Journal of Mathematics *Tạp chí Toán học* **21**(3 & 4)(1993), 63–84

BIFURCATION FROM THE ESSENTIAL SPECTRUM OF EQUATIONS DEPENDING ON A PARAMETER IN BANACH SPACES

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Abstract. *The purpose of this paper is to prove some new results on generalized implicit function theorems when the derivative has no bounded inverse and no eigenvalue, and apply these theorems to bifurcation problems concerning the essential spectrum of nonlinear equations depending on a parameter.*

1. INTRODUCTION

Let X and Y be a real Banach spaces, D be a neighbourhood of the origin in X. The closure of D will be denoted by $\overline{D}$. Let Λ be an open subset of a normed space and let R be the space of real numbers, with the usual absolute norm $|.|$. For the sake of simplicity of notation, we shall use the same symbol $\|.\|$ to denote the norms in X and in Y, respectively. The norm of the space containing Λ will be denoted by $|.|_\Lambda$. We consider the equations depending on a parameter of the form

$$F(\lambda, v) = 0, \quad (\lambda, v) \in \Lambda \times \overline{D}, \tag{1}$$

where $F : \Lambda \times \overline{D} \mapsto Y$ is, in general, a nonlinear mapping with $F(\lambda, 0) = 0$ for all $\lambda \in \Lambda$. Any point $(\lambda, 0) \in \Lambda \times \overline{D}$ is called a trivial solution of Equation (1). A point $(\overline{\lambda}, 0)$ is called a bifurcation point of Equation (1) if for any $\delta, \varepsilon > 0$ there exists a solution (λ, v) of (1) with $|\lambda - \overline{\lambda}| < \delta$ and $0 < \|v\| < \varepsilon$. In other words,

* The author was supported by a grant of the Alexander von Humboldt Foundation (Germany) at the Mathematical Institute of the University of Cologne. This work was supported in part by the Vietnam NCST Program "Applied Mathematics".

in every neighbourhood of $(\overline{\lambda},0)$ one can find a nontrivial solution of Equation (1). Usually assuming that F is a continuously Fréchet differentiable mapping, the implicit function theorems play several roles in the study of bifurcation points $(\overline{\lambda},0)$ when the null space, Ker $F_v(\overline{\lambda},0)$, of the partial Fréchet derivative $F_v(\overline{\lambda},0)$ of F with respect to $v \in D$, is nontrivial i.e., Ker $F_v(\overline{\lambda},0) \neq \{0\}$. Furthermore, under some sufficient assumptions on the mapping F, one can prove that the bifurcation of Equation (1) occurs only in the point $(\overline{\lambda},0) \in \Lambda \times \overline{D}$, where Ker $F_v(\overline{\lambda},0) \neq \{0\}$ (see [2], [3], [6], [9], [11], [14], etc.).

The purpose of this paper is to prove some new results on generalized implicit function theorems when the derivative has no bounded inverse and no eigenvalue, and to apply these theorems to bifurcation problems concerning the essential spectrum.

Section 2 is devoted to some definitions and preliminaries that will be used in the next sections.

In Section 3 we consider a local solvability of the equation

$$G(x) = 0, \; x \in X, \text{ given } G(a) = 0. \tag{2}$$

One usually assumes that the mapping G is continuously differentiable, with derivative $M = G'(a)$ surjective. There exist several versions of the implicit function theorems in Banach spaces, depending on the assumptions made on M (see Bartle [1], Craven and Nashed [5], Leach [8], Nashed [10], etc.). In [5] Craven and Nashed presented new generalized implicit function theorems in Banach spaces, only assuming that the mapping G is strongly Fréchet or strongly Hadamard differentiable at a, and the derivative M is approximately right (outer) invertible. In their results the mapping M is not supposed to have an exact bounded inverse, or bounded right inverse. However, they need the assumption that Ker $M \neq \{0\}$. In this section we improve these results, showing that the hypotheses on M can be weakened to assume that M satisfies the following condition (A): to any given $\epsilon > 0$ there exists $v \in X$ with $\|v\| = 1$ such that $\|Mv\| < \epsilon$.

In Section 4 we apply the results obtained in Section 3 to bifurcation problems. We consider the equations depending on a parameter of the form (1). Using the implicit function theorems and the Liapunov-Schmidt procedure, many authors obtained different results on the existence of bifurcation points of Equation (1). Given a point $(\overline{\lambda},0) \in \Lambda \times \overline{D}$, we need only to suppose that the mapping F is strongly Fréchet (or Hadamard) differentiable at $(\overline{\lambda},0)$ and, in general, the partial derivative $F_v(\overline{\lambda},0)$ is not bounded, and its range need not be closed. Moreover, $F_v(\overline{\lambda},0)$ is assumed to be approximately right (outer) invertible, and to satisfy condition (A). We emphasize that it can be here Ker $F_v(\overline{\lambda},0) = \{0\}$. The results in this section play an important role in the study of bifurcation problems concerning essential spectral points.

2. DEFINITIONS AND PRELIMINARIES

We begin this section by introducing the following definition.

Definition 1. Let X and Y be Banach spaces, and let $M : \Omega \mapsto Y$ be a linear mapping, where Ω is a dense set in X. We say that M satisfies Condition (A) if to any given $\epsilon > 0$ there exists a point $c \in \Omega$, $\|c\| = 1$, with $\|Mc\| < \epsilon$.

Remark 1.

1/ If $\operatorname{Ker} M \neq \{0\}$, then M satisfies Condition (A).

2/ Let $B : \Omega \mapsto X$ be a linear mapping. Let $\sigma(B) = \{\lambda \in R : B - \lambda\text{id}$ has no inverse$\}$, and $\sigma_o = \{\lambda \in R : Bv - \lambda v = 0$, for some $v \neq 0\}$. Then the set

$$\sigma_e(B) = \{\lambda \in \sigma(B) : R(B - \lambda\text{id}) \text{ is not closed}\}$$
$$\bigcup \{\lambda \in \sigma(B) : \lambda \text{ is a cluster point of } \sigma_o(B)\}$$
$$\bigcup \{\lambda \in \sigma(B) : \bigcup_{n \geq 1} \operatorname{Ker}(B - \lambda\text{id})^n \text{ is infinite dimensional}\},$$

is called an essential spectrum of B. Any point $\lambda \in \sigma_e(B)$ is said to be an essential spectral point of B. Suppose that the mapping B is self-adjoint mapping. Then the Weyl's Theorem shows that a point $\overline{\lambda}$ is an essential spectral point of the operator B if and only if there exists a sequence $f_n \in \Omega$, with $\|f_n\| = 1$, $f_n \rightharpoonup 0$ and $(B - \overline{\lambda}\text{id}) f_n \to 0$ (see, for example, in [12, pp.348]). It follows that if $\overline{\lambda}$ is an essential spectral of the self-adjoint mapping B, the mapping $(B - \overline{\lambda}\text{id})$ satisfies Condition (A).

Next, let X, Y and Ω be as in Definition 1. We recall the following definitions.

Definition 2. (see [5]) 1/ The mapping $M : \Omega \mapsto Y$ is called approximately right invertible if, for each $\mu \in (0, 1)$, there exists a norm $\|.\|_\mu$ on X, a bounded mapping $B_\mu : Y \mapsto X$, and a bound Γ, depending on μ, for which

$$(\forall y \in Y)\|MB_\mu y - y\| \leq \mu\|y\| \text{ and } \|B_\mu y\|_\mu \leq \Gamma\|y\|,$$

and

$$(\forall x \in X)\{\|x\|_\mu\} \nearrow \|x\| \text{ as } \mu \searrow 0.$$

Then each such B_μ is called an approximate right inverse of M, corresponding to the bound function $\Gamma(.)$. Denote by X_μ the completion of X in $\|.\|_\mu$.

2/ The mapping $M : \Omega \mapsto Y$ is called approximately outer invertible if, for each $\mu \in (0,1)$, there exists a bounded linear mapping $B_\mu : Y \mapsto X$, and a bound Γ, depending in μ, for which

$$(\forall y \in Y)\|(B_\mu M B_\mu - B_\mu)y\| \leq \mu\|B_\mu y\| \text{ and } \|B_\mu y\| \leq \Gamma\|y\|.$$

Then each B_μ is called an approximate outer inverse of M, with bound function $\Gamma(.)$.

3/ The mapping $M : \Omega \mapsto Y$ is called outer invertible if there exists a linear mapping $B : Y \mapsto X$ such that $BMB = B$. Then the mapping B is said to be an outer inverse of M.

Further, the notion of strongly Fréchet and Hadamard differentiability can be found in the paper of Craven and Nashed [5].

Remark 2. It is clear that if M is outer invertible with a bounded outer inverse, then it is approximately outer invertible. The Tikhonov's regularization (see, e.g. [10]) provides an approximate outer inverse, but not an outer inverse. The reader is refered to [5] for the further study of approximately right (outer) invertible mappings.

3. GENERALIZED IMPLICIT FUNCTION THEOREMS

In this section we formulate and prove some theorems on local solvability of Equation (2) for the case when the derivative $M = G'(a)$ does not necessarily have a bounded inverse and the null space $\operatorname{Ker} M$ can be a trivial space i.e., $\operatorname{Ker} M = \{0\}$. In what follows, let the mapping M satisfy Condition (A). For given $\varepsilon > 0$ we put

$$C_\varepsilon = \{c \in X \ : \ \|c\| = 1 \text{ and } \|Mc\| < \varepsilon/4\}.$$

The following theorem generalizes a result obtained by Craven and Nashed (Theorem 1 in [5]).

Theorem 1. *Let X and Y be Banach spaces and $a \in X$. Let the mapping $G : X \mapsto Y$ be strongly Fréchet differentiable at a, with $M = G'(a) : \Omega \mapsto Y$, where Ω is dense in X. Let $G(a) = 0$, and let the mapping M be approximately right invertible, with approximate right inverse B_μ and $\Gamma(\mu) = k_o\mu^{-\gamma}$ with $\gamma < 1$. In addition, assume that M satisfies Condition (A). Then there exists a positive real number ε_o such that for any ε, $0 < \varepsilon < \varepsilon_o$, one can find a neighbourhood I of* *zero in R and a mapping $\eta : I \times C_\varepsilon \mapsto X$ such that $x(t,c) = a + tc + \eta(t,c)$, $(t,c) \in I \times C_\varepsilon$, is a solution to the equation $G(x) = 0$, valid for all sufficiently small $t, x \neq a$. With approximate choice of $\mu = \mu(t)$, $\|\eta(t,c)\|_\mu = o(t)$ as $t \to 0$.*

Proof. The proof of this theorem proceeds similarly as the one of Theorem 1 in [5]. Only some changes need to be pointed out. Let $\tau < 1$, ε_0, γ be the same as in the proof of Theorem 1 in [5]. For any ε, $0 < \varepsilon < \varepsilon_0$, let $\psi = 2 + \Gamma + \mu$, where $\mu = \mu_{\min}(\varepsilon)$ as in [5]. We put $I = (-q(\varepsilon), q(\varepsilon))$, with $q(\varepsilon) = \frac{1}{2}\min\{1, \psi^{-1}\}\delta(\varepsilon)$ and $\delta(\varepsilon)$ obtained from the definition of the strongly Fréchet differentiability of G for a given $\varepsilon/2$. Further, for any $(t,c) \in I \times C_\varepsilon$, we define the iterative sequence $\{x_n(t,c)\}$ by $x_o(t,c) = a + tc$, and

$$x_n(t,c) = x_{n-1}(t,c) - B_\mu G(x_{n-1}(t,c)), \quad n = 1, 2, \ldots$$

In the sequel, for the sake of simplicity of notation, we write $x_n = x_n(t,c)$, $n = 1, 2, \ldots$. By the same arguments as in the proof of Theorem 1 in [5], we put

$$d_n = \|x_n - x_{n-1}\|_\mu, \quad \text{and } h_n = \|G(x_n)\|_\mu.$$

From the strong differentiability of G we have

$$\begin{aligned} d_1 = \|x_1 - x_0\|_\mu &= \| - B_\mu(tMc + \xi(a,tc))\|_\mu \\ &\leq \|B_\mu tMc\|_\mu + \|B_\mu \xi(a,tc)\|_\mu \\ &\leq \Gamma t\varepsilon/4 + \Gamma t\varepsilon/2 < \Gamma t\varepsilon \leq t\tau, \end{aligned}$$

$$\begin{aligned} h_1 = \|G(x_1)\| \leq \|G(x_0)\|_\mu + \|M(x_1 - x_0)\|_\mu + \|B_\mu\xi(a,tc)\|_\mu + \\ (\|\mathrm{id} - B_\mu)\xi(a,tc)\|_\mu \leq \varepsilon t + \Gamma\varepsilon t + (1+\mu)\varepsilon t = \psi\varepsilon t. \end{aligned}$$

By induction we obtain

$$d_n \leq \Gamma h_{n-1} \leq \Gamma\varepsilon\psi t\tau^{n-1} \leq \psi t\tau^n$$

and

$$h_n \leq \mu h_{n-1} + \varepsilon\Gamma h_{n-1} \leq \tau h_{n-1};$$

(see also the proof in [5]). Consequently, $\{\|x_n - x_{n-1}\|_\mu\} \to 0$ geometrically, and so $\{x_n\}$ converges to a limit $x(t,c) \in X_\mu$ with $\|x(t,c) - a\|_\mu \leq \delta(\varepsilon)$ and

$$\|G(x(t,c))\| = \lim_{n\to\infty} \|G(x_n(t,c))\| \leq \lim_{n\to\infty} \psi\varepsilon t\tau^n = 0.$$

Further, the proof proceeds exactly as the one of Theorem 1 in [5] by setting $\eta(t,c) = x(t,c) - a - tc$. Then, $x(t,c) \neq a$ for $t > 0$, $c \in C_\varepsilon$, and $\|\eta(t,c)\|_\mu = o(t)$ as $t \to 0$. This completes the proof of the theorem.

The following theorem is similar to Theorem 2 in [5]. It is also true for the case with a cone constraint $-G(x) \in S$, where S is a closed convex cone in Y. But, for the sake of simplicity, we only state the case $S = \{0\}$.

Theorem 2. *Let X and Y be real Banach spaces, with $a \in X$. Let the mapping $G : X \mapsto Y$ be restricted strongly Hadamard differentiable at a. Let $G(a) = 0$. Let the Hadamard derivative $M = G'(a) : X \mapsto Y$ be bounded approximately right invertible, with approximate right inverse B_μ and bound function $\Gamma(\mu) = k_o\mu^{-\gamma}$, with $\gamma < 1$. In addition, assume that M satisfies Condition (A). Then there exists a positive real number ε_o such that for any ε, $0 < \varepsilon < \varepsilon_o$, one can find a neighbourhood I of zero in R, and a mapping $\eta : I \times C_\varepsilon \mapsto X$ such that for any $(t,c) \in I \times C_\varepsilon$, $x(t,c) = a + tc + \eta(t,c) \in X_\mu$ is a solution to the equation $G(x) = 0$, valid for all sufficiently small $t < 0$, with $x \neq a$. With an approximate choice of $\mu = \mu(t) \to 0$ as $t \to 0$, $\|\eta(t,c)\|_\mu = o(t)$ as $t \to 0$.*

The following result is a generalized implicit function theorem for local solvability of Equation (2) concerning approximately outer invertible mappings.

Theorem 3. *Let X and Y be Banach spaces and $a \in X$. Let the mapping $G : X \mapsto Y$ be strongly Fréchet differentiable at a, with $G(a) = 0$. Let the Fréchet derivative $M = G'(a) : \Omega \mapsto Y$ be approximately outer invertible, with approximate outer inverse B_μ and bound function $\Gamma(\mu) = k_0\mu^{-\gamma}$, where Ω is dense in X and $\gamma < 1$. In addition, assume that M satisfies Condition (A). Then there exists a positive real number ε_o such that for any ε, $0 < \varepsilon < \varepsilon_o$, one can find a neighbourhood I of zero in R and a mapping $\eta : I \times C_\varepsilon \to X$, such that $x(t,c) = a + tc + \eta(t,c)$, $(t,c) \in I \times C_\varepsilon$ is a solution to the equation $B_\mu G(x) = 0$, for sufficiently small t, $x(t,c) \neq a$. With approximate choice of $\mu = \mu(t)$ as $t \to 0$ we have $\|\eta(t,c)\| = o(t)$ as $t \to 0$.*

Proof. The proof proceeds similarly as the ones of Theorem 1 above and Theorem 3 in [5], remarking that if the mapping $F_v(\bar{\lambda}, 0)$ is approximately outer invertible, with approximate outer inverse B_μ and bound function $\Gamma(\mu) = k_o\mu^{-\gamma}$, where $\gamma < 1$, then the mapping $M = -F_\lambda(\bar{\lambda}, 0)\bar{\lambda} + F_v(\bar{\lambda}, 0) : R \times X \mapsto Y$ is also approximately outer invertible with approximate outer inverse $\tilde{B}_\mu = (0, B_\mu)$, and the same bound function $\Gamma(\mu)$.

Theorem 4. *Let X and Y be Banach spaces and $a \in X$. Let the mapping G:*

$X \mapsto Y$ be strongly Fréchet differentiable at a, with $G(a) = 0$. Let the Fréchet derivative $M = G'(a) : \Omega \mapsto Y$ be outer invertible, with a bounded outer inverse B. In addition, assume that M satisfies Condition (A). Then there exists $\varepsilon_o > 0$ such that for any ε, $0 < \varepsilon < \varepsilon_o$, one can find a neighbourhood I of zero in R, and a mapping $\eta : I \times C_\varepsilon \mapsto X$ such that $x(t,c) = a + tc + \eta(t,c)$, $(t,c) \in I \times C_\varepsilon$, is a solution to the equation $BG(x) = 0$. Moreover, if $\tau = \|B\|\varepsilon < 1$, then

$$\|\eta(t_1,c_1) - \eta(t_2,c_2)\| \leq \frac{5\tau}{4(1-\tau)}\|t_1c_1 - t_2c_2\| \tag{3}$$

holds for all $t_1, t_2 \in I$, $c_1, c_2 \in C_\varepsilon$.

Proof. Let ε_o be as in the proof of Theorem 1. Given ε, $0 < \varepsilon < \varepsilon_o$, let τ, ψ, δ, I, and C_ε be as in the proof of Theorem 1. For any $(t,c) \in I \times C_\varepsilon$ we define the iterative sequence $\{x_n(t,c)\}$ by $x_0(t,c) = a + tc$;

$$x_n(t,c) = x_{n-1}(t,c) - BG(x_{n-1}(t,c)), \quad n = 1, 2, \ldots$$

By the same arguments as in the proofs of Theorem 1 above and Theorem 1 in [5], we conclude that $\{\|x_n(t,c) - x_{n-1}(t,c)\|\} \to 0$ geometrically, and so $\{x_n(t,c)\}$ converges to a limit $x(t,c) \in X$, with $\|x(t,c) - a\| \leq \delta(\varepsilon)$, and $BG(x(t,c)) = 0$.

Putting $\eta(t,c) = x(t,c) - tc$, $(t,c) \in I \times C_\varepsilon$ and using the same proof of Theorem 1 in [5], we obtain $\|\eta(t,c)\| = o(t)$ as $t \to 0$. Now, we prove (3). Indeed, let $t_1, t_2 \in I$ and $c_1, c_2 \in C_\varepsilon$. We have

$$\|\eta(t_1,c_1) - \eta(t_2,c_2)\| = \lim_{n\to\infty} \|x_n(t_1,c_1) - x_n(t_2,c_2) - t_1c_1 + t_2c_2\|. \tag{4}$$

Now, we can see

$$\begin{aligned} x_n(t_1,c_1) - x_n(t_2,c_2) &= x_{n-1}(t_1,c_1) - x_{n-1}(t_2,c_2) \\ &\quad - B(G(x_{n-1}(t_1,c_1)) - G(x_{n-1}(t_2,c_2))) \\ &= x_{n-1}(t_1,c_1) - x_{n-1}(t_2,c_2) - BM(x_{n-1}(t_1,c_1) - x_{n-1}(t_2,c_2)) \\ &\quad - B\xi(x_{n-1}(t_1,c_1), x_{n-1}(t_1,c_1) - x_{n-1}(t_2,c_2)) \\ &= x_{n-1}(t_1,c_1) - x_{n-1}(t_2,c_2) - BM(x_{n-2}(t_1,c_1) - x_{n-2}(t_2,c_2)) \\ &\quad - B(G(x_{n-2}(t_1,c_1)) - G(x_{n-2}(t_2,c_2))) \\ &\quad - B\xi(x_{n-1}(t_1,c_1), x_{n-1}(t_1,c_1) - x_{n-1}(t_2,c_2)) \\ &= x_{n-1}(t_1,c_1) - x_{n-1}(t_2,c_2) - B\xi(x_{n-1}(t_1,c_1), x_{n-1}(t_1,c_1) - x_{n-1}(t_2,c_2)) \\ &\quad + B\xi(x_{n-2}(t_1,c_1), x_{n-2}(t_1,c_1) - x_{n-2}(t_2,c_2)). \end{aligned}$$

Continuing this procedure, we obtain

$$\begin{aligned}
&x_n(t_1,c_1) - x_n(t_2,c_2)\\
&= x_0(t_1,c_1) - x_0(t_2,c_2) - BM(x_0(t_1,c_1) - x_0(t_2,c_2))\\
&\qquad - B\xi(x_{n-1}(t_1,c_1), x_{n-1}(t_1,c_1) - x_{n-1}(t_2,c_2))\\
&= t_1c_1 - t_2c_2 - BM(t_1c_1 - t_2c_2) - B\xi(x_{n-1}(t_1,c_1), x_{n-1}(t_1,c_1) - x_{n-1}(t_2,c_2)),
\end{aligned}$$

for all $n = 1, 2, \ldots$. Hence,

$$\begin{aligned}
&\|x_n(t_1,c_1) - x_n(t_2,c_2) - t_1c_1 + t_2c_2 + BM(t_1c_1 - t_2c_2)\|\\
&\le \tau\|x_{n-1}(t_1,c_1) - x_{n-1}(t_2,c_2)\|\\
&\le \tau(\|x_{n-1}(t_1,c_1) - x_{n-1}(t_2,c_2) - t_1c_1 + t_2c_2 + BM(t_1c_1 - t_2c_2)\|\\
&\qquad + \|-t_1c_1 + t_2c_2 + BM(t_1c_1 - t_2c_2)\|)\\
&\le \tau^2\|x_{n-2}(t_1,c_1) - x_{n-2}(t_2,c_2)\| + \tau\|-t_1c_1 + t_2c_2 + BM(t_1c_1 - t_2c_2)\|\\
&\ldots\\
&\le \tau^{n-1}\|x_1(t_1,c_1) - x_1(t_2,c_2)\| + \sum_{i=1}^{n-2}\tau^i\|-t_1c_1 + t_2c_2 + BM(t_1c_1 - t_2c_2)\|,
\end{aligned}$$

with $\tau = \|B\|\varepsilon_0 < 1$. Letting $n \to \infty$ and using (4), we conclude that

$$\|\eta(t_1,c_1) - \eta(t_2,c_2) + BM(t_1c_1 - t_2c_2)\| \le \frac{\tau}{1-\tau}\|t_1c_1 - t_2c_2 - BM(t_1c_1 - t_2c_2)\| \quad (5)$$

holds for all $t_1, t_2 \in I$, $c_1, c_2 \in C_\varepsilon$. Further, let X_ε be the space spanned by the set C_ε. Since $c_1, c_2 \in X_\varepsilon$, then $t_1c_1 - t_2c_2 \in X_\varepsilon$. Assume that $t_1c_1 - t_2c_2 = \alpha c$ for some $\alpha \in R$ and $c \in C_\varepsilon$. We have $\|t_1c_1 - t_2c_2\| = |\alpha|$ and

$$\|BM(t_1c_1 - t_2c_2)\| = \|BM\alpha c\| \le |\alpha|\,\|B\|\frac{\varepsilon_0}{4} \le \frac{\tau}{4}\|t_1c_1 - t_2c_2\|.$$

Therefore, it follows from (5) that

$$\|\eta(t_1,c_1) - \eta(t_2,c_2)\| \le \frac{5\tau}{4(1-\tau)}\|t_1c_1 - t_2c_2\|$$

holds for all $t_1, t_2 \in I$, $c_1, c_2 \in C_\varepsilon$. This completes the proof of the theorem. □

4. BIFURCATION THEOREMS

In this section we prove some new theorems on bifurcation points of Equation (1) concerning mappings which satisfy Condition (A). Using Remark 1, we also obtain new results on bifurcation from essential spectrum. In the sequel, we only prove these theorems for the case $\overline{\lambda} \neq 0$. They are certainly also valid for the case $\overline{\lambda} = 0$, provided that the normed space containing Λ is complete. The first result on bifurcation points of Equation (1) involving Fréchet differentiability and approximately right invertible mappings can be formulated as follows

Theorem 5. *Let $\overline{\lambda} \in \Lambda$ and the mapping F be strongly Fréchet differentiable at $(\overline{\lambda}, 0) \in \Lambda \times \overline{D}$. Let the mapping $F_v(\overline{\lambda}, 0)$ be approximately right invertible, with approximate right inverse B_μ and $\Gamma(\mu) = k_0\mu^{-\gamma}$ with $\gamma < 1$. In addition, assume that $F_v(\overline{\lambda}, 0)$ satisfies Condition (A). Then $(\overline{\lambda}, 0)$ is a bifurcation point of Equation (1). More precisely, to given δ, $\varepsilon > 0$ and for sufficiently small μ, there exists a neighbourhood I of zero in R, and two mappings $\alpha : I \times C_\varepsilon \mapsto R$, $\varphi : I \times C_\varepsilon \mapsto X_\mu$, $|\alpha(t,c)| + \|\varphi(t,c)\| = o(t)$ as $t \to 0$ such that $(\lambda(t,c), v(t,c))$, $(t,c) \in I \times C_\varepsilon$, with*

$$\lambda(t,c) = \frac{\overline{\lambda}}{1+\alpha(t,c)} \tag{6}$$

and

$$v(t,c) = tc + \varphi(t,c) \tag{7}$$

satisfies Equation (1) and $|\lambda(t,c) - \overline{\lambda}|_\Lambda < \delta$; $0 < \|v(t,c)\| < \varepsilon$, for $t \neq 0$. (Such a family $(\lambda(t,c), v(t,c))$, $(t,c) \in I \times C_\varepsilon$, is called a parameter family of nontrivial solutions in a neighbourhood of $(\overline{\lambda}, 0)$).

Proof. We set $J = \{\beta \in R : \frac{\overline{\lambda}}{1+\beta} \in \Lambda\}$, and define the mapping $G : J \times D \mapsto Y$ by

$$G(\alpha, v) = F\Big(\frac{\overline{\lambda}}{1+\alpha}, v\Big), \quad (\alpha, v) \in J \times \overline{D}, \tag{8}$$

It is easy to verify that the mapping G is strongly Fréchet differentiable at the point $(0,0) \in J \times D$ and $M = G'(0,0)$ is given by

$$M(\beta, u) = -F_\lambda(\overline{\lambda}, 0)\overline{\lambda}\beta + F_v(\overline{\lambda}, 0)u, \quad (\beta, u) \in R \times X.$$

Next, we define the norm $\|.\|^*$ on the product space $R \times X$ by $\|(\alpha, v)\|^* = |\alpha| + \|v\|$ and claim that the mapping M is approximately right invertible, and it satisfies

Condition (A). Indeed, since the mapping $F_v(\overline{\lambda},0)$ is approximately right invertible, there exist, for each $\mu \in (0,1)$, a norm $\|.\|_\mu$ on X, a bounded mapping $B_\mu : Y \mapsto X$, and a bound Γ, depending on μ, for which

$$(\forall y \in Y)\|F_v(\overline{\lambda},0)B_\mu y - y\| \leq \mu\|y\| \text{ and } \|B_\mu y\|_\mu \leq \Gamma\|y\|$$

and

$$(\forall x \in X)\{\|x\|_\mu\} \nearrow \|x\| \text{ as } \mu \searrow 0.$$

Therefore, for each $\mu \in (0,1)$, we put

$$\|(\alpha, v)\|^*_\mu = |\alpha| + \|v\|_\mu \text{ for } (\alpha, v) \in R \times X,$$

and define the mapping $\tilde{B}_\mu : Y \mapsto R \times X$, by

$$\tilde{B}_\mu y = (0, B_\mu y), \quad y \in Y.$$

It then follows that

$$(\forall y \in Y)\|M\tilde{B}_\mu y - y\| = \|F_v(\overline{\lambda},0)B_\mu y - y\| \leq \mu\|y\|$$

and

$$\|\tilde{B}_\mu y\|^*_\mu = \|B_\mu y\|_\mu \leq \Gamma\|y\|,$$

and

$$(\forall (\alpha, v) \in R \times X)\{\|(\alpha, v)\|^*_\mu\} \nearrow \|(\alpha, v)\|^* \text{ as } \mu \searrow 0.$$

This shows that the mapping M is approximately right invertible, with the approximate right inverse $\tilde{B}_\mu$ as above and with the same bound function as $F_v(\overline{\lambda},0)$. Now, let $\varepsilon > 0$ be given. Since $F_v(\overline{\lambda},0)$ satisfies Condition (A), for given $\varepsilon/4$, there exists $v \in X$, $\|v\| = 1$, with $\|F_v(\overline{\lambda},0)v\| < \varepsilon/4$. Taking $c = (0, v)$, we deduce $c \in C^*_\varepsilon = \{d \in R \times X \ : \ \| d \|^* = 1 \text{ and } \|Md\| < \varepsilon/4\}$. Thus, the mapping M also satisfies Condition (A).

Further, we apply Theorem 1 to the equation $G(x) = 0$, $x = (\alpha, v) \in \Lambda \times \overline{D}$, given $G(0,0) = 0$, to conclude that there exists $\varepsilon_o > 0$ such that for any ε, $0 < \varepsilon < \varepsilon_o$, one can find a neighbourhood I of zero in R and a mapping $\overline{\eta}$: $I \times C^*_\varepsilon \mapsto X$ such that $\overline{x}(t,c^*) = tc^* + \overline{\eta}(t,c^*)$, $(t,c^*) \in I \times C^*_\varepsilon$, is a solution to the equation $G(x) = 0$, valid for all sufficiently small t, $\overline{x}(t,c^*) \neq 0$. For any $(t,c) \in I \times C_\varepsilon$, we put $\eta(t,c) = \overline{\eta}(t,c^*) = (\alpha(t,c), \varphi(t,c))$, with $c^* = (0,c)$. Then, $x(t,c) = (\alpha(t,c), tc + \varphi(t,c))$ satisfies Equation (8). Consequently, $(\lambda(t,c), v(t,c))$, with $\lambda(t,c)$ and $v(t,c)$ being as in (6) and (7), respectively, satisfies Equation

(1), for all $(t, c) \in I \times C_\varepsilon$. To complete the proof, it remains to use the fact $|\alpha(t,c)| + \|\varphi(t,c)\| = o(t)$ as $t \to 0$, and $v(t,c) \neq 0$ for $t \neq 0$. □

Next, we state the second result on bifurcation points of Equation (1) concerning Hadamard differentiability and approximately right invertible mappings.

Theorem 6. *Let the mapping $F : \Lambda \times \overline{D} \mapsto Y$ be restricted strongly Hadamard differentiable at the point $(\overline{\lambda}, 0)$. Let the partial Hadamard derivative $F_v(\overline{\lambda}, 0)$ of F with respect to $v \in D$ be bounded linear, with approximate right inverse B_μ and bound function $\Gamma(\mu) = k_0\mu^{-\gamma}$, where $\gamma < 1$. In addition, assume that the mapping $F_v(\overline{\lambda}, 0)$ satisfies Condition (A). Then the conclusions of Theorem 4 continue to hold.*

Proof. The proof of this theorem proceeds exactly as the proof of Theorem 5, remarking that instead of applying Theorem 1 to Equation (8), we use Theorem 2. □

The first result on bifurcation points of Equation (1) involving Fréchet differentiability and approximately outer invertible mappings can be stated as

Theorem 7. *Let the mapping F be strongly Fréchet differentiable at the point $(\overline{\lambda}, 0)$. Let the Fréchet derivative $F_v(\overline{\lambda}, 0) : X \mapsto Y$ be approximately outer invertible, with approximate outer inverse B_μ and bound function $\Gamma(\mu) = k_0\mu^{-\gamma}$, where $\gamma < 1$. In addition, assume that $F_v(\overline{\lambda}, 0)$ satisfies Condition (A). Then for sufficiently small μ, $(\overline{\lambda}, 0)$ is a bifurcation point of the equation*

$$B_\mu F(\lambda, v) = 0, \quad (\lambda, v) \in \Lambda \times \overline{D}. \tag{9}$$

More precisely, to given δ, $\varepsilon > 0$, for sufficiently small μ, one can find a neighbourhood I of zero in R and two mappings $\alpha : I \times C_\varepsilon \mapsto R$, $\varphi : I \times C_\varepsilon \mapsto X$, $|\alpha(t,c)| + \|\varphi(t,c)\| = o(t)$ as $t \to 0$ such that $(\lambda(t,c), v(t,c))$, with $\lambda(t,c)$ and $v(t,c)$ being as in (6) and (7), respectively, satisfies Equation (9), and $|\lambda(t,c) - \overline{\lambda}|_\Lambda < \delta$, $0 < \|v(t,c)\| < \varepsilon$ for all $(t,c) \in I \times C_\varepsilon$, $t \neq 0$.

Proof. The proof proceeds similarly as the one of Theorem 5, remarking that if the mapping $F_v(\overline{\lambda}, 0)$ is approximately outer invertible with approximate outer inverse B_μ and bound function $\Gamma(\mu) = k_0\mu^{-\gamma}$, where $\gamma < 1$, then the mapping $M = -F_\lambda(\overline{\lambda}, 0)\overline{\lambda} + F_v(\overline{\lambda}, 0) : R \times X \mapsto Y$ is also approximately outer invertible with approximate outer inverse $\tilde{B}_\mu = (0, B_\mu)$, and the same bound function $\Gamma(\mu)$ and that instead of applying Theorem 1, we use Theorem 3. □

Remark 3. In Theorem 7, if for sufficiently small μ, $\operatorname{Ker} B_\mu = \{0\}$, then $(\overline{\lambda}, 0)$ is a bifurcation point of Equation (1).

Next, we apply Theorem 4 to prove some new results on bifurcation points of equations of the form

$$T(v) = L(\lambda, v) + H(\lambda, v) + K(\lambda, v), \quad (\lambda, v) \in \Lambda \times \overline{D}. \tag{10}$$

where Λ, $\overline{D}$ are as before, T and $L(\lambda, .)$, for any fixed $\lambda \in \Lambda$, are linear mappings (not necessarily bounded) from dense subsets of X into Y, H and K are nonlinear mappings from $\Lambda \times \overline{D}$ into Y, with $H(\lambda, 0) = K(\lambda, 0) = 0$ for all $\lambda \in \Lambda$. We now assume that $\overline{\lambda} \in \Lambda$ is such that the mapping $M = T - L(\overline{\lambda}, .)$ satisfies Condition (A), and is outer invertible, with bounded outer inverse B, and the mapping $E = H + K$ is strongly Fréchet differentiable at the point $(\overline{\lambda}, 0)$, with the Fréchet derivative $E'(\overline{\lambda}, 0) = 0$. Here, we emphasize that the linear mapping M, in general, is not continuous and whose range is not closed, the null space, $\operatorname{Ker} M$, can be trivial. Therefore, the usual implicit function theorems do not work to prove the existence of bifurcation at the point $(\overline{\lambda}, 0)$. However, we shall see that the application of Theorem 4 is effective to show, under additional hypotheses on the mappings T, L, H and K, that $(\overline{\lambda}, 0)$ is a bifurcation point of Equation (10). Moreover, we can also describe nontrivial solutions of (9) in a neighbourhood of $(\overline{\lambda}, 0)$ in an analytical form. In case $\operatorname{Ker} B = \{0\}$, we apply Theorem 7 to conclude that $(\overline{\lambda}, 0)$ is a bifurcation point of Equation (10). In the sequel, we only consider the case $\operatorname{Ker} B \neq \{0\}$. We make the following hypotheses:

Hypothesis 1. *There exists a real number b such that $\alpha L(\lambda, v) = L(\alpha^b \overline{\lambda}, v)$ holds for all $\alpha \in [0, 1]$ and $v \in \overline{D}$.*

Hypothesis 2. *There exists a close subspace Y_1 of Y such that $M(X) \subset Y_1$ and $Y = Y_0 \oplus Y_1$, where $Y_0 = \operatorname{Ker} B$ and the symbol $\oplus$ denotes the topological direct sum.*

The projectors of Y into Y_0 and Y_1 will be denoted by P_Y and Q_Y, respectively.

Hypothesis 3. *The mapping $P_Y H$ is compact and there exists a real number $a \geq 2$ such that*

i/ $P_Y H(\lambda, tv) = t^a P_Y H(\lambda, v)$ holds for all $t > 0$, $v \in \overline{D}$.

ii/ The mapping $P_Y K$ is Lipschitz continuous and $\alpha^{-a} P_Y K(\lambda, \alpha v) \to 0$ as $\alpha \to 0$ uniformly on (λ, v) from any bounded subset.

Let X_ε be the smallest closed subspace of X containing the set $C_\varepsilon = \{c \in X : \|c\| = 1 \text{ and } \|Mc\| < \varepsilon/4\}$, for a given $\varepsilon > 0$. We need:

Hypothesis 4. *$\|P_Y T\| < \infty$ and for a given $\varepsilon > 0$ there exists a closed subspace X'_ε of X_ε such that the mapping $\Phi = P_Y T / X'_\varepsilon$ has a continuous inverse Φ^{-1} from*

Y_0 into X'_ε.

The restricted norm of X to X'_ε, and X_ε will be also denoted by the same symbol $\|.\|$.

Hypothesis 5. *For $q = 1$ or $q = -1$ there exists a point $w^q \in X'_\varepsilon$, $w^q \neq 0$, and a bounded neighbourhood U_q of w^q in X'_ε with $0 \notin \overline{U}_q$ and $\varepsilon\|B\|(3(3+2s)\|BM\|+1) < 1$, where $s = \sup\{\|u\| \,:\, u \in U_q\}$, such that the Leray-Schauder topological degree, $\deg(\mathrm{id} - q\Phi^{-1}P_Y H(\overline{\lambda}, .), U_q, 0)$, of the mapping $\mathrm{id} - q\Phi^{-1}P_Y H(\overline{\lambda}, .)$ over U_q with respect to zero in X'_ε, is defined and different from zero.*

We now prove the following theorem on bifurcation points of Equation (10), which is an extension of the result obtained by the author in [14] for the case where the mapping $M = T - L(\overline{\lambda}, .)$ is not continuous and whose range is not closed. This also generalizes some well-known results obtained by McLeod and Sattinger [9], Buchner, Marsden and Schecter [2]. In what follows, for a given $\varepsilon > 0$, we put $C^*_\varepsilon = \{(\alpha, w) \in R \times X'_\varepsilon \,:\, |\alpha| + \|w\| = 1 \text{ and } \|Mw\| < \varepsilon/4\}$.

Theorem 8. *Under Hypotheses 1 - 5, $(\overline{\lambda}, 0)$ is a bifurcation point of Equation (10). More precisely, to any δ, $\varepsilon > 0$, there exists a neighbourhood I of zero in R, and three mappings $n : I_+ \times C^*_\varepsilon \mapsto R$, $\varphi : I_+ \times C^*_\varepsilon \mapsto X$, $u : I_+ \mapsto X'_\varepsilon$ with $|n(t, \alpha, w)| + \|\varphi(t, \alpha, w)\| = o(t)$ as $t \to 0$ for all $(\alpha, w) \in C^*_\varepsilon$, such that the pair $(\lambda(t), v(t))$ $t \in I_+$, with*

$$\lambda(t) = \frac{\overline{\lambda}}{\left(1 + q\left(\frac{t}{1 + \|u(t)\|} + n\left(t, \frac{1}{1 + \|u(t)\|}, \frac{u(t)}{1 + \|u(t)\|}\right)\right)^{a-1}\right)^b} \tag{11}$$

and

$$v(t) = \frac{tu(t)}{1 + \|u(t)\|} + \varphi\left(t, \frac{1}{1 + \|u(t)\|}, \frac{u(t)}{1 + \|u(t)\|}\right) \tag{12}$$

satisfies Equation (10); $|\lambda(t) - \overline{\lambda}|_\Lambda < \delta$ and $0 < \|v(t)\| < \varepsilon$ for all $t \in I_+$, $t > 0$.

Proof. For the sake of simplicity of notation we only prove the theorem for the case $q = 1$, (the proof of the case $q = -1$ proceeds similarly). We put $J = \left\{\alpha \in R \,:\, \alpha \geq 0 \text{ and } \frac{\overline{\lambda}}{(1 + \alpha^{a-1})^b} \in \Lambda\right\}$ and define the mapping $G : J \times \overline{D} \mapsto Y$ by

$$G(\alpha, v) = T(v) - L\left(\frac{\overline{\lambda}}{(1 + \alpha^{a-1})^b}, v\right) - E\left(\frac{\overline{\lambda}}{(1 + \alpha^{a-1})^b}, v\right), \quad (\alpha, v) \in J \times \overline{D}.$$

Then $G(0,0) = 0$ and the mapping G is strongly Fréchet differentiable at the point $(0,0)$. Furthermore, by a simple calculation, we can see

$$G'(0,0) = T - L(\overline{\lambda}, .),$$

which is also an outer invertible mapping, with the bounded outer inverse $\tilde{B} = (0, B)$, and $\|\tilde{B}\|^* = \|B\|$; (we recall that the norm $\|.\|^*$ is the norm of the product space $R \times X$ defined as in the proof of Theorem 5). Further, we apply Theorem 4 to the equation

$$G(x) = 0, \quad x = (\alpha, v) \in J \times \overline{D} \tag{13}$$

to conclude that there exists $\varepsilon_o > 0$, (without loss of generality we may assume that $\|\tilde{B}\|\varepsilon_o < 1$,) such that for any ε, $0 < \varepsilon < \varepsilon_o$, one can find a neighbourhood I of zero in R and a mapping $\eta : I_+ \times C^*_\varepsilon \mapsto X$, satisfying (3) such that $x(t,c) = tc + \eta(t,c)$, $(t,c) \in I_+ \times C^*_\varepsilon$, is a solution to Equation (11). Let $c = (\alpha, u)$, $\eta(t,c) = (n(t,\alpha,u), \varphi(t,\alpha,u))$. Then, $|n(t,\alpha,u)| + \|\varphi(t,\alpha,u)\| = o(t)$ as $t \to 0$ for all $(\alpha, u) \in R \times X'_\varepsilon$, $|\alpha| + \|u\| = 1$, and

$$x(t,c) = (m(t,\alpha,u), \xi(t,\alpha,u)),$$

where

$$m(t,\alpha,u) = t\alpha + n(t,\alpha,u)$$

and

$$\xi(t,\alpha,u) = tu + \varphi(t,\alpha,u),$$

satisfies

$$BG(m(t,\alpha,u), \xi(t,\alpha,u)) = 0, \quad \text{for all } t \in I_+,\ (\alpha,u) \in C^*_\varepsilon.$$

It then follows that

$$BT(\xi(t,\alpha,u)) = BL\Big(\frac{\overline{\lambda}}{(1+(m(t,\alpha,u))^{a-1})^b}, \xi(t,\alpha,u)\Big) + BE\Big(\frac{\overline{\lambda}}{(1+(m(t,\alpha,u))^{a-1})^b}, \xi(t,\alpha,u)\Big).$$

Hence,

$$Q_Y T(\xi(t,\alpha,u)) - L\Big(\frac{\overline{\lambda}}{(1+(m(t,\alpha,u))^{a-1})^b}, \xi(t,\alpha,u)\Big) - E\Big(\frac{\overline{\lambda}}{(1+(m(t,\alpha,u))^{a-1})^b}, \xi(t,\alpha,u)\Big) = 0 \tag{14}$$

for all $t \in I_+$, $(\alpha, u) \in C^*_\varepsilon$.

Let U_1 exist by Hypothesis 5, (for $q = 1$). One easily verify that if $u \in U_1$, then $\left(\frac{1}{1+\|u\|}, \frac{u}{1+\|u\|}\right) \in C^*_\varepsilon$. Given $t \in I_+$, $u \in U_1$, we set

$$h(t,u) = \begin{cases} \dfrac{1+\|u\|}{t} n\Big(t, \dfrac{1}{1+\|u\|}, \dfrac{u}{1+\|u\|}\Big), & \text{if } t > 0; \\ 0, & \text{if } t = 0, \end{cases} \tag{15}$$

$$g(t,u) = 1 + \Big(m\Big(t, \frac{1}{1+\|u\|}, \frac{u}{1+\|u\|}\Big)\Big)^{a-1}, \tag{16}$$

$$\gamma(t,u) = \begin{cases} \dfrac{1+\|u\|}{t} \varphi\Big(t, \dfrac{1}{1+\|u\|}, \dfrac{u}{1+\|u\|}\Big), & \text{if } t > 0; \\ 0, & \text{if } t = 0, \end{cases} \tag{17}$$

and

$$\sigma(t,u) = \xi\Big(t, \frac{1}{1+\|u\|}, \frac{u}{1+\|u\|}\Big), \tag{18}$$

where n, m, φ, and ξ are as above. Further, for arbitrary u^1, $u^2 \in U_1$, one can easily verify from (3) that

$$\begin{aligned} |h(t,u^1) - h(t,u^2)| &\le \Big| \frac{n\big(t, \frac{1}{1+\|u^1\|}, \frac{u^1}{1+\|u^1\|}\big)}{t} \Big| \|u^1 - u^2\| \\ &\quad + (1+\|u^2\|) \Big| \frac{n\big(t, \frac{1}{1+\|u^1\|}, \frac{u^1}{1+\|u^1\|}\big)}{t} - \frac{n\big(t, \frac{1}{1+\|u^2\|}, \frac{u^2}{1+\|u^2\|}\big)}{t} \Big| \\ &\le \frac{\tau}{1-\tau} \big(1 + 2(1+s)\big)\kappa \|u^1 - u^2\|, \end{aligned} \tag{19}$$

where $\tau = \|B\|\varepsilon$ and $s = \sup\{\|u\|, u \in U_1\}$. Analogously,

$$\|\gamma(t,u^1) - \gamma(t,u^2)\| \le \frac{\tau}{1-\tau} \big(1 + 2(1+s)\big)\kappa \|u^1 - u^2\| \tag{20}$$

$$|g(t,u^1) - g(t,u^2)| \le \kappa_1 t \big(1 + 2(1+s)\big) \frac{\tau}{1-\tau} \|u^1 - u^2\|, \tag{21}$$

for some $\kappa_1 > 0$, independent on $u^1, u^2 \in U_1$.

Further, we define the mappings $N_i : I_+ \times U_1 \mapsto X'_\varepsilon$, $i = 1, \dots, 5$ by

$$N_1(t,u) = \Phi^{-1}\Big(g(t,u) P_Y H\Big(\frac{\bar{\lambda}}{(g(t,u))^b}, u + \gamma(t,u)\Big)\Big),$$

$$N_2(t,u) = \big(1 - (1 + h(t,u))^{a-1}\big)u,$$
$$N_3(t,u) = -\Phi^{-1}\big((1 + h(t,u))^{a-1} P_Y T(\gamma(t,u))\big)$$

and

$$N_4(t,u) = \Phi^{-1}\Big(g(t,u)\Big(\frac{t}{1+\|u\|}\Big)^{-a} P_Y K\Big(\frac{\overline{\lambda}}{(g(t,u))^b}, \sigma(t,u)\Big)\Big), \quad (t,u) \in I_+ \times U_1,$$

where h, g, γ, σ are from (15) - (18), respectively. It then follows from (19) - (21) and Hypothesis 3 that, for ε_o sufficiently small, there exist constants κ_2, κ_3, and $\kappa_4 > 0$, with $\kappa_2 + \kappa_3 + \kappa_4 \leq 1$, such that

$$\|N_i(t,u^1) - N_i(t,u^2)\| \leq \kappa_i \|u^1 - u^2\|, \quad i = 2, \ldots, 4$$

hold for all $t \in I_+$, $u^1, u^2 \in U_1$. Since, for sufficiently small t, $g(t,.)$, $\gamma(t,.)$ are bounded mappings, $P_Y H(\overline{\lambda},.)$ is a compact mapping from X into Y_0, and Φ^{-1} is a continuous mapping from Y_0 into X'_ε, we conclude that, for any fixed $t \in I_+$, $N_1(t,.)$ is a compact mapping from U_1 into X'_ε.

Next we define the mapping $\Gamma : I_+ \times U_q \mapsto X'_\varepsilon$ by

$$\Gamma(t,u) = \sum_{i=1}^{4} N_i(t,u), \quad (t,u) \in I_+ \times U_q. \tag{22}$$

Then, for any $t \in I_+$, $\Gamma(t,.)$ is a $(\beta)k$-contraction mapping from U_1 into X'_ε (see, for example, in [13]). We now assume that for all $t \in I_+$, $u \in \partial U_1$, $\Gamma(t,u) \neq u$. Thus, the topological degree, $\deg_\beta(\mathrm{id} - \Gamma(t,.), U_1, 0)$, of the mapping id $-\Gamma(t,.)$ over U_1 with respect to the origin in X'_0 is defined in the sense given in [13]. For any fixed $t \in I_+$, we define the mapping $\Omega : [0,1] \times \overline{U}_1 \mapsto X'_\varepsilon$ by

$$\Omega(\beta,u) = \Gamma(\beta t, u), \quad (\beta,u) \in [0,1] \times \overline{U}_1.$$

One can easily verify that id $-\Omega$ is a homotopy between id $-\Gamma(t,.)$ and id $-\Gamma(0,.)$. Therefore,

$$\begin{aligned}\deg_\beta(\mathrm{id} - \Omega(1,.), U_1, 0) &= \deg_\beta(\mathrm{id} - \Gamma(t,.), U_1, 0) = \deg_\beta(\mathrm{id} - \Omega(0,.), U_1, 0) \\ &= \deg\,(\mathrm{id} - \Phi^{-1}(P_Y H(\overline{\lambda},.), U_1, 0)) \neq 0.\end{aligned}$$

It follows that for any $t \in I_+$, $t > 0$, there exists $u(t) \in \overline{U}_1$, $u(t) \neq 0$, (because $0 \notin \overline{U}_1$) such that $\Gamma(t, u(t)) = u(t)$. Consequently,

$$u(t) = \sum_{i=1}^{4} N_i(t, u(t)) \quad t \in I_+,\ t \neq 0,$$

and so

$$P_Y\Big\{(1+h(t,u(t)))^{a-1}T(u(t)+\gamma(t,u(t)))$$

$$-\,g(t,u(t))\Big(\frac{t}{1+\|u(t)\|}\Big)^{-a}E\Big(\frac{\overline{\lambda}}{(g(t,u(t)))^b},\sigma(t,u(t))\Big)\Big\}=0. \tag{23}$$

Multiplying both the sides of (23) with $\Big(\frac{t}{1+\|u(t)\|}\Big)^a$, we obtain

$$P_Y\Big\{\Big(m\Big(t,\frac{1}{1+\|u(t)\|},\frac{u(t)}{1+\|u(t)\|}\Big)\Big)^{a-1}T(v(t))-g(t,u(t))E(\lambda(t),v(t))\Big\}=0 \tag{24}$$

with $\lambda(t)$, $v(t)$ being as in (11) and (12). Since $(T-L(\overline{\lambda},.))(X)\subset Y_1$, it then implies $P_Y(T(v(t))-L(\overline{\lambda},v(t)))=0$. Together with (24) we deduce

$$P_Y\Big\{g(t,u(t))T(v(t))-L(\overline{\lambda},v(t))-g(t,u(t))E(\lambda(t),v(t))\Big\}=0,$$

or

$$P_Y\Big\{T(v(t))-L(\lambda(t),v(t))-E(\lambda(t),v(t))\Big\}=0. \tag{25}$$

On the other hand, since $\Big(\frac{1}{1+\|u(t)\|},\frac{u(t)}{1+\|u(t)\|}\Big)\in C^*_\varepsilon$ for all $t\in I_+$, it follows from (14) that

$$Q_Y\Big\{T(v(t))-L(\lambda(t),v(t))-E(\lambda(t),v(t))\Big\}=0, \tag{26}$$

noticing that

$$\lambda(t)=\frac{\overline{\lambda}}{\Big(1+\Big(m\Big(t,\frac{1}{1+\|u(t)\|},\frac{u(t)}{1+\|u(t)\|}\Big)\Big)^{a-1}\Big)^b}$$

and

$$v(t)=\xi\Big(t,\frac{1}{1+\|u(t)\|},\frac{u(t)}{1+\|u(t)\|}\Big).$$

A combination of (25) and (26) yields

$$T(v(t))=L(\lambda(t),v(t))+E(\lambda(t),v(t)),\quad t\in I_+. \tag{27}$$

To get $|\lambda(t)-\overline{\lambda}|<\delta$ and $0<\|v(t)\|<\varepsilon$, $t>0$, for any given $\delta,\varepsilon>0$, it remains to choose ε'_o smaller, if necessary, observing that $\lambda(t)\to\overline{\lambda}$ and $v(t)\to 0$ as $t\to 0$, $v(t)\neq 0$ for $t\neq 0$.

In the case there exist $t \in I_+$, $t > 0$ and $u(t) \in \partial U_0$ with $\Gamma(t, u(t)) = u(t)$, we use the same proof as above to obtain (25) and together with (14) to get (27). This completes the proof of the theorem. □

Remark 4. In the case $\dim Y_0 < +\infty$ and $\dim X_\varepsilon \geq \dim Y_0$, Hypothesis 4 can be dropped and Hypothesis 5 is replaced by:

Hypothesis 5'. *For $q = 1$ or $q = -1$ there exists a closed subspace X'_ε of X_ε with $\dim X'_\varepsilon = \dim Y_0$, a point $w^q \in X'_\varepsilon$ and a bounded neighbourhood U_q of w^q with $0 \notin U_q$ such that the Leray - Schauder topological degree $deg\,(P_Y(T - H(\overline{\lambda}, .))/X'_\varepsilon, U_q, 0)$ of the mapping $P_Y(T - H(\overline{\lambda}, .))/X'_\varepsilon$ over U_q with respect to zero in X'_ε, is defined and different from zero.*

Then the conclusions of Theorem 8 continue to hold.

Remark 5. In the case there exists a closed subset C of X'_ε, not containing the origin, such that the mapping $\Phi^{-1}P_Y H(\overline{\lambda}, .)$ is k-contraction from C into itself, with $0 < k \leq 1$, the Hypothesis 5 can be dropped. Indeed, if $\Phi^{-1}P_Y H(\overline{\lambda}, .)$ is a k-contraction mapping from C into itself, with $0 < k < 1$, then by the Banach contraction principle it possesses a fixed point, say u_0, in C. Taking a number $d > 0$ small enough, we set $C_0 = \{u \in C \ : \ \|u - u_0\| \leq d\}$. Then, C_0 is also closed subset in X'_ε. By a simple proof we can verify that, for sufficiently small t, the mapping $\Gamma(t, .)$ defined as in (22) is also a k'-contraction mapping, with some k', $0 < k' < 1$, and it maps C_0 into itself. Applying the Banach contraction principle again, we conclude that, for sufficiently small t, there exists a fixed point $u(t)$ in C_0 of the mapping $\Gamma(t, .)$, i.e. $\Gamma(t, u(t)) = u(t)$. Further, by the same arguments as in the proof of Theorem 8, it follows that $(\lambda(t), v(t))$, with $\lambda(t)$, $v(t)$ being as in (11) and (12), is a solution of Equation (10).

Next, we assume that $\overline{\lambda} \in \Lambda$ is such that the mapping $T - L(\overline{\lambda}, .)$ satisfies Condition (A), with a bounded outer inverse B such that $Y_0 = \operatorname{Ker} B$ is a one-dimensional space. Further, let $Y_0 = [\varphi^1]$. By the Hahn-Banach theorem, there exists a continuous functional $\psi^1 \in Y^*$ such that $\langle \varphi^1, \psi^1 \rangle = 1$ and $Y = Y_0 \oplus Y_1$, with $Y_1 = \{y \in Y \ : \ \langle y, \psi^1 \rangle = 0\}$ and $\langle , \rangle$ denoting the pairing between elements of Y and Y^*. It is clear that $P_Y(y) = 0$ if and only if $\langle y, \psi^1 \rangle = 0$.

In this special case, we obtain the following corollary, which is an extension of a result obtained by the author (see, [14, Corollary 10]).

Corollary 9. *Let T, L, H and K satisfy Hypotheses 2, 3 and $(T - L(\overline{\lambda}, .))(X) \subset Y_1$. Let Y_0 and ψ^1 be as above. In addition, for a given $\varepsilon > 0$ assume that there exists an element $v^1 \in X_\varepsilon$ such that $\gamma = \langle T(v^1), \psi^1 \rangle . \langle H(\overline{\lambda}, v^1), \psi^1 \rangle \neq 0$, where, as before, X_ε is the smallest closed subspace of X containing the set $C_\varepsilon = \{c \in X$:*

$\|c\| = 1$ and $\|Tc - L(\overline{\lambda}, c)\| < \varepsilon/4\}$. Then the conclusions of Theorem 8 continue to hold for $q = \operatorname{sign}\gamma$.

Proof. To prove this corollary, it suffices to show that Hypotheses 4 and 5 for $q = \operatorname{sign}\gamma$ are satisfied.

Since $\langle T(v^1), \psi^1\rangle \neq 0$, it follows that $P_Y T(u) \neq 0$ for all $u \in X'_\varepsilon = [v^1]$, $u \neq 0$. Hence, the mapping $\Phi = P_Y T/X'_\varepsilon$ is one-to-one from X'_ε into Y_0 and so it has a bounded inverse. Thus, Hypothesis 4 satisfied.

Now, let

$$\alpha_q = q\left|\frac{\langle T(v^1), \psi^1\rangle}{\langle H(\overline{\lambda}, v^1), \psi^1\rangle}\right|^{\frac{1}{a-1}}$$

and $w_q = \alpha_q v^1$. By a simple calculation, one can see that

$$w_q - q\Phi^{-1}(P_Y H(\overline{\lambda}, w_q)) = 0$$

and the mapping $\operatorname{id} - q\Phi^{-1}(P_Y H_v(\overline{\lambda}, w_q))$ is one-to-one from X'_ε into itself. Moreover, w_q is an isolated nonzero solution to the equation $u - q\Phi^{-1}(P_Y H(\overline{\lambda}, u)) = 0$. Hence, there exists a neighbourhood U_q of w_q in X'_ε, $0 \notin U_q$, such that for all $u \in U_q$, $u \neq w_q$, $u - q\Phi^{-1}(P_Y H(\overline{\lambda}, u)) \neq 0$. It then follows that the Leray-Schauder topological degree, $\deg(\operatorname{id} - q\Phi^{-1}(P_Y H(\overline{\lambda}, .)), U_q, 0)$, of $\operatorname{id} - q\Phi(P_Y H(\overline{\lambda}, .)$ over U_q with respect to the origin is defined and different from zero. Thus, Hypothesis 5 is also satisfied. This completes the proof of the corollary. □

The following theorem is a generalization of the results on bifurcation from simple eigenvalues obtained by Crandall and Rabinowitz [3] and by the author [14] for the case where the linear mapping $T - L(\overline{\lambda}, .)$ is not continuous and whose range need not be closed. This also includes the case $\operatorname{Ker}(T - L(\overline{\lambda}, 0)) = \{0\}$.

Theorem 10. *Let Hypotheses 1, 2 and 3 be satisfied and let Y_0, ψ^1 be as above. In addition, for a given $\varepsilon > 0$, assume that there exists an element $v^1 \in X_\varepsilon$, $\|v^1\| = 1$, such that $\langle T(v^1), \psi^1\rangle \neq 0$. Then $(\overline{\lambda}, 0)$ is a bifurcation point of Equation (10). More precisely, for $q = 1$ or $q = -1$ and to any δ, $\varepsilon > 0$, there exist a neighbourhood I of zero in R, and three mappings $n : I_+ \times R \times X'_\varepsilon \mapsto R$, $\varphi : I_+ \times R \times X'_\varepsilon \mapsto X$, with $X'_\varepsilon = [v^1]$, and $\alpha : I_+ \mapsto R$, with $|n(t, \beta, u)| + \|\varphi(t, \beta, u)\| = o(t)$ as $t \to 0$ for all $t \in I_+$, $(\beta, u) \in R \times X'_\varepsilon$, $|\beta| + \|u\| = 1$ such that $(\lambda(t), v(t))$, $t \in I_+$, with*

$$\lambda(t) = \frac{\overline{\lambda}}{\left(1 + q\left(\frac{t\alpha(t)}{1 + |\alpha(t)|} + n\left(t, \frac{\alpha(t)}{1 + |\alpha(t)|}, \frac{v^1}{1 + |\alpha(t)|}\right)\right)\right)^b} \tag{28}$$

and

$$v(t) = \frac{tv^1}{1+|\alpha(t)|} + \varphi\Big(t, \frac{\alpha(t)}{1+|\alpha(t)|}, \frac{v^1}{1+|\alpha(t)|}\Big) \tag{29}$$

satisfies Equation (10); $|\lambda(t) - \overline{\lambda}|_\Lambda < \delta$, *and* $0 < \|v(t)\| < \varepsilon$ *for all* $t \in I_+$, $t > 0$.

Proof. As before, without loss of generality, we only prove the case $q = 1$. Let $\varepsilon_o > 0$ be given and let I, m, n, ξ, φ exist as in the proof of Theorem 8, with

$$G(\alpha, v) = F\Big(\frac{\overline{\lambda}}{(1+\alpha)^b}, v\Big) \quad (\alpha, v) \in J \times \overline{D}.$$

For any $t \in I$, $\alpha \in R$ we put

$$\overline{h}(t,\alpha) = \begin{cases} \dfrac{1+|\alpha|}{t} n\Big(t, \dfrac{\alpha}{1+|\alpha|}, \dfrac{v^1}{1+|\alpha|}\Big), & \text{if } t > 0; \\ 0, & \text{if } t = 0, \end{cases}$$

$$\overline{g}(t,\alpha) = 1 + \Big(m\Big(t, \frac{\alpha}{1+|\alpha|}, \frac{v^1}{1+|\alpha|}\Big)\Big),$$

$$\overline{\gamma}(t,\alpha) = \frac{1+|\alpha|}{t} \varphi\Big(t, \frac{\alpha}{1+|\alpha|}, \frac{v^1}{1+|\alpha|}\Big),$$

$$\overline{\sigma}(t,\alpha) = \xi\Big(t, \frac{\alpha}{1+|\alpha|}, \frac{v^1}{1+|\alpha|}\Big),$$

and define the mappings $\overline{N}_i : I_+ \times R \mapsto R$, $i = 1, \ldots, 4$ by

$$\overline{N}_1(t,\alpha) = -\frac{\overline{g}(t,\alpha)}{\langle T(v^1), \psi^1\rangle}\Big(\frac{t}{1+|\alpha|}\Big)^{a-2} \langle H\Big(\frac{\overline{\lambda}}{(\overline{g}(t,\alpha))^b}, v^1 + \overline{\gamma}(t,\alpha)\Big), \psi^1\rangle,$$

$$\overline{N}_2(t,\alpha) = \alpha + \overline{h}(t,\alpha),$$

$$\overline{N}_3(t,\alpha) = \frac{(\alpha + \overline{h}(t,\alpha))}{\langle T(v^1), \psi^1\rangle} \langle T(\overline{\gamma}(t,\alpha)), \psi^1\rangle,$$

and

$$\overline{N}_4(t,\alpha) = -\frac{\overline{g}(t,\alpha)}{\langle T(v^1), \psi^1\rangle}\Big(\frac{t}{1+|\alpha|}\Big)^{-2} \langle K\Big(\frac{\overline{\lambda}}{(\overline{g}(t,\alpha))^b}, \overline{\sigma}(t,\alpha)\Big), \psi^1\rangle.$$

Let

$$\alpha_1 = \begin{cases} \Big(\dfrac{\langle H(\overline{\lambda}, v^1), \psi^1\rangle}{\langle T(v^1), \psi^1\rangle}\Big), & \text{if } a = 2; \\ 0, & \text{if } a > 2, \end{cases}$$

and let Ω be a neighbourhood of the point α_1 in R. We define the mapping $\overline{\Gamma} : I_+ \times \Omega \mapsto R$ by

$$\overline{\Gamma}(t,\alpha) = \sum_{i=1}^{4} \overline{N}_i(t,\alpha), \quad (t,\alpha) \in I_+ \times \Omega.$$

As before, we first assume that $\overline{\Gamma}(t,\alpha) \neq 0$ holds for all $t \in I_+$, $\alpha \in \partial\Omega$. It then follows that the Leray-Schauder topological degree, $\deg(\overline{\Gamma}(t,.),\Omega,0)$, of $\overline{\Gamma}(t,.)$ over Ω with respect to zero is defined and

$$\deg(\overline{\Gamma}(t,.),\Omega,0) = \deg(\overline{\Gamma}(0,.),\Omega,0) \neq 0,$$

because

$$\overline{\Gamma}(0,\alpha) = \begin{cases} \alpha - \dfrac{\langle H(\overline{\lambda},v^1),\psi^1\rangle}{\langle T(v^1 - BM(v^1)),\psi^1\rangle}, & \text{if } a = 2, \\ \alpha, & \text{if } a > 2. \end{cases}$$

Consequently, for any sufficiently small t, there exists $\alpha(t) \in \Omega$ such that $\overline{\Gamma}(t,\alpha(t)) = 0$. Hence,

$$\begin{aligned} &(\alpha(t) + \overline{h}(t,\alpha(t)))\langle T(v^1) + \overline{\gamma}(t,\alpha(t)),\psi^1\rangle \\ &- \overline{g}(t,\alpha(t))\Big(\frac{t}{1+|\alpha(t)|}\Big)^{a-2}\langle H\Big(\frac{\overline{\lambda}}{(\overline{g}(t,\alpha(t)))^b}, v^1 + \overline{\gamma}(t,\alpha(t))\Big),\psi^1\rangle \\ &- \overline{g}(t,\alpha(t))\Big(\frac{t}{1+|\alpha(t)|}\Big)^{-2}\langle K\Big(\frac{\overline{\lambda}}{(\overline{g}(t,\alpha(t)))^b}, \overline{\sigma}(t,\alpha(t))\Big),\psi^1\rangle = 0. \end{aligned} \tag{30}$$

Multiplying both sides of (30) with $\Big(\dfrac{t}{1+|\alpha(t)|}\Big)^2$ and using the fact

$$\langle T(v(t)) - L(\lambda(t),v(t)),\psi^1\rangle = 0,$$

we obtain

$$\langle T(v(t)) - L(\lambda(t),v(t)) - E(\lambda(t),v(t)),\psi^1\rangle = 0.$$

Hence,

$$P_Y(T(v(t)) - L(\lambda(t),v(t)) - E(\lambda(t),v(t))) = 0,$$

with $\lambda(t)$, $v(t)$ being as in (28) and (29), respectively. Further, by the same arguments as in the proof of Theorem 8, we conclude

$$T(v(t)) = L(\lambda(t),v(t)) + E(\lambda(t),v(t)).$$

This completes the proof of the theorem. □

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Received November 28, 1992