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ON FINITE CODIMENSIONAL SUBALGEBRAS OF ASSOCIATIVE ALGEBRAS

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Abstract. *In this note we consider the mutual influence of properties of algebras and their subalgebras of finite codimension. For an infinite dimensional algebra R with finite codimensional subalgebra A we show: R is semiprime if and only if A is semiprime and either R has an essential ideal I which is semiprime as a ring and $I \cap A \neq 0$, or $R = K \oplus I$ with ideals $I, K \subset R$, such that K is a finite dimensional semiprime ring and $I \subset A$.*

R is right primitive if and only if A is right primitive and for every ideal of R contained in A the right annihilator is zero.

If A is semiprime with Krull dimension, then R is a Goldie ring.

1. PRELIMINARIES

A subalgebra A of an algebra R over a field F is said to have *finite codimension* if the vector space $(R/A, +)$ is finite dimensional.

Similarly, a subring A of a ring R is said to have *finite index* if the abelian group $(R/A, +)$ is finite.

Properties of finite codimensional left ideals were considered in [1] and [3]. Subrings of finite index were investigated in [2], subalgebras of finite codimension were studied in [4]. In [1], A.I. Malcev shows that if an algebra R over the field F has a left ideal H of finite codimension and H is representable by matrices over some field, then R is also representable by matrices over a field. In [3], S.A. Amitsur and L.W. Small show that in an affine PI-algebra the finite codimensional property of a left ideal is equivalent to the algebra R being algebraic over the left ideal.

The finite codimensional property of an algebra has connections with properties of representability and finite dimensional (finite) approximability of algebras (rings).

Proposition 1.1. *Let R be an algebra over a field F. Then there is an R-module M which is finite dimensional as an F-vector space if and only if R has subalgebras of finite codimension.*

One part of the Proposition is well-known, the converse follows from [4].

From this Proposition it follows, for example, that the simple F-algebra R has an R-module M which is finite dimensional as an F-vector space if and only if R is a finite dimensional algebra over F.

If R is an infinite dimensional simple algebra over a field F, then every R-module M is infinite dimensional as an F-vector space.

An algebra may also have many finite codimensional subalgebras which are not approximable by finite dimensional algebras (i.e., are not representable as a subdirect product of finite dimensional algebras).

For example, let $R = F\langle x_1, \ldots, x_n, \ldots \rangle$ be a relatively free algebra over the field F, with the T-ideal $T(R) = \{xyz\}^T$. Recall that an ideal I in the free associative algebra $F[X]$ with $X = \{x_1, \ldots, x_n, \ldots\}$ is called *fully invariant* or a *T-ideal* if it is invariant under all algebra endomorphisms of $F[X]$. Consider the F-ideal M generated by

$$\{x_i^2, i \in \boldsymbol{N}; \text{ all finite sums } \sum \alpha_{ij} x_i x_j, \text{ where } \alpha_{ij} \in F \text{ with } \sum \alpha_{ij} = 0\}.$$

Then the factor algebra $\overline{R} = R/M$ also satisfies the identity $xyz = 0$ and every element of $\overline{R}$ has the form $\overline{a} = \sum \alpha_i \overline{x_i} + \beta \overline{x_1}\,\overline{x_2}$. For every $m \in \boldsymbol{N}, m > 1$, the algebra $\overline{R}$ has an ideal $\overline{A_m}$ with $\dim \overline{R}/\overline{A_m} = m$ and $\overline{R}$ is not finitely approximable because every ideal of $\overline{R}$ contains the element $\overline{x_1}\,\overline{x_2}$.

Notice that $F[X]$ has many finite codimensional subalgebras and approximable finite dimensional algebras.

2. FINITE CODIMENSIONAL SUBALGEBRAS

We now consider the mutual influence of some properties of algebras and their finite codimensional subalgebras.

Lemma 2.1. *Let R be an infinite dimensional algebra over a field with finite codimensional subalgebra A. Then:*

(1) If R is prime, then A is a prime algebra.

(2) If R is semiprime, then A is a semiprime algebra.

Proof.

(1) Let $xAy = 0$, where $x, y \in A$. By [4], A contains an ideal I of the algebra R of finite codimension. Then we have $xIy = 0$. From this it follows that $RxIyR = 0$ and $0 = RxIyR \supset RxRIyR$. R is prime, consequently we have $RxR = 0$ or $IyR = 0$. From this relation we get $x = 0$ or $y = 0$. This shows that A is a prime algebra.

(2) Let $R = \prod_{i \in I} R/P_i$, $\bigcap P_i = 0$, where the P_i are prime ideals of the algebra R. Let $\overline{R_i} = R/P_i$, $i \in I$, and consider the subalgebras

$$\overline{A_i} = (A + P_i)/P_i \simeq A/(A \cap P_i)$$

of the algebras $\overline{R_i}$. If $A + P_i = R$ for some $i \in I$, then

$$R/P_i \cong (A + P_i)/P_i \cong A/(A \cap P_i)$$

is a prime algebra. If $A + P_i \neq R$ then R/P_i has a subalgebra

$$(A + P_i)/P_i \cong A/(P \cap A_i) = A_i$$

of finite codimension. By (1), A_i is prime.

Note that $\bigcap_{i \in I} (P_i \cap A) = (0)$, and $\overline{A_i} = A/(P \cap A_i)$ are prime algebras for all $i \in I$, consequently A is semiprime.

The converse statements of this lemma are not true. For finite dimensional algebras the assertions of Lemma 2.1 are also false in general. □

Example 1. The matrix algebra $M_n(F)$ is prime, but the subalgebra of the upper triangular matrices is not prime.

Example 2. Let A be an infinite dimensional semiprime (prime) algebra and B be a finite dimensional nilpotent algebra. Then the direct sum $R = A \oplus B$ shows that semiprimeness (primeness) of finite codimensional subalgebras does not imply semiprimeness (primeness) of the algebra R.

Theorem 2.2. *Let R be an infinite dimensional algebra over a field F and A a subalgebra of finite codimension.*

(1) R is a prime algebra if and only if A is a prime algebra and contains an essential two-sided ideal of the algebra R.

2) R is semiprime if and only if A is semiprime and either

R has an essential ideal which is semiprime as a ring and has a nontrivial intersection with A, or

R has the form $R = K \oplus I$ where K is a finite dimensional semiprime ring such that $I, K \lhd R$, and $I \subset A$.

Proof. (1) If R is prime, then A is prime by Lemma 2.1.

By [4], A contains a finite codimensional ideal I of the algebra R. Since R is prime it is clear that I is essential.

Conversely, let A be prime and $I \lhd R$ such that $I \subseteq A$ and I is essential in R. Let $M, H \lhd R$ and $M \cdot H = (0)$. Then $M \cap I = M_1 \neq (0)$ and $H_1 = H \cap I \neq (0)$ are ideals in A and $M_1 H_1 = (0)$. This is a contradiction to A being prime. Consequently $M = (0)$ or $H = (0)$ and R is a prime algebra.

(2) Assume R is semiprime. Then by Lemma 2.1, A is semiprime and by [4], A contains an ideal I of R which has finite codimension. If I is not an essential ideal of R, then there is an ideal L_1 of R such that $L_1 \cap I = (0)$. Then it is clear that $L_1 I = I L_1 = (0)$ and R contains the ideal $L_1 \oplus I = I_1$. If the ideal I_1 is not essential, we continue the process. After a finite number of steps we obtain an ideal

$$K = L_m \oplus L_{m-1} \oplus \cdots \oplus L_1 \oplus I,$$

where for every $i = 1, 2, \ldots, m$, $\dim_F L_i < \infty$ and either K is essential or $R = K$, because $\dim_F R/I$ is finite.

In the first case, K is essential and $K \cap A \neq (0)$ and K is semiprime as a ring. In the second case, R has the form $R = L \oplus I$ where $L = L_1 \oplus \cdots \oplus L_m$, I is an ideal contained in A and of finite codimension, and L is an ideal of the algebra R which is finite dimensional over F.

Conversely, let A be a semiprime subalgebra of R. Then by [4], A contains an ideal I of R which is also of finite codimension. By Lemma 2.1, I is a semiprime ring. If R has the form $R = K \oplus I$ where $K \lhd R$, K is finite dimensional over the field F, then R is semiprime. Let R have a non-trivial essential ideal I_0 which is semiprime as a ring and intersects with A non-trivially. Suppose R is not semiprime. Then R has a nilpotent ideal N such that $N \neq (0)$. Since I_0 is essential, $I_0 \cap N$ is non-zero and also a nilpotent ideal of the semiprime ring I_0, a contradiction showing that R is semiprime. □

Theorem 2.3. *Let R be an infinite dimensional algebra over a field with finite codimensional subalgebra A.*

Then R is right primitive if and only if A is a right primitive algebra, and *for every ideal K of R contained in A, the right annihilator is zero, i.e.,*

$$Ann_r K = \{r \in R \mid Kr = 0\} = 0.$$

Proof. Let R be right primitive and M a faithful irreducible R-module. By [4], A contains an ideal I of R, and it is clear that M is a faithful A- and I-module. It is well known that M is an irreducible I-module, consequently M is an irreducible A-module and A is right primitive.

Moreover, for every non-zero ideal $K \triangleleft R$, $M = MK = mK$ for some $m \in M$. If there exists an non-zero $x \in R$ with $Kx = 0$, then $Mx = mKx = 0$ and so the R-module M is not faithful. This contradicts the primitivity of the algebra, consequently $\mathrm{Ann}_r(K) = 0$.

Conversely, let A be a right primitive subalgebra of the algebra R and M a faithful, irreducible A-module. By [4], A contains an non-zero ideal I of the algebra R (of finite codimension). As an ideal of A, I is primitive and, for some $m \in M$, $M = MI = mI$ is a faithful irreducible I-module. From the condition $\mathrm{Ann}_r(I) = (0)$ it follows that $M = mI$ is a faithful R-module. Irreducibility follows from the condition $A \subset R$. Consequently M is a faithful irreducible R-module and R is right primitive. □

Note that the condition $\mathrm{Ann}_r(K) \neq (0)$ is necessary: Consider an algebra $R = A \oplus Fe$, where A is a right primitive algebra and Fe is a nilpotent ideal, one-dimensional over F. Then R is not primitive, and $\mathrm{Ann}_r(A) \neq 0$.

Corollary 2.4. *Let $R = F[x_1, \ldots, x_n]$, $n \geq 2$, be a finitely generated free associative algebra. Then*

(i) every finite codimensional subalgabra of R is primitive, and

(ii) every finite codimensional subalgabra of R contains a T-ideal of R.

Proof. Statement (i) follows from [7], [4] and Theorem 2.3, while (ii) is also a consequence of [4] and properties of polynomial identities. □

Now we consider the influence of subalgebras of finite codimension with Krull dimension. We denote the Krull dimension of an R-module M by $\mathcal{K}(M)$. Note that

$$\mathcal{K}(M) = \sup\{\mathcal{K}(N), \mathcal{K}(M/N)\}$$

for any submodule $N \subset M$ (see [5]).

Theorem 2.5. *Let R be an infinite dimensional algebra over the field F and A a subalgebra of finite codimension.*

If A is semiprime and has Krull dimension, then R is a Goldie ring and R can be embedded in an artinian algebra.

Proof. Let the A-module A have Krull dimension. By [4], A contains an ideal I of the algebra R with finite codimension. Then

$$\mathcal{K}(A) = \sup\{\mathcal{K}(A/I), \mathcal{K}({}_AI)\}.$$

Since $\dim_F A/I < \infty$, we have $\mathcal{K}(A/I) = 0$ and $\mathcal{K}(A) = \mathcal{K}({}_AI)$. Consider I as an R-module. Then $\mathcal{K}(R) = \sup\{\mathcal{K}(R/I), \mathcal{K}({}_RI)\}$. Since $\dim R/I < \infty$, we have $\mathcal{K}(R) = \mathcal{K}({}_RI)$.

Every R-submodule of the R-module I is an A-module, consequently $\mathcal{K}({}_RI)$ exists, and it is well known that $\mathcal{K}({}_RI) \leq \mathcal{K}({}_AI)$.

Since A is a semiprime algebra with Krull dimension, then by [5], A is a semiprime Goldie ring.

Assume R be a semiprime algebra. Since it has Krull dimension, R is a semiprime Goldie, and an order in an artinian ring.

Now consider the case when R is not semiprime. Then R has a nontrivial nil radical $N(R)$, and by [5], $N(R)$ is a nilpotent algebra over the field F. Being a semiprime algebra as an ideal, I is semiprime as a ring, consequently $N(R) \cap I = (0)$. From $\dim_F R/I < \infty$ follows $\dim_F N(R) < \infty$. Then R is a subdirect sum of the two algebras $R_1 = R/N(R)$, which is a semiprime algebra, and $R_2 = R/I$ which is a finite dimensional algebra. Consequently R is a subalgebra of a direct sum of algebras, $\overline{R} = R/N \oplus R/I$.

It is not difficult to show that the algebra $\overline{R}$ has the ACC(Ann). Then the subalgebra R also has the ACC(Ann) [6]. It is well known, if an R-module has Krull dimension, then it has finite uniform dimension [5]. Consequently, the algebra R is a Goldie ring. R/N is a semiprime Goldie algebra, hence it is an order in an artinian algebra. R/I is finite dimensional, therefore it is artinian. From this it follows that R can be embedded in an artinian algebra. □

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