EXTENSION PROBLEM FOR GENERALIZED MULTI-MONOGENIC FUNCTIONS IN CLIFFORD ANALYSIS

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Abstract. *The main purpose of this paper is to extend some properties of multi-monogenic functions, which are a generalization of monogenic functions in higher dimensions, for a class of functions satisfying Vekua-type generalized Cauchy-Riemann equations in Clifford analysis. It is proved that the Hartogs theorem is valid for these functions.*

1. INTRODUCTION

In [1] the theory of functions taking values in a Clifford algebra was studied. It is proved that many important properties of holomorphic functions of one complex variable may be extended to monogenic functions which are solution of the generalized Cauchy-Riemann equations and play an important role in theoretical physics.

Following this way, in [6] Le Hung Son introduced a version of multi-monogenic functions which are a generalization of monogenic functions in higher dimensions and proved some properties of these functions; among them there are the Hartogs extension theorems.

The purpose of this paper is to extend these results for functions satisfying Vekua-type generalized Cauchy-Riemann equations, which are a generalization of multi-monogenic functions for Vekua-type in Clifford analysis. It is proved that the Hartogs extension theorem is valid for these functions. This is a generalization of some results in [4], [5], [6], [7].

2. PRELIMINARIES AND NOTATIONS

Let $\mathcal{A}$ be the universal Clifford algebra constructed over a real m-dimensional vector space V with orthonormal basic $e_1, e_2, \dots, e_m$ (see [1]). A basic for $\mathcal{A}$ is given by

$$\{e_A : A = (h_1, \dots, h_r) \in \mathcal{P}\{1, \dots, m\}, \quad 1 \le h_1 < \dots < h_r \le m\},$$

where $e_\emptyset = e_0 = 1$ is the identity element.

Multiplication in $\mathcal{A}$ is defined by the following rule for the basic elements

$$e_i e_j + e_j e_i = -2\delta_{ij} e_0.$$

For $n \le m$, R^{n+1} is naturally imbedded into $\mathcal{A}$. Hence $x = (x_0, x_1, \dots, x_n) \in R^{n+1}$ will be identified with

$$x = x_0 + \sum_{j=1}^{n} e_j x_j := x_0 + \vec{x}.$$

The involution of x is denoted by

$$\bar{x} := x_0 - \vec{x}.$$

The norm on $\mathcal{A}$ is defined by

$$\|a\|_0 = 2^{m/2}\Big(\sum_A a_A^2\Big)^{1/2}$$

for the element $a = \sum_A a_A e_A$. Then $\mathcal{A}$ turns out to be a Banach algebra of dimension 2^m.

In what follows, we shall denote an open subset of the Euclidean space $R^{n+1} \times R^{k+1}$ by $G = G_1 \times G_2$ where $1 \le n < m$, $1 \le k < m$, $n + k < m$. We shall consider functions defined on G and taking values in the Clifford algebra $\mathcal{A}$. These functions are given by the following form:

$$f : G \to \mathcal{A},$$

$$(x_0, \dots, x_n; y_0, \dots, y_k) \mapsto f(x, y) = \sum_A f_A(x, y) e_A,$$

where $f_A(x,y)$ are functions taking real values.

We introduced the following operators

$$D_x = \sum_{i=0}^{n} e_i \frac{\partial}{\partial x_i}, \qquad D_y = \sum_{j=0}^{k} e_j \frac{\partial}{\partial y_j},$$

$$J_{G_1} f(x,y) = -\int_{G_1} E_{n+1}(u,x) f(u,y)du \quad \text{for each fixed } y \in G_2,$$

$$J_{G_2} f(x,y) = -\int_{G_2} E_{k+1}(v,y) f(x,v)dv \quad \text{for each fixed } x \in G_1,$$

where $E_{n+1}(u,x) = \dfrac{1}{\omega_{n+1}} \dfrac{\overline{u}-\overline{x}}{|u-x|^{n+1}}$ and ω_{n+1} is the area of the unit sphere in R^{n+1}. Then the following relation is valid (see [3]):

$$D_x J_{G_1} f(x,y) = f(x,y) \quad \text{for} \quad f(x,y) \in L^1(G_1). \tag{1}$$

For the operators D_y and J_{G_2} the analogous relation also holds.

If $D_x f = 0$, we say f is (left-)monogenic with respect to x (see [1]). If for each fixed $y \in G_2$ f is monogenic with respect to x and for each fixed $x \in G_1$ f is monogenic with respect to y, then f is called multi-monogenic on G (shortly speaking $D_x f = D_y f = 0$) (see [6]). The set of multi-monogenic functions on G is denoted by $\mathcal{M}(G)$.

Following [3], we say that an $\mathcal{A}$-valued function w belongs to $L^{p,s}(R^s)$ if $|w|$ and $|w^{(s)}|$ belong to $L^p(\Delta_s)$ where $w^{(s)}(x) = |x|^{-s} w\left(\dfrac{1}{x}\right)$ and Δ_S is the unit ball in R^s

The space $L^{p,s}(R^s)$ is normed by

$$\|A\|_{p,s} = \|A, \Delta_s\|_p + \|A^{(s)}, \Delta_s\|_p.$$

If Ω is a boundary domain of R^n, we have

$$C^k(\overline{\Omega}) \subset L^q(\overline{\Omega}) \subset L^{p,n}(R^n) \quad \text{for every } k, q \in N.$$

In this paper we shall consider the system

$$\begin{cases} D_x w(x,y) - w(x,y)A(x) &= 0 \quad \text{for each fixed } y \in G_2, \\ D_y w(x,y) - w(x,y)B(y) &= 0 \quad \text{for each fixed } x \in G_1, \end{cases} \tag{2}$$

where

$$A \in L^{p,n+1}(\mathbf{R}^{n+1}), \quad B \in L^{q,k+1}(\mathbf{R}^{k+1}), \quad p > n+1, \quad q > k+1. \tag{3}$$

From the Fredholm theory the system (2) relates to the following integral equations

$$w(x,y) - J_{G_1}\big(w(x,y)A(x)\big) = 0, \tag{4}$$

$$w(x,y) - J_{G_2}\big(w(x,y)B(y)\big) = 0. \tag{5}$$

In the next section we need the following known results.

Lemma 1 ([3], Theorem 4.29). *Under the assumption (3), the operators $T_1 w := J_{G_1}(wA)$ and $T_2 w := J_{G_2}(wB)$ are compact in the spaces $C(\mathbf{R}^{n+1})$ and $C(\mathbf{R}^{k+1})$ resp., where $C(\mathbf{R}^{n+1})$ denotes the space of bounded continuous functions in $\mathbf{R}^{n+1}$.*

Furthermore,

$$|T_1 w| \le M |A|_{p,n+1} |w|_\infty, \tag{6}$$

$$|T_2 w| \le N |B|_{q,k+1} |w|_\infty, \tag{7}$$

where M and N are constants depending on p and q only, and

$$|w|_\infty = \sup_{R^{n+1}\times R^{k+1}} |w|.$$

Corollary 1. *Assume that*

$$|A| < M^{-1}, \quad |B| < N^{-1}, \tag{8}$$

where M and N are the constants in (6) and (7). Then the equations (4) and (5) possess only trival solutions.

In the sequel we only consider solutions of the system (2) in which the given functions A and B satisfy the conditions (3) and (8).

3. GENERALIZED MULTI-MONOGENIC FUNCTIONS

Definition. Under the assumptions (3) and (8), a function $w \in C^1(G)$ is called generalized multi-monogenic if it satisfies the system (2). The system (2) is called generalized Cauchy-Riemann equations.

The set of generalized multi-monogenic functions on G is denoted by $\mathcal{H}(G)$.

The Fredholm theory showed that if the equation (4) (resp. the equation (5)) possesses only the trival solution, the inhomogeneous equation

$$w(x,y) - T_1 w(x,y) = \varphi(x,y)$$

(resp. $w(x,y) - T_2 w(x,y) = \varphi(x,y)$) possesses an unique solution.

Hence we have the following

Lemma 2. *For a given monogenic function φ on G_1 there exists an unique function w which is a C^1-solution of the equation*

$$w(x) - J_{G_1}(w(x)A(x)) = \varphi(x), \tag{9}$$

where G_1 is a (bounded) simply connected domain with smooth boundary in R^{n+1}.

Conversely, we also have

Lemma 3. *Let w be a C^1-solution on G_1, continuous on $\overline{G}_1$ of the equation*

$$D_x w(x) - w(x)A(x) = 0.$$

Then the function

$$\varphi(x) = \frac{1}{\omega_{n+1}} \int_{\partial G_1} \frac{\bar{t} - \bar{x}}{|t-x|^{n+1}} d\sigma_t w(t) \tag{10}$$

is monogenic on G_1, continuous on $\overline{G}_1$, and satisfying the equation (9).

Proof. We observe that the function

$$Tw(x) = -\frac{1}{\omega_{n+1}} \int_{G_1} \frac{\bar{t} - \bar{x}}{|t-x|^{n+1}} w(t)A(t)dt$$

is monogenic in the exterior of $\overline{G}_1$ and equals to zero at ∞. The function $E_{n+1}(u,x)$ is also (left- and right-)monogenic with respect to u in the exterior of $\overline{G}_1$ and equals to zero at ∞ for every $x \in G_1$.

Thus, by Corollary 9.3 in [1], we have

$$\int_{\partial G_1} E_{n+1}(u,x)d\sigma_u Tw(u) = 0. \tag{11}$$

By (1), $w(x) - Tw(x)$ is monogenic on G_1. Hence we can apply Cauchy's integral formula ([1], Corollary 9.6) to obtain

$$\begin{aligned} w(x) - Tw(x) &= \int_{\partial G_1} E_{n+1}(t,x)d\sigma_t\big(w(t) - Tw(t)\big) \\ &= \int_{\partial G_1} E_{n+1}(t,x)d\sigma_t w(t) = \varphi(x). \end{aligned}$$

As w is continuous in $\overline{G}_1$ and Tw is continuous in $\boldsymbol{R}^{n+1}$, it follows that φ is continuous in $\overline{G}_1$. $\square$

In view of the preceding results we have

Theorem 1. *There is a '1-1' correspondence between the set* $\mathcal{M}(G) \cap C(\overline{G})$ *and the set* $\mathcal{H}(G) \cap C(\overline{G})$.

Proof. At first, let $\varphi \in \mathcal{M}(G) \cap C(\overline{G})$. For each fixed $y \in G_2$ by Lemma 2 there exists an unique function v which is a C^1-solution of the equation

$$v(x,y) - J_{G_1}\big(v(x,y)A(x)\big) = \varphi(x,y). \tag{12}$$

We observe that v is monogenic with respect to the variable y. Indeed, from (12) it follows that

$$\begin{aligned} &D_y v(x,y) - J_{G_1} D_y\big(v(x,y)A(x)\big) \\ &= D_y v(x,y) - J_{G_1}\big(D_y v(x,y)\big)A(x) = D_y\varphi(x,y) = 0, \end{aligned}$$

and hence, by Corollary 1, $D_y v(x,y) = 0$.

Again using Lemma 2, for each fixed $x \in G_1$ there exists an unique function w stisfying

$$w(x,y) - J_{G_2}\big(w(x,y)B(y)\big) = v(x,y). \tag{13}$$

We shall show that w is generalized multi-monogenic. Indeed, from (13),

$$D_x w(x,y) - J_{G_2}\big(D_x w(x,y)B(y)\big) = D_x v(x,y).$$

On the other hand, by (1) and (12),

$$\begin{aligned} D_x v(x,y) &= v(x,y)A(x) \\ &= \big[w(x,y) - J_{G_2}\big(w(x,y)B(y)\big)\big]A(x). \end{aligned}$$

Thus,

$$\big(D_x w(x,y) - w(x,y)A(x)\big) - J_{G_2}\big(D_x w(x,y) - w(x,y)A(x)\big)B(y) = 0$$

and therefore, by Corollary 1, one has

$$D_x w(x,y) - w(x,y)A(x) = 0.$$

Obviously $D_y w(x,y) - w(x,y)B(y) = 0$ (by (13) and (1)).

Notice that w is clearly continuous in $\overline{G}$ and thereby $w \in \mathcal{H}(G) \cap C(\overline{G})$.

Conversely, let $w \in \mathcal{H}(G) \cap C(\overline{G})$. Consider the function

$$T_2 w(x,v) = -\int_{G_2} E_{k+1}(t,v)w(x,t)B(t)\,dt.$$

By a method analogous to that used to prove Lemma 3 we have

$$\int_{\partial G_2} E_{k+1}(v,y)\,d\sigma_v T_2(x,v) = 0 \quad \text{for each } (x,y) \in G. \tag{14}$$

Applying Cauchy's integral formula ([1], Corollary 9.6) for the monogenic (with respect to x) function $w(x,y) - T_2 w(x,y)$ and using (14) we obtain

$$\begin{aligned} w(x,y) - T_2 w(x,y) &= \int_{\partial G_2} E_{k+1}(v,y)\,d\sigma_v \big(w(x,v) \\ &\quad - T_2 w(x,y)\big) \int_{\partial G_2} E_{k+1}(v,y)\,d\sigma_v w(x,v) := g(x,y). \end{aligned}$$

Clearly, g is continuous in $\overline{G}$ and $D_y g(x,y) = 0$.

In a similar way we also have

$$\int_{\partial G_1} E_{n+1}(u,x) d\sigma_u T_1 w(u,y) = 0 \quad \text{for each } (x,y) \in G$$

and

$$g(x,y) - T_1 g(x,y) = \int_{\partial G_1} E_{n+1}(u,x) d\sigma_u (g(u,y) - T_1 g(u,y))$$

$$= \int_{\partial G_1 \times \partial G_2} E_{n+1}(u,x) d\sigma_u E_{k+1}(v,y) d\sigma_v w(u,v) := f(x,y).$$

It is easy to check that f is multi-monogenic in G and continuous in $\overline{G}$. This completes the proof of the theorem. □

Theorem 2. (Uniqueness theorem). *If w is generalized multi-monogenic in G and $w = 0$ in a non-empty open subset $\sigma \subset G$, then $w = 0$ in G.*

Proof. We show first that, for each fixed $y \in G_2$, if w is a solution of the equation

$$D_x w(x,y) - w(x,y) A(x) = 0$$

and $w = 0$ in a non-empty open subset $\sigma_1 \subset G_1$, then $w = 0$ in G_1.

Indeed, putting

$$\varphi(x,y) = \int_{\partial G_1} E_{n+1}(u,x) d\sigma_u w(u,y)$$

and using Lemma 3 we have

$$\varphi(x,y) = w(x,y) - \int_{G_1} E_{n+1}(t,x) w(t,y) A(t) dt$$

$$= w(x,y) - \int_{G_1 \setminus \sigma_1} E_{n+1}(t,u) w(t,y) A(t) dt.$$

The second term of this representation is a monogenic function with respect to x in σ_1. Thus by Cauchy's theorem ([1], Corollary 9.3) we obtain

$$\int_{\partial\sigma_1} E_{n+1}(u,x)d\sigma_u \int_{G_1\setminus\sigma_1} E_{n+1}(t,u)w(t,y)A(t)dt = 0. \tag{15}$$

Applying the Cauchy's integral formula ([1], Corollary 9.6) for the monogenic function φ and using (15) we get

$$\varphi(x,y) = \int_{\partial\sigma_1} E_{n+1}(u,x)d\sigma_u\varphi(u,y) = \int_{\partial\sigma_1} E_{n+1}(u,x)d\sigma_u w(u,y) = 0$$

for each $x \in \sigma_1$.

Finally, from the identity theorem for the monogenic function (see [1], Theorem 11.3.9) we conclude that $\varphi(.,y) = 0$ in G_1 and, therefore, by Corollary 1, $w(.,y) = 0$ in G_1.

Now we may without restriction assume that

$$\sigma = \sigma_1 \times \sigma_2,$$

where σ_1 and σ_2 are polydisks in R^{n+1} and R^{k+1} resp.

For each fixed $y \in \sigma_2$, $w(.,y)$ is a solution of the equation

$$D_x w(x,y) - w(x,y)A(x) = 0,$$

and equals to zero in $\sigma_1 \times \{y\}$. This implies that $w = 0$ in $G_1 \times \{y\}$ and as y is chosen arbitrarily in σ_2, $w = 0$ in $G_1 \times \sigma_2$.

Next take arbitrary $x \in G_1$, then $w(x,.)$ is a solution of the equation

$$D_y w(x,y) - w(x,y)B(y) = 0.$$

In a similar way we also show $w = 0$ in $\{x\} \times G_2$ and as x is also chosen arbitrarily in G_1, this yields $w = 0$ in $G_1 \times G_2$. □

The following theorem is the main results of this paper.

Theorem 3. (Hartogs extension theorem). *Let $G = G_1 \times G_2$ where G_1 and G_2 are domains in R^{n+1} and R^{k+1} resp.. Let Σ be an open neighbourhood of*

∂G. Then for every generalized multi-monogenic function w in Σ, there exists an unique generalized multi-monogenic function W in $G \cup \Sigma$ such that $W = w$ in Σ.

Proof. Without loss of generality we may assume that the given generalized multi-monogenic function w is continuous in $\overline{\Sigma}$. Then from Theorem 1 there exists an unique function φ multi-monogenic in Σ and continuous in $\overline{\Sigma}$.

By [6, Theorem 4.1], there exists an unique multi-monogenic function Φ in $G \cup \Sigma$ such that $\Phi = \varphi$ in Σ.

Using Theorem 1 it follows that there exists an unique generalized multi-monogenic function W satisfying the system

$$\begin{cases} v(x,y) - J_{G_1}\big(v(x,y)A(x)\big) = \varphi(x,y) & \text{for each fixed } y \in G_2, \\ W(x,y) - J_{G_2}\big(W(x,y)B(y)\big) = v(x,y) & \text{for each fixed } x \in G_1, \end{cases} \tag{16}$$

where v is defined as in Theorem 1.

Both W and w satisfy the system (16) in Σ. From Corollary 1 and the Fredholm theory it follows $W = w$ in Σ.

Finally, by Theorem 2, W is the unique extension, and the proof is complete. □

Remark. We observe that the condition (8) may be replaced by a larger one. In fact, it suffices to require that the functions A and B are given such that the integral equations

$$v(x,y) - J_{G_1}\big(v(x,y)A(x)\big) = 0 \quad \text{for each fixed } y \in G_2,$$
$$w(x,y) - J_{G_2}\big(w(x,y)B(y)\big) = 0 \quad \text{for each fixed } x \in G_1,$$

possess only trival solutions. Then the above results remain valid.

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