{Ker}\, A^m$ is (LNC) (see Lemma 2.2 in [50]) and on the use of the implication $(LNC) \overset{(d)}{\underset{(c)}{\longrightarrow}} (LC)$, as well as Theorem 3.7 stated above. Next, let denote by S'_k the controllable set in time k of the system $(A, B, \mathrm{int}\,\Omega)$, i.e.

$$S'_k = \Big\{ x \in X : -A^k x \in \sum_{i=1}^{k} A^{k-i} B(\mathrm{int}\,\Omega) \Big\}.$$

The following theorem gives a sufficient condition for (LNC).

Theorem 3.12 [50]. *Let Ω be as in Theorem 3.11. Then the system (A, B, Ω) is (LNC) if*

a) $\exists k \in \mathbf{N} : S'_k \neq \emptyset$ and (3.6) holds and

b) (3.11) holds.

The proof is based on the factorization theorem due to Douglas [18] and Lemma 3.5. We note that if the operator A is onto then the first condition of a) in the above theorem is trivially satisfied while the second one is replaced by the condition (3.5). Thus we obtain

Corollary 3.13 [52]. *Suppose that Ω is a convex set with nonempty interior in U, $0 \in \Omega$ and that the operator A is onto. Then, for the system (A, B, Ω) to be (LNC) it is necessary and sufficient that the conditions (3.5) and (3.11) are satisfied.*

It is worth noticing that this result can be also derived directly from Theorem 3.7 and the equivalence $(LC) \Longleftrightarrow (LNC)$ which is valid under the above assumptions (see the diagram drawn in the previous subsection).

3.3. Criteria of global controllability.

We consider now the property (GC) of the system with constrained controls (A, B, Ω). We note first that if Ω is a cone in U (with vertex at the origin) then reachable set R is also a cone in X. Therefore, in this case, the conceps (LC) and (GC) coincide and hence Theorem 3.7 and Corollary 3.8 give also the criteria for (GC). The same is true also for null-controllability. For example, the following result is immediate from Corollary 3.10.

Corollary 3.14. *The system in $\boldsymbol{R}^n$ with a single non-negative input $x_{k+1} = Ax_k + bu_k$, $u_k \geq 0$ is (GNC) iff* $\operatorname{rank}[b, Ab, \ldots, A^{n-1}b] = \operatorname{rank}[b, Ab, \ldots, A^{n-1}b, A^n]$ *and the matrix A has no real positive eigenvalues.*

Analogously, the main result of Evans and Murthy [20] on global controllability of the system with a single non-negative input in $\boldsymbol{R}^n$ is a simple consequence of our Theorem 3.7. The cases where Ω is not a cone, however, is more interesting from practical points of view (for example, Ω is a bounded set or a polyhedral set in $\boldsymbol{R}^n$ defined by a finite number of linear inequalities). To deal with these cases, we need recall the following notions from convex analysis (see, e.g. Rockafellar [70]).

Definition 3.15. Let Γ be a convex subset of $\boldsymbol{R}^n$ containing 0. The set

$$C_b(\Gamma) = \left\{ f \in \boldsymbol{R}^n : \sup_{x \in \Gamma} \langle f, x \rangle < \infty \right\}$$

is said to be the barrier cone of Γ and the set

$$C_r(\Gamma) = \cap \{\lambda\Gamma : \lambda > 0\}$$

is said to be the recession cone of Γ.

To derive the criteria for (GNC) we first observe that (GNC) implies (LNC) and therefore our task is reduced to find such a property $(*)$ that $(GNC) = (LNC) + (*)$. Indeed, for the finite-dimensional case, we have the following result (compare with Theorem 3.9).

Theorem 3.16 [51]. *Let Ω be a convex set of $\boldsymbol{R}^m$ satisfying (3.2). Suppose that*

a) the system (3.9) is (LNC), and

b) the transpose matrix A^* has neither complex eigenvectors with eigenvalues $\lambda \in \boldsymbol{C}$, $|\lambda| > 1$, orthogonal to $C_r(B\Omega)$ nor real eigenvectors with eigenvalues $\lambda \in \boldsymbol{R}$, $\lambda > 1$, belonging to $(C_r(B\Omega))^0$.

Then the linear discrete-time system (3.9) is (GNC). Conversely, if the system (3.9) is (GNC) and, in addition,

$$(C_r(B\Omega))^0 = C_b(B\Omega), \tag{3.12}$$

then the conditions a), b) hold.

We remark that if Ω is a cone then $C_b(B\Omega) = (B\Omega)^0$ and $C_r(B\Omega) = B\Omega$. Therefore, the condition b) in the above theorem is included, in view of Theorem 3.9, in the condition a). In fact, in this case, (LNC) is equivalent to (GNC), as we have already noted in the begining of this subsection. If Ω is bounded, then $C_r(B\Omega) = \{0\}$, $C_b(B\Omega) = R^n$ and hence (3.12) is satisfied. Therefore, as an immediate consequence of the above theorem we obtain the following criterion for (GNC) of the systems with *bounded controls.*

Corollary 3.17. *Let the control set Ω be convex, bounded and satisfy (3.2). The linear discrete-time system (3.9) is (GNC) iff it is (LNC) and $|\lambda| \leq 1$ for any $\lambda \in \sigma(A)$.*

Note that the condition (3.12) is not so restrictive as it seems. In [51] some examples of the control set Ω satisfying this condition are given. In particular, when Ω is a polyhedral set in R^m defined as the intersection of a finite number of half-spaces $\{u \in R^m : \langle f, u \rangle \leq \alpha_i\}$, $i = \overline{1,k}$, this condition is satisfied. Moreover, in this case, the above result leads to an easily checkable test. Theorem 3.16 has an equivalent formulation which does not involve the cones $C_b(B\Omega)$ and $C_r(B\Omega)$, but we omit it here. See also [49] for the continuous-time analogue to the above results. In [50] and [53], these results are extended to the case of infinite-dimensional systems (3.1)-(3.2) under the assumption that the operator A satisfies the following additional condition.

Hypothesis H. *There exists a positive number $r < 1$ such that the set $\sigma_1 \equiv \{\lambda \in C : \lambda \in \sigma(A),\ |\lambda| \geq r\}$ consists of a finite number of eigenvalues of A with finite multiplicities.*

Note that the above Hypothesis is clearly satisfied for any compact operator $A \in L(X)$. We formulate, for example, the criterion of (GNC) with bounded controls.

Theorem 3.18 [53]. *Let the control set Ω be bounded and satisfy (3.2) and let the operator A satisfy the Hypothesis H. The infinite-dimensional discrete-time system (3.1) is (GNC) iff it is (LNC) and $|\lambda| \leq 1$ for any $\lambda \in \sigma(A)$.*

We note that in the above theorem, the control set Ω is not necessarily assumed to be convex. Regarding the property (GC) we have the following negative result.

Theorem 3.19 [53]. *If the state space X is infinite-dimensional and the operator A satisfies the Hypothesis H, then the system (3.1) with bounded control set Ω is not (GC).*

For the finite-dimensional case, however, (GC) can occur and a test for this property is given by the following (compare with Corollary 3.17).

Corollary 3.20. *Let the control set Ω be convex, bounded and satisfy (3.2). The system (3.9) is (GC) iff it is (LC) and $|\lambda| \geq 1$ for any $\lambda \in \sigma(A)$.*

Finally, we shall say that the system (3.9) is *completely controllable* if it is (GC) and (GNC) simultaneously. Consequently, Theorem 3.9 and Corollaries 3.8, 3.17, 3.20 yield together the following

Corollary 3.21. *Let Ω be as in Corollary 3.20. The system (3.9) is completely controllable if and only if*

a) $(y \in C^n,\ A^*y = \lambda y,\ \langle y, Bu\rangle = 0\ \forall u \in \Omega) \Rightarrow y = 0$;

b) $(y \in R^n,\ A^*y = \lambda y,\ \lambda > 0,\ \langle y, Bu\rangle = 0\ \forall u \in \Omega) \Rightarrow y = 0$;

c) $\lambda \in \sigma(A) \Rightarrow |\lambda| = 1$.

3.4. Approximate controllability

We have been sofar concerned mainly with the property of exact controllability which is known as quite a restrictive concept for infinite-dimensional systems. It has been shown, for example, that linear evolutionary systems with compact semigroups in infinite-dimensional spaces are never exactly controllable in finite time [82]. On the other hand, it is well known that the semigroup $S(t)$ generated by FDE is compact for t large enough [73]. That is why, for the systems governed by FDE, approximate controllability is a more interesting problem and is more widely examined. Motivated by the purpose of application of discrete-time models to this class, we treated in [54], [55] the problem of approximate controllability of the infinite-dimensional system

$$x_{k+1} = Ax_k + Bu_k, \quad x_k \in X, \quad u_k \in \Omega \subset U, \quad k = 0, 1, \ldots \tag{3.13}$$

where $A \in L(X)$ is a compact operator, $B \in L(U, X)$ and the control set Ω is a cone (with the vertex at the origin). We recall that the system is said to be (GAC) if its reachable set R is dense in the state space $X : \overline{R} = X$.

From the spectral analysis of compact operators, it is well known that for every $r > 0$, the space X can be decomposed into the direct sum of two closed subspaces invariant w.r.t. A such that

$$X = P_r \oplus Q_r, \tag{3.14}$$

where P_r is finite-dimensional and the spectrum of the restriction of A to P_r (resp., to Q_r) coincides with the part of $\sigma(A)$ which lies outside (resp., inside) of the disk $\{\lambda \in \boldsymbol{C} : |\lambda| < r\}$. This implies, by Genfald-Beurling spectral radius formula, that $\|(A/r)^k x\| \to 0$ as $k \to \infty$ for any $x \in Q_r$. Let us define the set

$$X_\infty = \Big\{ x \in X : \lim_{k\to\infty} \|(A/r)^k x\| = 0, \quad \forall r > 0 \Big\}. \tag{3.15}$$

Then, from the above, we have $\cap\{Q_r : r > 0\} \subset X_\infty$. Moreover, clearly, $\operatorname{Ker} A^i \subset X_\infty, \forall i \in \boldsymbol{N}$.

Definition 3.22. The operator $A \in L(X)$ is said to satisfy the small solution condition (SSC) if there exists $j \in \boldsymbol{N}$ such that $X_\infty = \operatorname{Ker} A^j$.

The (SSC) means that each solution of the linear difference equation $x_{k+1} = Ax_k$ which tends to the origin more rapidly than any r^k as $k \to \infty$ ("small solution") must be vanished identically after certain finite time j. We need the following two auxiliary lemmas.

Lemma 3.23. *Let $A \in L(X)$ be a compact operator such that the adjoint operator A^* satisfies the (SSC). Then there exists $j \in \boldsymbol{N}$ such that*

$$\overline{\operatorname{Im} A^j} = \overline{\operatorname{span} \cup \{P_r : r > 0\}}.$$

The above result can be considered as a discrete-time analogue to the known result due to Henry [31, Cor.2] and Manitius [42, Lemma 4.3]. Now, let us denote by π_r the projection on P_r (along Q_r) associated with the space decomposition (3.14).

Lemma 3.24. *Suppose $A \in L(X)$ is a compact operator and $B\Omega$ is a convex cone in X. Let R be the reachable set of the discrete-time system (3.13). If, for some positive number r, $\pi_r \overline{R} = P_r$, then*

$$P_r \subset \overline{R}. \tag{3.16}$$

The proof of the above lemma is based on the generalized open mapping theorem due to Robinson [69]. We state now the main result.

Theorem 3.25 [54]. *Let $A \in L(X)$ be a compact operator such that A^* satisfies the (SSC). We assume that* $\operatorname{int}\Omega \neq \emptyset$ *and* $\overline{B\Omega}$ *is a convex cone. Then the infinite-dimensional system (3.13) is (GAC) if and only if*

$$\exists m \in \boldsymbol{N} : \operatorname{Im} A^m \subset \overline{\operatorname{span}}\{BU, ABU, A^2BU, \ldots\} \tag{3.17}$$

and

$$\operatorname{Ker}(A^* - \lambda I^*) \cap (B\Omega)^0 = \{0\}, \quad \forall \lambda \geq 0. \tag{3.18}$$

The main points of the proof follow: Let (3.17) and (3.18) are satisfied. Taking $r > 0$ and projecting the above relations on the finite-dimensional subspaces P_r we obtain that the projection of the system (3.13) on P_r satisfies the conditions a) and b) of Theorem 3.7 and is therefore (GC). This implies $\pi_r(\overline{R}) = P_r$. By Lemma 3.24 we have $P_r \subset \overline{R}$ and, since r is arbitrary, $\overline{\operatorname{span}} \cup \{P_r : r > 0\} \subset \overline{R}$. This shows, by virtue of Lemma 3.23 that $\exists j \in \boldsymbol{N} : \overline{\operatorname{Im} A^j} \subset \overline{R}$. From the last relation, (GAC) of the system is easily derived from the contrary.

4. APPLICATIONS

We are going to describe some applications of the results presented in the previous sections to the following problems: the optimal control problem for the abstract evolutionary system with state and control constraints; the exact controllability of nonautonomous linear systems with periodic coefficients; and the approximate controllability of the general retarded system, governed by a linear functional differential equation, with positive controls. As another application of our above abstract results, in [57] we considered the approximate controllability for a class of systems described by partial differential equations with positive controls. Those results, however, are not surveyed here to avoid lengthy presentation.

4.1. Optimal control

Let us consider the evolutionary system (2.1) with the control constraint (2.2) and the state constraint (2.4). Assume that all hypotheses stated in Section 2.1, regarding this system hold. Given $x_0 \in X$ and $T > 0$, let us denote, as in Section 2.1, by $L(x_o)$ the set of all admissible trajectories of the system on $[0, T]$, by $L_0(x_0)$ the set of all admissible interior trajectories and by $R(x_0)$ the reachable set of the system in time $T : R(x_0) = \{x(T) : x(.) \in L(x_0)\}$. We consider the following optimal control problem (P):

$$\varphi(x_0, x(.)) \to \inf$$
$$x(.) \in L(x_0),$$

where the cost functional $\varphi : X \times C(0,T,X) \to \mathbf{R}$ is assumed to be convex and continuous. Then, it is clear that a trajectory $x(.) \in L(x_0)$ is a solution of the above problem (P) if and only if it is also a solution of the following problem (P'):

$$\varphi(x_0, x(.)) \to \inf$$
$$x(.) \in \overline{L(x_0)}.$$

We recall that, by Theorem 2.3, the set $\overline{L(x_0)}$ is convex if the set of admissible interior trajectories $L_0(x_0)$ is nonempty. Let denote by $\partial\varphi(x(.))$ the subgradient of the convex function $x(.) \to \varphi(x_0, x(.))$. Then from the above observation and the known results of the optimization theory (see, e.g. [33]) we obtain the following.

Theorem 4.1 [48]. *Suppose that the set $L_0(x_0)$ is nonempty. Then for a trajectory $\hat{x}(.) \in L(x_0)$ to be a solution of the problem (P), it is necessary and sufficient that there exists a functional $f \in \partial\varphi(\hat{x}(.))$ such that*

$$\langle f, \hat{x}(.)\rangle = \min\{\langle f, x(.)\rangle : x(.) \in \overline{L(x_0)}\}.$$

There remains, of course, the problem of translation of the above optimality criterion into the concrete conditions expressed in terms of the original system (for example, in the form of the Pontriaghin maximum principle), but we do not treat this here, noting only that this kind of problems is widely explored within the framework of the classical theory of optimal control.

Corollary 4.2. *Let $h : X \to \mathbf{R}$ be a convex and continuous cost functional and $L_0(x_0)$ be nonempty. A trajectory $\hat{x}(.) \in L(x_0)$ is a solution of the terminal control problem*

$$h(x(T)) \to \inf$$
$$x(.) \in L(x_0)$$

if and only if there exists a functional $f \in \partial h(\hat{x}(T))$ such that

$$\langle f, \hat{x}(T)\rangle = \min\{\langle f, x\rangle : x \in \overline{R(x_0)}\}.$$

Let us denote by $J(x_0)$ the Bellman function of the problem (P), i.e.

$$J(x_0) = \inf\{\varphi(x_0, x(.)) : x(.) \in L(x_0)\}.$$

The property of continuity and differentiability of the Bellman function is known to play a remarkable role in some control problems (in particular, in the method of dynamic programming) and has been the object of research in a number papers, for instance, [7], [62]. For the evolutionary system (2.1) with state and control constraints we have the following

Theorem 4.3 [48]. *Let X, U be reflexive separable Banach spaces. The Bellman function of the optimal control problem (P) is a continuous function of the initial state x_0.*

The proof is based on the Theorem 2.4 and on some fundamental results about semicontinuity of convex functionals due to Aubin [2]. We remark that a similar problem was considered in [56] for optimization problem on the solutions set of a nonlinear differential inclusion, where the lower semicontinuity of the Bellman function $J(x_0)$ has been proved.

4.2. Controllability of linear periodic systems

Controllability of nonautonomous systems has been extensively considered in the literature for the case with unconstrained controls. The situation is less satisfactory for constrained controllability. There has been only some treatment done by Kerimov [36] and Schmitendorf [77], [83] for finite dimensional linear systems, where the controllability conditions are given in the form of maximization of certain linear functionals. In [52] we have adopted quite a different approach which is based on the discretization of the process and the use of controllability criteria for the associated linear discrete-time system. This method is proved to be particularly effective for nonautonomous systems with periodic coefficients: we are able not only to deal with infinite-dimensional systems but also to obtain explicit verifiable tests. Let consider the following linear system $(A(t), B(t), \Omega)$

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad x(t) \in X, \quad u(t) \in \Omega \subset U, \quad t \geq 0, \tag{4.1}$$

where X, U are Banach spaces, $A(t)$ and $B(t)$ are, for each $t \geq 0$, linear bounded operators and continuous in t, Ω is a nonempty subset containing 0. For each $T \geq 0$, the set of admissible controls Ω_T consists of functions $u(.) \in L_\infty(0, T, U)$ with values in Ω a.e. on $[0.T]$. For each admissible control $u(.)$ and each $x_0 \in X$, the unique solution of (4.1) initiated at x_0 is given by

$$x(t) = \Phi(t)x_0 + \Phi(t)\int_0^t \Phi^{-1}(s)B(s)u(s)ds,$$

where, for each t, $\Phi(t) \in L(X)$, is invertible and satisfies the equation $\dot{\Phi}(t) = A(t)\Phi(t)$, $\Phi(0) = I$. Clearly, the above system is a special case of the abstract evolutionary models (2.1). Let denote by S the null-controllable set of this system, i.e. the set of all initial states $x_0 \in X$ for which one can find $T \geq 0$ and a control $u(.) \in \tilde{\Omega}_T$, such that the corresponding solution of the system satisfies the condition $x(T) = 0$. The system (4.1) is called locally null-controllable, or (LNC), if $0 \in \operatorname{int} S$ and globally null-controllable, or (GNC) if $S = X$.

Assume now that $A(t)$ and $B(t)$ are periodic functions with period $\omega > 0$: $A(t+\omega) = A(t)$ and $B(t+\omega) = B(t)$, for all $t > 0$. Let define the linear bounded operators $A : X \to X$ and $B : L_\infty(0, \omega, U) \to X$ by setting

$$Ax = \Phi(\omega)x,$$

$$Bu(.) = \Phi(\omega)\int_0^\omega \Phi^{-1}(t)B(t)u(t)dt.$$

To the system (4.1) we associate the following discrete-time system

$$x_{k+1} = Ax_k + Bu_k, \quad x_k \in X, \quad u_k \in \tilde{\Omega}_\omega \subset L_\infty(0, \omega, U). \tag{4.2}$$

The following result is easily derived from the definition, by using the properties of the resolution operator $\Phi(t)$.

Lemma 4.4. *The nonautonomous system (4.1) with $A(t)$ and $B(t)$ being ω-periodic functions is (LNC) (respectively, (GNC)) if and only if the associated discrete-time system (4.2) is (LNC) (respectively, (GNC)).*

The above lemma makes it possible to use the criteria of controllability for discrete-time systems presented in Section 3. We note that in the case under consideration the operator A is invertible, so that the concepts of controllability and null-controllability of the system (4.2) are equivalent. Therefore, from Theorem 3.7 and Lemma 4.4 we can easily establish the following result.

Theorem 4.5 [52]. *Let Ω be a convex set with nonempty interior and $0 \in \Omega$. Let $A(t)$ and $B(t)$ be ω-periodic functions with $\omega > 0$. The system (4.1) is (LNC) iff*

a) the system with unconstrained controls $(A(t), B(t), U)$ is (GC);

b) the adjoint monodromy operator $\Phi^*(\omega)$ has no eigenvectors f with a positive eigenvalue such that $\langle f, \Phi^{-1}(t)B(t)u\rangle \leq 0$, $\forall u \in \Omega$, $\forall t \in [0, \omega]$.

Analogously, Theorem 3.18 together with Lemma 4.4 yields

Theorem 4.6 [52]. *Let Ω be convex bounded set containing 0. Let $A(t)$ and $B(t)$ be ω-periodic functions with $\omega > 0$ and with the monodromy operator $\Phi(\omega)$ satisfying Hypothesis H. Then the system (4.1) is (GNC) if and only if*

a) this system is (LNC) and

b) $|\lambda| \leq 1, \quad \forall\lambda \in \sigma(\Phi_A(\omega))$.

We note that, for nonautonomous linear systems with unconstrained controls $(A(t), B(t), U)$, criteria of controllability have been established by different authors and are well known in the literature. Thus, we can combine them with our above results to obtain complete characterizations of controllability for the class of periodic systems under consideration.

4.3. Approximate controllability of linear retarded systems with positive controls

We shall be concerned with the linear autonomous retarded system (L, B_0, Ω) described by the following functional differential equation

$$\dot{z}(t) = L(z_t) + B_0 u(t), \quad z(t) \in \mathbf{R}^n, \quad u(t) \in \Omega \subset \mathbf{R}^m, \tag{4.3}$$

where z_t denotes the function $\theta \to z_t(\theta) = z(t+\theta)$, $\theta \in [-h, 0]$.

We shall suppose the following:

1) L is a bounded linear functional from $C(-h, 0, \mathbf{R}^n)$ into $\mathbf{R}^n$ given by

$$L(\varphi) = \int_{-h}^{0} d\eta(\theta)\varphi(\theta),$$

where $\eta(.)$ is a $n \times n$ real matrix function of bounded variation such that $\eta(\theta) = 0$ for $\theta \geq 0$, $\eta(\theta) = \eta(-h)$ for $\theta \leq -h$ and η is left-sided continuous on $(-h, 0)$, the integral being understood in the Stieltjes sense.

2) Ω is a cone in $\mathbf{R}^m$ such that $\operatorname{int}\operatorname{co}\Omega \neq \emptyset$. For every $T > 0$, the set of admissible control $\tilde{\Omega}_T$ consists of functions $u(.)$ in the space $L_p(0, T, \mathbf{R}^m)$ with values in Ω a.e. on $[0, T]$.

3) B_0 is $n \times m$ real matrix.

It is a characterizing feature for retarded systems that, in the systems of this type, both trajectory value $z(t)$ and its behavior on the whole interval of retardation are of interest for applications. That is why function z_t is usually considered as a full state of the systems. The function space controllability of the systems of the form (4.3) has been extensively studied during the last decades (see, e.g. [4, 43, 44, 58, 60, 74]) but most of works were dealing with the case of unconstrained controls. One of the most successful approach to this problem is based on the use of strongly continuous semigroups theory in the state space $X = \mathbf{R}^n \times L_p(-h, 0, \mathbf{R}^n)$ for representing the solutions and studying their properties. We shall denote by (φ^0, φ^1) an element in the space X. It is well-known that the homogeneous equation $\dot{z}(t) = L(z_t)$ induces a strongly continuous semigroup $S(t)$, $t \geq 0$ of bounded linear operators in X. Let $z(t)$ be the solution of (4.3) corresponding to the initial condition

$$z(0) = \varphi^0, \quad z(\theta) = \varphi^1(\theta), \quad \theta \in [-h, 0), \quad \varphi = (\varphi^0, \varphi^1) \in X$$

and some admissible control $u(.) = \tilde{\Omega}_T$. Then $x(t) := (z(t), z_t)$ is the mild solution of the following abstract differential equation in X:

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = \varphi, \quad t \in [0, T],$$

where A is the infinitesimal generator of $S(t)$, B is a bounded linear operator from $\mathbf{R}^m$ to X defined as $Bu = (B_0u, 0)$.

For a given $T > 0$ the reachable set (in the state space X, from $\varphi = 0$) in time T of the system (L, B_0, Ω) is

$$R_T = \left\{ \int_0^T S(T-t)Bu(t)dt : u(.) \in \tilde{\Omega}_T \right\}$$

and the approximately null-controllable set in time T is $C_T = \{\varphi \in X : -S(T)\varphi \in \overline{R}_T\}$. We define $R = \cup\{R_T : T > 0\}$ and $C = \cup\{C_T : T > 0\}$. We say that the system (L, B_0, Ω) is globally approximately controllable, or (GAC), if $\overline{R} = X$ and globally approximately null-controllable, or $(GANC)$, if $\overline{C} = X$. Our task is to derive the necessary and sufficient conditions of (GAC) for the system (L, B_0, Ω). Note that (GAC) means that for every $\varepsilon > 0$, $\varphi^0 \in \mathbf{R}^n$ and $\varphi^1 \in L_p(-h, 0, \mathbf{R}^n)$, there exist $T > 0$ and an admissible control $u(.) \in \tilde{\Omega}_T$ such that the corresponding solution of (4.3) (with the null initial condition) satisfies $z(T) = \varphi^0$, $\left\| z_T - \varphi^1 \right\|_{L_p} < \varepsilon$. As in the previous subsection, let us define the operators

$$P : X \to X, \quad Px = S(h)x, \tag{4.4}$$

$$Q : U \to X, \quad Qu(.) = \int_0^h S(h-t)Bu(t)dt, \tag{4.5}$$

where $U = L_p(0,h,\mathbb{R}^m)$, and let consider the associated discrete-time system $(P,Q,\tilde{\Omega}_h)$:

$$x_{k+1} = Px_k + Qu_k; \quad x_k \in X, \quad u_k \in \tilde{\Omega}_h \subset U, \quad k = 0,1,\ldots \tag{4.6}$$

Using the transitive property of $S(t)$ it is not difficult to show that the system (L, B_0, Ω) is (GAC) if and only if the associated discrete-time system $(P,Q,\tilde{\Omega}_h)$ is (GAC). The same is true for the property $(GANC)$. It is a remarkable feature of the above discrete-time system that the operator P is compact and its adjoint P^* satisfies the (SSC), specified by the Definition 3.22. Therefore, Theorem 3.25 applies and we are led to the following result.

Theorem 4.7 [54], [55]. *Let Ω be a cone in $\mathbb{R}^m$ such that* $\operatorname{int} \operatorname{co} \Omega \neq \emptyset$. *The retarded system (L, B_0, Ω) is (GAC) if and only if*

a) the system with unconstrained controls $(L, B_0, \mathbb{R}^m)$ is $(GANC)$,

b) $\operatorname{Ker} P^* \cap (Q\tilde{\Omega}_h)^0 = \{0\}$,

c) $\operatorname{Ker}(P^* - \lambda I^*) \cap (Q\tilde{\Omega}_h)^0 = \{0\}, \forall \lambda > 0$,

where the operator P and Q are defined by the formula (4.4), (4.5).

Of course, the above criterion is not an easily verifiable test for controllability and we have to translate it into a more explicit statement. It is known that the translation from abstract results into verifiable results for concrete classes of systems is not always an easy problem: even for simple systems, this leads quickly to cumbersome calculations. Fortunately, for the case under consideration, this translation is successfully done in [55], relying on the properties of the *structural operators* F and G associated with the FDE $\dot{z}(t) = L(z_t)$. As resutls we obtained the characterizations of (GAC) expressed completely in terms of the original system data $\{L, B_0, \Omega\}$. We note that the structural operators were introduced first by Bernier and Manitius in [6] for linear differential equations with delays and were extensively studied in many subsequent publications [42, 14, 73]. They are proved to be an extremely useful tool in the qualitative analysis of FDE of both retarded and neutral types, as well as in the problems of controllability and observability for these classes of systems. To avoid lengthy notations we formulate here only some particular results, which have been obtained from the above theorem with the aid of the structural operators.

Assume η is of the form

$$\eta(\theta) = -A_0\chi_{(-\infty,0)}(\theta) - \sum_{i=1}^{N} A_i\chi_{(-\infty,-h_i)}(\theta),$$

where χ_I denotes the characteristic function of the interval $I \subset \boldsymbol{R}$, A_i are $n \times n$ real matrices and $0 = h_0 < h_1 < \cdots < h_N = h$. In this case, the system (L, B_0, Ω) takes the form

$$\dot{z}(t) = A_0 z(t) + \sum_{i=1}^{N} A_i z(t-h_i) + B_0 u(t), \; z(t) \in \boldsymbol{R}^n, \; u(t) \in \Omega \subset R^m. \quad (4.7)$$

The characteristic matrix of this system is defined as

$$\Delta(\lambda) = \sum_{i=1}^{N} A_i e^{-\lambda h_i} - \lambda I.$$

Theorem 4.8 [55]. *Consider the retarded system (4.7). Let Ω be a cone with int co $\Omega \neq \emptyset$. Then this system is (GAC) iff*

a) rank $[\Delta(\lambda), B_0] = n, \forall \lambda \in \boldsymbol{C}$;

b) $(y \in \boldsymbol{R}^n, A^* y = 0, \langle y, B_o u\rangle \leq 0, \forall u \in \Omega) \Rightarrow y = 0$;

c) $(y \in \boldsymbol{R}^n, \Delta^*(\lambda) y = 0, \lambda \in \boldsymbol{R}, \langle y, B_0 u\rangle \leq 0, \forall u \in \Omega) \Rightarrow y = 0$.

Corollary 4.9. *Consider the retarded system with a single non-negative input*

$$\dot{z}(t) = A_0 z(t) + \sum_{i=1}^{N} A_i z(t-h_i) + bu(t), \quad z(t) \in \boldsymbol{R}^n, \quad u(t) \geq 0.$$

This system is (GAC) iff

a) rank $[\Delta(\lambda), b] = n, \forall \lambda \in \boldsymbol{C}$;

b) det $A_N \neq 0$;

c) det $\Delta(\lambda) \neq 0, \forall \lambda \in \boldsymbol{R}$.

We note that the condition a) in the above two criteria are the necessary and sufficient conditions for approximate null-controllability of the corresponding retarded systems with unconstrained controls (see, e.g. [60]).

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