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RINGS WITH CERTAIN MODULES CS

PATRICK F. SMITH

Abstract. *Let R be a ring with identity and M a unital left R-module. Then M is a CS-module if every submodule is essential in a direct summand of M. Our concern is to study when certain classes of left R-modules consist of CS-modules. In particular, we consider when all finitely (respectively, countably) generated projective left R-modules are CS, when all projective left R-modules are CS, when all singular left R-modules are CS and when all finitely generated left R-modules are CS. To help the reader, we have included background material, a guide to the literature and an extensive bibliography.*

Throughout, all rings are associative with identity and all modules are unital left modules (unless stated otherwise). Let R be a ring and M a left R-module. A submodule K of M is called *closed (in M)* provided K has no proper essential extension in M. The module M is called a *CS-module* provided every closed submodule is a direct summand.

Let R be a ring (with identity). For any (left) R-module M, let $[M]$ denote the isomorphism class of M. Let $\mathbf{C}$ be a class of left R-modules, i.e. $\mathbf{C}$ is a collection of modules such that ${}_R0 \in \mathbf{C}$ and $[M] \subseteq \mathbf{C}$ whenever $M \in \mathbf{C}$, where ${}_R0$ is the zero R-module. Any module belonging to the class $\mathbf{C}$ will be called a $\mathbf{C}$-*module.* We are interested in the following general question:

'for which rings R, is every module in a given class $\mathbf{C}$ CS?'

We shall consider this question in the following particular cases:

(i) $\mathbf{C} = [{}_RR] \cup [{}_R0]$,

(ii) $\mathbf{C}$ is the class of finitely generated projective left R-modules,

(iii) $\mathbf{C}$ is the class of countably generated projective left R-modules,

(iv) $\mathbf{C}$ is the class of all projective left R-modules,

(v) $\mathbf{C}$ is the class of singular left R-modules, and

(vi) C is the class of all left R-modules.

As we shall see, it is natural to consider some of these cases when the ring R is left nonsingular.

Before embarking on this investigation, we make two comments. The first is that if R is *any* ring and C the class of injective (more generally, quasi-injective) left R-modules then, of course, every C-module is CS. Secondly, what about nonsingular modules? Our first result shows that in the case of left nonsingular R, nonsingular modules will be taken care of in the programme outlined above. Let M be a left R-module. Then M is called Σ-CS (respectively, *finitely* Σ-CS, *countably* Σ-CS) if every (finite, countable) direct sum of copies of M is CS.

Proposition 1. *Let R be any ring. Consider the following statements.*

(i) ${}_RR$ is (finitely, countably) Σ-CS.

(ii) Every (finitely generated, countably generated) projective left R-module is CS.

(iii) Every (finitely generated, countably generated) left R-module is a direct sum of a projective module and a singular module.

(iv) Every (finitely generated, countably generated) nonsingular left R-module is projective.

(v) Every (finitely generated, countably generated) nonsingular left R-module is CS.

Then (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v). If, in addition R is left nonsingular then the above statements are equivalent.

Proof. We shall prove this result in the general case. The proofs for finitely generated modules and for countably generated modules are similar.

(i) $\Rightarrow$ (ii). Because any direct summand of a CS-module is also a CS-module (see, for example, [32, Proposition 2.7]).

(ii) $\Rightarrow$ (iii). Let M be any R-module. There exists a free R-module F and an epimorphism $\varphi : F \to M$. Let $K = \ker\varphi$. There exist submodules P, P' of F such that $F = P \oplus P'$ and K is an essential submodule of P. Now

$$M = \varphi(F) = \varphi(P) \oplus \varphi(P'),$$

where $\varphi(P) \cong P/K$, so that $\varphi(P)$ is singular, and $\varphi(P') \cong P'$, so that $\varphi(P')$ is projective.

(iii) $\Rightarrow$ (i). Let F be any free R-module. Let K be a closed submodule of F. By hypothesis, $F/K = (P/K) \oplus (S/K)$, for some submodules P, S of F, containing

K, such that P/K is projective and S/K is singular. Because, $F/P \cong S/K$, it follows that F/P is singular, and hence P is an essential submodule of F. Therefore $K = P \cap S$ is an essential submodule of S, and hence $K = S$. It follows that $F/K = P/K$, so that F/K is projective, and K is a direct summand of F. Thus F is a CS-module.

(iii) $\Rightarrow$ (iv). Clear.

(iv) $\Rightarrow$ (v). Let G be a nonsingular left R-module. Let H be a proper submodule of G. Let N denote the submodule of G containing H such that N/H is the singular submodule of G/H. It can easily be checked that H is essential in N and that the module G/N is nonsingular, hence projective. It follows that N is a direct summand of G. Thus G is a CS-module.

If R is a left nonsingular ring then it is clear that (v) implies (i).

In Proposition 1, (v) $\Rightarrow$ (i) fails in general. It is easy to give an example. Let R be a commutative local ring with unique maximal ideal J and suppose that J is nilpotent. Then R has no non-zero nonsingular modules, so (iv) and (v) hold vacuously. On the other hand, it is clear that the R-module R is a CS-module if and only if it is uniform. For a particular example, take K to be any field, let $S = K \oplus K$ and let R denote the subring of the ring of all 2×2 matrices over S consisting of all matrices of the form

$$\begin{bmatrix} k & s \\ 0 & k \end{bmatrix}$$

with k in K and s in S. Then R satisfies (iv) but not (i).

We now begin our investigation outlined above by examining when the left R-module R is CS; it is natural to say that the ring R is a *left CS-ring* in this situation. Let R be a ring and M a left (right) R-module. Then we shall write

$$\ell_R(m) = \{r \in R \ : \ rm = 0\} \quad \text{(respectively,} \quad r_R(m) = \{r \in R \ : \ mr = 0\}).$$

When there is no ambiguity we write $\ell(m)$ or $r(m)$. First we prove a lemma due essentially to Utumi [50].

Lemma 2. *Let R be a left nonsingular ring. Then every closed left ideal of R is a left annihilator if and only if every non-essential left ideal of R has non-zero right annihilator.*

Proof. Suppose first that every closed left ideal is a left annihilator. Let A be a non-essential left ideal of R. There exists a closed left ideal B such that A is essential in B. Clearly, $B \neq R$. By hypothesis, $B = \ell r(B)$ and hence $r(B) \neq 0$. Thus $r(A) \neq 0$.

Conversely, suppose that every non-essential left ideal of R has non-zero right annihilator. Let C be a closed left ideal of R. Suppose that $C \neq \ell r(C)$. Then C is not essential in $\ell r(C)$, and hence $C \cap D = 0$ for some non-zero left ideal D in $\ell r(C)$. Let E be a complement of D in R such that $C \subseteq E$. Clearly E is non-essential and hence, by hypothesis, $r(E) \neq 0$. Let $0 \neq x \in r(E)$. Then $Ex = 0$ implies $Cx = 0$. Now $x \in r(C) = r\ell r(C)$ implies that $Dx = 0$. But $D \oplus E$ is an essential left ideal of the left nonsingular ring R and $(D \oplus E)x = 0$, so that $x = 0$, a contradiction. Thus $C = \ell r(C)$, as required.

Recall that a ring R is called a *Baer ring* if every left annihilator is generated by an idempotent, equivalently, every right annihilator is generated by an idempotent.

Proposition 3. *A ring R is a left nonsingular left CS-ring if and only if R is a Baer ring such that every non-essential left ideal has non-zero right annihilator.*

Proof. Suppose first that R is a Baer ring such that every non-essential left ideal has non-zero right annihilator. Clearly R is left nonsingular and hence R is a left CS-ring by Lemma 2. Conversely, suppose that R is a left nonsingular left CS-ring. Then every closed left ideal of R is a left annihilator, and Lemma 2 then gives that every non-essential left ideal has non-zero right annihilator. Let A be any left annihilator of R. There exists an idempotent e in R such that A is essential in Re. Let $b \in r(A)$. There exists an essential left ideal L such that $Le \subseteq A$ and hence $Leb = 0$. But R is left nonsingular, so that $eb = 0$ and hence $b \in (1-e)R$. Thus $r(A) = (1-e)R$. It follows that $A = \ell r(A) = Re$. Thus R is a Baer ring.

Recall that a ring R is called a *left PP-ring* provided every principal left ideal of R is projective. It is rather clear that Baer rings are left (and right) PP-rings and left PP-rings are left nonsingular. Moreover, a ring R is a left PP-ring if and only if for each a in R there exists an idempotent e in R such that $\ell(a) = Re$. The next result is a companion to Proposition 3.

Proposition 4. *Let R be a left nonsingular right PP-ring such that every non-essential left ideal has non-zero right annihilator. Then every closed left ideal of finite uniform dimension is generated by an idempotent. In particular, if in addition R is a left Goldie ring then R is a left CS-ring.*

Proof. Let L be a closed left ideal of R of finite uniform dimension. To prove that L is a direct summand of ${}_RR$ it is sufficient (by induction) to prove this in the case that L is uniform. Suppose that L is uniform and let $0 \neq a \in L$. Because R is left nonsingular and Ra is essential in L it follows that $r(L) = r(a)$. But $r(a) = eR$ for some idempotent e, because R is right PP. By Lemma 2, $L = \ell r(L) = \ell r(a) = R(1-e)$. The result follows.

Next we give a sufficient condition for every closed left ideal of a left nonsingular ring to be a left annihilator. Recall that if R is a left nonsingular ring then the injective hull $E({}_RR)$ has a unique ring structure compatible with its left R-module structure, and, as usual, we call this ring the *maximal left quotient ring* of R. Denote the maximal left quotient ring of R by Q. Let M be a nonsingular left R-module. Then the injective hull $E({}_RM)$ of M can be given the structure of a left Q-module (see [20, Theorem 2.2]). Let L be a left ideal of R. Then $K = E({}_RL)$ is a direct summand of ${}_RQ$, say $Q = K \oplus K'$, for some R-submodule K' of Q. Because ${}_RQ$ is nonsingular, K is essential in QK and hence $K = QK$. Similarly, $K' = QK'$. Thus $K = Qe$ for some idempotent e in Q. We shall generalize this fact in Lemma 12.

Lemma 5. *Let R be a left nonsingular ring with maximal left quotient ring Q. Let X and Y be left Q-modules with ${}_RY$ nonsingular. Then*

(i) $\mathrm{Hom}_Q(X,Y) = \mathrm{Hom}_R(X,Y)$.

(ii) If $X = Y \oplus Z$ for some R-submodule Z of X then Z is a Q-submodule of X.

Proof.

(i) Let $\varphi \in \mathrm{Hom}_R(X,Y)$. Let $q \in Q$, $x \in X$. There exists an essential left ideal L of R such that $Lq \subseteq R$. Let $a \in L$. Then $a\varphi(qx) = \varphi(aqx) = aq\varphi(x)$. It follows that $L(\varphi(qx) - q\varphi(x)) = 0$. But Y is a nonsingular R-module. Thus $\varphi(qx) = q\varphi(x)$. It follows that $\varphi \in \mathrm{Hom}_Q(X,Y)$.

(ii) Now suppose that $X = Y \oplus Z$. Let $\pi : X \to Y$ denote the projection. Then $\pi \in \mathrm{Hom}_R(X,Y) = \mathrm{Hom}_Q(X,Y)$, so that $Z = \ker \pi$ is a Q-submodule of X.

Proposition 6.

(i) Let R be a left nonsingular ring with maximal left quotient ring Q. Then Q is a regular left self-injective ring.

(ii) Let R be a subring of a regular left self-injective ring S such that R is an essential submodule of the left R-module S. Then R is left nonsingular and $S = Q$.

Proof.

(i) Consider any diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d, "\beta"'] & B & \text{exact} \\ & Q & & \end{tikzcd}$$

of left Q-modules. Since Q is an injective R-module there exists an R-homomorphism $\gamma : B \to Q$ such that $\beta = \alpha\gamma$. By Lemma 5(i), γ is also a Q-homomorphism. It follows that ${}_QQ$ is injective.

Let L be a finitely generated left ideal of Q. There exists a Q-epimorphism $\varphi : X \to L$, where $X = Q(n)$ for some positive integer n. Let $Y = \ker \varphi$. Then ${}_R(X/Y)$ is nonsingular and hence Y is a direct summand of the R-module X. By Lemma 5(ii), there exists a Q-submodule Y' of X such that $X = Y \oplus Y'$. Then $L \cong Y'$ (as Q-modules) so that ${}_QL$ is injective, and hence L is a direct summand of Q. Thus Q is a regular ring.

(ii) Let A be an essential left ideal of R. Then A is an essential submodule of ${}_RS$. Thus SA is an essential left ideal of S. Thus S left nonsingular implies that R is left nonsingular. Again let Q denote the maximal left quotient ring of R. The inclusion mapping $R \to Q$ can be lifted to an R-monomorphism $\alpha : S \to Q$. Let $t \in S$. Define a mapping $\beta : S \to Q$ by

$$\beta(s) = \alpha(st) - \alpha(s)\alpha(t) \quad (s \in S).$$

Clearly β is an R-homomorphism and $\beta(R) = 0$. Because ${}_RQ$ is nonsingular, it follows that $\beta(S) = 0$. Thus $\alpha : S \to Q$ is a ring monomorphism. We identify S with $\alpha(S)$, so that S is a subring of Q. Now S is an essential injective submodule of ${}_SQ$, so that $S = Q$.

Lemma 7. *Let R be a left nonsingular ring with maximal left quotient ring Q. Suppose further that R_R is an essential submodule of Q_R. Then every closed left ideal of R is a left annihilator.*

Proof. Let L be a non-essential left ideal of R. Then L is essential in a direct summand K of ${}_RQ$. There exists an idempotent $1 \neq e \in Q$ such that $K = Qe$ by the remarks before Lemma 5. Note that $r_Q(L) = r_Q(Qe) = (1-e)Q \neq 0$, because the left R-module Q is nonsingular. Now R_R essential in Q_R implies that $r_R(L) = R \cap r_Q(L) \neq 0$. By Lemma 2, every closed left ideal of R is a left annihilator.

Let R be a left and right nonsingular ring R with the property that the maximal left and right quotient rings coincide. Then Lemma 7 tells us that every closed left ideal is a left annihilator and every closed right ideal is a right annihilator. In fact, the converse is also true (see [20, Theorem 2.38]). Thus we have the following result.

Corollary 8. *A ring R is a left and right nonsingular left and right CS-ring if and only if R is a Baer ring for which the left and right maximal quotient rings coincide.*

Let R be a left CS-ring. What about the left R-module $R \oplus R$, is it a CS-module? The answer is "no" in general. For any positive integer n and left R-module M, let ${}_RM^{(n)}$, or simply $M^{(n)}$, denote the left R-module $M \oplus \cdots \oplus M$ (n copies) and let $M_n(R)$ denote the ring of all $n \times n$ matrices with entries in R.

Lemma 9. *Let R be a ring and n a positive integer. Then the left R-module $R^{(n)}$ is CS if and only if the ring $M_n(R)$ is left CS.*

Proof. Let $T = M_n(R)$. Let e_{11} denote the matrix with $(1,1)$ entry 1 and all other entries 0. Let S denote the subring $e_{11}Te_{11}$ of T and note that $R \cong S$. Note further that $T = Te_{11}T$. Let M be a left T-module and let K be a T-submodule of M. Then it is easy to check the following facts:

(i) K is a closed submodule of M if and only if $e_{11}K$ is a closed submodule of the left S-module $e_{11}M$.

(ii) K is a direct summand of M if and only if $e_{11}K$ is a closed submodule of $e_{11}M$, and

(iii) ${}_TM$ is CS if and only if ${}_S(e_{11}M)$ is CS.

The result now follows by taking $M = T$.

Corollary 10. *Let R be a left Ore domain. Then*

(i) ${}_RR$ is a CS-module, and

(ii) ${}_R(R \oplus R)$ is a CS-module only if R is right Ore.

Proof.

(i) The module ${}_RR$ is uniform and hence CS.

(ii) Suppose that $R \oplus R$ is CS. Then the matrix ring $T = M_2(R)$ is a left nonsingular left CS-ring by Lemma 9. Let $0 \neq a, b \in R$. Suppose that $aR \cap bR = 0$. Let α denote the matrix

$$\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$$

in T. It is clear that $r(\alpha) = 0$ and hence $T\alpha$ is essential in T, by Proposition 3. However, it is easy to check that $Te_{11} \cap T\alpha = 0$, a contradiction. Thus $aR \cap bR \neq 0$. It follows that R is a right Ore domain.

It is not difficult to give examples of left Ore domains which are not right Ore and thus examples of left CS-rings R such that the left R-module $R \oplus R$ is not CS (see, for example, [33, 1.2.11 Example (ii)]). However, there are easier examples of such rings R as the following result shows.

Corollary 11. *The following statements are equivalent for a commutative domain R.*

(i) R is a Prüfer domain.

(ii) ${}_R(R \oplus R)$ is a CS-module.

(iii) ${}_RR$ is finitely Σ-CS.

Proof.

(i) $\Rightarrow$ (iii). Let n be any positive integer. Then $M_n(R)$ is a prime left and right Goldie left and right PP-ring. By Lemma 7, $M_n(R)$ is a left CS-ring. Now $R^{(n)}$ is a CS-module, by Lemma 9. Thus ${}_RR$ is finitely Σ-CS.

(iii) $\Rightarrow$ (ii). Clear.

(ii) $\Rightarrow$ (i). By Lemma 9, $M_2(R)$ is a left CS-ring, and hence a left PP-ring (Proposition 3). Thus every 2-generator ideal of R is projective. By [18, Theorem 22.1], R is a Prüfer domain.

For example, the polynomial ring $R = Z[x]$, in the indeterminate x with integer coefficients, is a commutative Noetherian domain such that the R-module $R \oplus R$ is not a CS-module. We shall extend Corollary 11 in Theorem 18 below. Our next concern is with rings R such that the left R-module R is finitely Σ-CS. Recall that we know, by Proposition 1, that ${}_RR$ is finitely Σ-CS if and only if every finitely generated projective left R-module is CS. First we prove the following result.

Lemma 12. *Let R be a left nonsingular ring with maximal left quotient ring Q. Let M be a finitely generated nonsingular left R-module with R-injective hull E. Then E is a left Q-module, $E = QM$ and E embeds in $Q^{(n)}$ for some positive integer n.*

Proof. We have already remarked that E is a left Q-module. In particular, $QM \subseteq E$. Clearly QM is a finitely generated left Q-module. There exists a positive integer n and a Q-epimorphism $\varphi : Q^{(n)} \to QM$. Let $X = Q^{(n)}$ and $Y = \ker \varphi$. Note that $X/Y \cong QM$, so that X/Y is a nonsingular left R-module. Thus Y is a closed submodule of the injective left R-module X. Hence Y is a direct summand of X, so that $X \cong Y \oplus QM$. The result now follows easily.

Corollary 13. *Let R be a left semihereditary ring with maximal left quotient ring Q such that Q is a projective left R-module. Then $_RR$ is finitely Σ-CS.*

Proof. Any finitely generated nonsingular left R-module M embeds in the projective left R-module $Q^{(n)}$ for some positive integer n, by Lemma 12. Because, R is left semihereditary, M is projective. Now apply Proposition 1.

Now we consider the following situation. Let R be a subring of a ring S (with the same identity). Let M be a left R-module. Suppose that M is contained in a left S-module N. Because S is a right R-module we can consider the left S-module $S \otimes_R M$. There exists a natural mapping $\mu : S \otimes_R M \to N$ defined by $\mu(\Sigma_i s_i \otimes m_i) = \Sigma_i s_i m_i$, for all finite sets of elements s_i in S and m_i in M. It is easy to check that μ is an S-homomorphism.

Lemma 14. *Let R be a subring of a ring S such that R is an essential submodule of the left R-module S. Let M be a left R-module such that M is contained in a left S-module N. Suppose further that $_R(SM)$ is nonsingular. Then there exists a natural S-epimorphism $\mu : S \otimes_R M \to SM$ with kernel $Z_R(S \otimes_R M)$.*

Proof. Let $Z = Z_R(S \otimes_R M)$. It is clear that $Z \subseteq \ker \mu$, with μ as above. On the other hand let $x = s_1 \otimes m_1 + \cdots + s_n \otimes m_n \in \ker \mu$, for some positive integer n and elements $s_i \in S$, $m_i \in M$ $(1 \leq i \leq n)$. There exists an essential left ideal L of R such that $Ls_i \subseteq R$ $(1 \leq i \leq n)$. Let $a \in L$. Then

$$ax = as_1 \otimes m_1 + \cdots + as_n \otimes m_n = 1 \otimes \Sigma_i(as_i m_i) = 1 \otimes \mu(ax) = 1 \otimes 0 = 0.$$

Thus $Lx = 0$ and $x \in Z$.

Let R be a ring. An R-module M is called *finitely presented* if there exists an exact sequence $G \to F \to M \to 0$ with F and G both finitely generated free

R-modules. Recall that a ring R is *left coherent* if every finitely generated left ideal is finitely presented. For any R-module M, the singular submodule of M will be denoted $Z_R(M)$, or simply $Z(M)$.

Corollary 15. *Let R be a left nonsingular ring with maximal left quotient ring Q. Let M be a nonsingular left R-module. Then there exists a natural Q-epimorphism $\mu_M : Q \otimes_R M \to E(M)$ with kernel $Z_R(Q \otimes_R M)$. If, in addition, M is finitely presented then μ_M is an isomorphism.*

Proof. The first part follows by Lemma 14. Now suppose that M is finitely presented. There exists a finitely generated free left R-module F, a finitely generated submodule K of F and an exact sequence

$$0 \to K \to F \to M \to 0.$$

Form the diagram:

$$\begin{array}{ccccccc} Q \otimes_R K & \to & Q \otimes_R F & \to & Q \otimes_R M & \to & 0 \\ \mu_K \downarrow & & \mu_F \downarrow & & \mu_M \downarrow & & \\ E(K) & \to & E(F) & \to & E(M) & & \end{array}$$

By Lemma 12, μ_K is onto. Also μ_F is an isomorphism. By a standard diagram chase, μ_M is a monomorphism and hence an isomorphism.

Corollary 16. *Let R be a left nonsingular ring with maximal left quotient ring Q. Suppose further that R is left coherent. Then Q is a flat right R-module.*

Proof. For any finitely generated left ideal L of R, the multiplication map $\mu : Q \otimes_R L \to QL$ is a monomorphism by Corollary 15. Thus Q_R is flat.

Let R be a ring and let M be a left (right) R-module. Let $m \in M$ and let N be a submodule of M. Then we shall denote

$$Nm^{-1} = \{r \in R \ : \ rm \in N\} \quad (m^{-1}N = \{r \in R \ : \ mr \in N\}).$$

Lemma 17. *Let R be a subring of a ring S. Then the following statements are equivalent.*

(i) The natural S-homomorphism $\mu : S \otimes_R M \to S$ is a monomorphism for every submodule M of the left R-module S.

(ii) The right R-module S is flat and the natural S-homomorphism $S \otimes_R S \to S$ is a monomorphism.

(iii) $S = S(Rs^{-1})$ for all s in S.

Proof.

(i) $\Rightarrow$ (ii). Clear.

(ii) $\Rightarrow$ (iii). Let $s \in S$. Let $A = Rs^{-1}$. Define a mapping $\alpha : R/A \to S/R$ by

$$\alpha(r + A) = rs + R.$$

Clearly α is an R-monomorphism. Because S_R is flat we obtain another monomorphism $S \otimes_R (R/A) \to S \otimes_R (S/R)$. However, if $j : R \to S$ denotes the inclusion map then

$$S \otimes_R R \xrightarrow{1 \otimes j} S \otimes_R S \xrightarrow{\mu} S$$

is just the canonical isomorphism $S \otimes {}_R R \cong S$. Thus $1 \otimes j$ is an isomorphism. But S_R flat gives the exact sequence

$$0 \to S \otimes_R R \xrightarrow{1 \otimes j} S \otimes_R S \to S \otimes_R (S/R) \to 0.$$

Thus $S \otimes_R (S/R) = 0$. It follows that $S \otimes_R (R/A) = 0$, and hence $S = SA$.

(iii) $\Rightarrow$ (i). Let M be any submodule of ${}_R S$. Suppose that $x \in \ker \mu$. There exist a positive integer n and elements $s_i \in S$ $(1 \leq i \leq n)$, $m_i \in M$ $(0 \leq i \leq n)$ such that $x - 1 \otimes m_0 + s_1 \otimes m_1 + \cdots + s_n \otimes m_n$. If $n = 0$ then $m_0 = \mu(x) = 0$ and hence $x = 0$. Suppose that $n \geq 1$. Let $r \in Rs_n^{-1}$. Then

$$rx = 1 \otimes (rm_0 + rs_n m_n) + (rs_1) \otimes m_1 + \cdots + (rs_{n-1}) \otimes m_{n-1} \quad \text{and} \quad rx \in \ker \mu.$$

By induction on n, it follows that $rx = 0$. Hence $(Rs_n^{-1})x = 0$. But $S = S(Rs_n^{-1})$, and hence $x = 0$. It follows that μ is a monomorphism.

Theorem 18. *Let R be a left nonsingular ring with maximal left quotient ring Q. Then the following statements are equivalent.*

(i) ${}_R R$ is finitely Σ-CS.

(ii) R is left semihereditary and the left R-module $R^{(2)}$ is CS.

(iii) R is left semihereditary and the right R-module Q satisfies $Q = (q^{-1}R)Q$ for all q in Q.

(iv) R is left semihereditary, the left R-module Q is flat and the left R-module $Q \otimes_R Q$ is nonsingular.

In this case, R is also right semihereditary.

Proof.

(i) $\Rightarrow$ (ii). Let L be any finitely generated left ideal of R. Suppose that L is generated by n elements. By Lemma 9, ${}_RR^{(n)}$ CS implies that the matrix ring $M_n(R)$ is left nonsingular left CS and hence left and right PP (Proposition 3). It follows by a standard argument that L is projective. Hence R is left semihereditary. This proves (ii). Similarly R is right semihereditary.

(ii) $\Rightarrow$ (iii). Let $q \in Q$. Then $M = R + Rq$ is nonsingular and hence projective (adapt the proof of Proposition 1). Thus there exists a monomorphism $\varphi : M \to F = R_1 \oplus R_2$, where $R_j = R (j = 1, 2)$. For $j = 1, 2$, let $\pi_j : F \to R_j$ denote the projection and $\imath_j : R_j \to F$ the inclusion mappings. Note that $R \subseteq M$ and we let $x_j = (1\varphi)\pi_j \in R (j = 1, 2)$. Let $j = 1, 2$. For all r in R, $rx_j = r(1\varphi)\pi_j = r\varphi\pi_j$, and hence $mx_j = m\varphi\pi_j$ for all $m \in M$, because R is essential in the nonsingular module M; in particular $qx_j = q\varphi\pi_j \in R$ and $x_j \in q^{-1}R$. Thus $x_j \in q^{-1}R (j = 1, 2)$.

Now consider the diagram

$$
\begin{array}{ccccc}
 & \varphi^{-1} & & & \\
M\varphi & \to & M & \to & Q \\
\downarrow & & & & \\
F & & & &
\end{array}
$$

where $M \to Q$ is inclusion. There exists a mapping $\theta : F \to Q$ such that the above diagram commutes. For $j = 1, 2$, let $q_j = (1\imath_j)\theta \in Q$. It is not difficult to show that $1 = x_1q_1 + x_2q_2$. It follows that $Q = (q^{-1}R)Q$.

(iii) $\Rightarrow$ (iv). By Corollary 15 and Lemma 17.

(iv) $\Rightarrow$ (i) Let n be any positive integer and let F denote the free left R-module $R^{(n)}$. Let K be any closed submodule of F and let $M = F/K$. Then M is a finitely generated nonsingular let R-module. By Lemma 10, there exists a positive integer k such that $F \subseteq {}_RQ^{(k)}$. To prove that K is a direct summand of F we need to show that M is projective and, because R is left semihereditary, it is sufficient to prove that M can be embedded in a free left R-module. To prove this last fact we can suppose without loss of generality that $M \subseteq Q$.

Suppose that $M = Rx_1 + \cdots + Rx_t$ for some positive integer t and elements $x_i \in Q$ $(1 \le i \le t)$. For each $1 \le i \le t$, let $A_i = x_i^{-1}R$ and let

$$A = \{r \in R \ : \ Mr \subseteq R\} = A_1 \cap \cdots \cap A_t.$$

Note that $Q = A_iQ(1 \le i \le t)$ by Lemma 17. Thus $(R/A_i) \otimes_R Q = 0$ $(1 \le i \le t)$. Now there exists a monomorphism $R/A \to (R/A_1) \oplus \cdots \oplus (R/A_t)$ which gives a monomorphism

$$(R/A) \otimes_R Q \to (\oplus_{1 \le i \le t}(R/A_i)) \otimes_R Q \cong \oplus_{1 \le i \le t}((R/A_i) \otimes_R Q) = 0.$$

Thus $(R/A) \otimes_R Q = 0$, and hence $Q = AQ$. There exist a positive integer s and elements $a_i \in A_i$, $q_i \in Q$ $(1 \le i \le s)$ such that $1 = a_1q_1 + \cdots + a_sq_s$. Define $\theta \ : M \to {}_RR^{(s)}$ by $\theta(m) = (ma_1, \ldots, ma_s)$ for all m in M. It is clear that θ is a monomorphism.

Note that one consequence of Theorem 18 is that, for a left semihereditary ring R, if the left R-module $R \oplus R$ is CS then the left R-module $R^{(n)}$ is CS, for every positive integer n. In particular, this holds if R is (von Neumann) regular. The question arises for which rings R does it follow that ${}_R(R \oplus R)$ CS implies that ${}_RR$ is finitely Σ-CS.

Let R be a left nonsingular ring with maximal left quotient ring Q. Let c be a *regular* element of R; by this, we mean that $cr \neq 0$ and $rc \neq 0$ for all $0 \neq r \in R$. It is well known that if X is an injective left R-module then $X = cX$. In particular, $Q = cQ$. Thus, by Theorem 18, if R is a left semihereditary ring such that for each q in Q there exists a regular element c in R such that $qc \in R$ then ${}_RR$ is finitely Σ-CS. This is true in particular in R is a semiprime (left and right) Goldie ring. This gives:

Corollary 19. *Let R be a semiprime left and right Goldie ring. Then the following statements are equivalent.*

(i) ${}_RR$ is finitely Σ-CS.

(ii) R_R is finitely Σ-CS.

(iii) R is left semihereditary.

(iv) R is right semihereditary.

Let R be a left nonsingular ring such that ${}_RR$ is finitely Σ-CS. Proposition 1 shows that in this case every finitely generated nonsingular (projective) left R-module is CS. In general, it does not follow that every countably generated nonsingular (projective) left R-module is CS. For example, if $\mathbf{Z}$ is the ring of rational integers, then ${}_{\mathbf{Z}}\mathbf{Z}$ is finitely Σ-CS (see for example Corollary 11) but ${}_{\mathbf{Z}}\mathbf{Z}$ is not countably Σ-CS by a result of Kamal and Muller [28, Theorem 5] (or see [32, p.19]). This fact is a consequence of the following result.

Lemma 20. *Let R be a left nonsingular left Goldie ring such that ${}_RR$ is countably Σ-CS. Then R is left Artinian.*

Proof. By Theorem 18, R is left semihereditary. By [22, p. 563 Theorem] it is sufficient to prove that any regular element of R is a unit. Let c be any regular element in R. Let F be a countable direct sum of copies of ${}_RR$. Let Q denote the maximal left quotient ring of R and recall that c is a unit in Q. Define a map $\varphi : F \to Q$ by

$$\varphi(r_1, r_2, r_3, \dots) = r_1 + r_2c^{-1} + r_3c^{-2} + \dots,$$

for all $(r_1, r_2, r_3, \dots)$ in F. It is easy to check that φ is an R-homomorphism. Let $K = \ker \varphi$. Then $F/K \cong \operatorname{im} \varphi \subseteq {}_RQ$. Now the left R-module Q is nonsingular and hence K is a closed submodule of F. By hypothesis, F is a CS-module and hence K is a direct summand of F, say $F = K \oplus K'$ for some submodule K' of F. Clearly, $K' \cong \operatorname{im} \varphi$ and hence K' has finite uniform dimension.

There exists a finitely generated submodule K'' of K' such that K'' is essential in K'. There is a positive integer n such that

$$K'' \subseteq G = R \oplus \dots \oplus R \oplus 0 \oplus 0 \dots (n\ R'\ s).$$

Thus $K' \subseteq G$. Consider the element $e_{n+1} = (0, \dots, 0, 1, 0, 0, \dots)$ of F, with $(n+1)st$ entry 1. Since $F = K \oplus K'$ there exist $a = (a_1, a_2, a_3, \dots) \in K$, $a' \in K'$ such that $e_{n+1} = a + a'$. Note that $a_{n+1} = 1$, $a_t = 0$ $(t \geq n+1)$ and

$$a_1 + a_2c^{-1} + a_3c^{-2} + \dots + a_nc^{-n+1} + c^{-n} = 0.$$

This implies that $xc = 1$ where $x = -a_1c^{n-1} - \dots - a_n \in R$. Moreover, $(1-cx)c = 0$ gives $cx = 1$. Thus c is a unit in R.

We shall prove a stronger version of Lemma 20 later. Now we show that for regular rings R the concepts of finitely Σ-CS and countably Σ-CS coincide. Recall that a regular ring is characterised by the fact that all modules are flat.

Theorem 21. *The following statements are equivalent for a regular ring R with maximal left quotient ring Q.*

(i) ${}_RR$ is countably Σ-CS.

(ii) ${}_RR$ is finitely Σ-CS.

(iii) $Q \otimes_R Q$ is a nonsingular left R-module.

(iv) The left R-module Q is projective.

(v) R is left self-injective.

Proof.

(i) $\Rightarrow$ (ii). Clear.

(ii) $\Rightarrow$ (iii). Suppose that $z \in Z(Q \otimes_R Q)$, where $z = x_1 \otimes y_1 + \cdots + x_n \otimes y_n$, for some positive integer n and elements $x_i, y_i \in Q$ $(1 \leq i \leq n)$. Let $Y = Ry_1 + \cdots + Ry_n$. Then Y is a projective left R-module by Proposition 1 and hence $R^{(n)} \cong Y \oplus Y'$ for some module Y'. Thus

$$Q^{(n)} \cong Q \otimes_R R^{(n)} \cong (Q \otimes_R Y) \oplus (Q \otimes_R Y').$$

It follows that the left R-module $Q \otimes_R Y$ is nonsingular, and hence $z = 0$. Thus $Q \otimes_R Q$ is nonsingular.

(iii) $\Rightarrow$ (iv), (v). By Corollary 15 the multiplication map $\mu_Q : Q \otimes_R Q \to Q$ is an isomorphism. Now if $\imath : R \to Q$ is the inclusion mapping then

$$Q \otimes_R R \xrightarrow{1 \otimes \imath} Q \otimes_R Q \xrightarrow{\mu_Q} Q$$

is the natural isomorphism $\Sigma_i q_i \otimes r_i \to \Sigma_i q_i r_i$. Thus $1 \otimes \imath$ is an isomorphism.

Consider the exact sequence

$$0 \to R \to Q \to Q/R \to 0 \qquad (1)$$

Because Q_R is flat, we obtain

$$0 \to Q \otimes_R R \to Q \otimes_R Q \to Q \otimes_R (Q/R) \to 0 \quad \text{exact.}$$

Thus $Q \otimes_R (Q/R) = 0$. But ${}_R(Q/R)$ is flat, and hence (1) gives the following exact sequence

$$0 \to R \otimes_R (Q/R) \to Q \otimes_R (Q/R).$$

Thus $R \otimes_R (Q/R) = 0$, and hence $Q/R = 0$. This proves that $R = Q$, as required.

(iv) $\Rightarrow$ (iii). Suppose that the left R-module Q is projective. Then there exists a positive integer n such that $R^{(n)} \cong Q \oplus P$ for some R-module P. Now $Q^{(n)} \cong Q \otimes_R R^{(n)} \cong (Q \otimes_R Q) \oplus (Q \otimes_R P)$, and hence the left R-module $Q \otimes_R Q$ is nonsingular.

(v) $\Rightarrow$ (i). Suppose that R is left self-injective. Let M be a countably generated nonsingular left R-module, say $M = Rm_1 + Rm_2 + Rm_3 + \ldots$. It is rather easy to see that any cyclic submodule of M is isomorphic to a direct summand of R and hence is both injective and projective. In particular, $M = Rm_1 \oplus N_1$, for some submodule N_1 of M. Let $\pi : M \to N_1$ denote the projection mapping. Then $M = Rm_1 + R\pi(m_2) + R\pi(m_3) + \ldots$. Moreover, $N_1 = R\pi(m_2) \oplus N_2$, for some submodule N_2, and hence $M = Rm_1 \oplus R\pi(m_2) \oplus N_2$. Repeating this argument, it follows that M is a direct sum of (a countable number of) cyclic submodules, and hence M is projective. By Proposition 1, ${}_RR$ is countably Σ-CS.

Let R be any ring. Recall that a left R-module M is called (countably) Σ-injective if every direct sum of (a countable number of) copies of M is injective. It is well known that for any ring R, a left R-module M is Σ-injective if and only if M is countably Σ-injective. Moreover, the following statements are equivalent for a ring R:

(i) ${}_RR$ is (countably) Σ-injective,

(ii) R_R is (countably) Σ-injective,

(iii) The ring R is QF,

(see [15] or [16, Proposition 20.3A, Theorem 24.20]). We show next by example that the corresponding results for Σ-CS modules are false in general.

First, recall that an ideal I of a ring R is called *left T-nilpotent* if, for any sequence $a_1, a_2, a_3, \ldots$ of elements of I, there exists a positive integer n such that $a_1 a_2 \ldots a_n = 0$. A ring R with Jacobson radical J is called *left perfect* if J is left T-nilpotent and the ring R/J is semiprime Artinian.

Lemma 22. *The following statements are equivalent for a ring R.*

(i) R is left perfect.

(ii) R has DCC on principal right ideals.

(iii) Every flat left R-module is projective.

Lemma 22 is due to Bass (see, for example, [16, 22.29 and 22.31A] or [20, Theorem 5.7]) and the next lemma is due to Chase (see, for example, [16, 22.31B] or [20, Theorem 5.15]).

Lemma 23. *The following statements are equivalent for a ring R.*

(i) Every direct product of projective left R-modules is projective.

(ii) The ring R is left perfect and right coherent.

Example 24.

(i) The exists a commutative regular ring R which is (left) CS such that ${}_RR$ is not finitely Σ-CS.

(ii) There exists a regular ring R such that ${}_RR$ is countably Σ-CS but R_R is not countably Σ-CS and ${}_RR$ is not Σ-CS.

Proof.

(i) Let F be any field having a proper subfield K. Let $F_n = F$ $(n \geq 1)$ and let S denote the commutative self-injective regular ring $\Pi_n\ F_n$. Let R denote the subring of S consisting of all sequences $\{a_n\}$ in S with $a_n \in K$ for all but a finite number of elements $n \geq 1$. Note that $S = E({}_RR)$, so that R is commutative regular but not self-injective. By Theorem 21, ${}_RR$ is not finitely Σ-CS. On the other hand, let A be any ideal of R. Then SA is an ideal of S and hence there exists an idempotent e in S such that SA is essential in Se, because ${}_SS$ is injective. It is clear that $e \in R$ and it is not difficult to check that A is essential in Re. It follows that R is a CS-ring.

(ii) Let F be any field and V any infinite dimensional vector space over F. Let $R = \text{End}\ ({}_FV)$. It is well known that R is a left self-injective regular ring which is not right self-injective (see, for example, [20, Proposition 2.23]). By Theorem 21, ${}_RR$ is countably Σ-CS but R_R is not countably Σ-CS.

Now suppose that ${}_RR$ is Σ-CS. By Proposition 1, it follows that every nonsingular left R-module is projective. In particular, because R is left nonsingular, every direct product of projective left R-modules is projective. By Lemma 23, R is left perfect. But this implies that R is Artinian, a contradiction. Thus ${}_RR$ is not Σ-CS.

In contrast to this example, we shall show for a given ring R:

(1) R is left nonsingular and ${}_RR$ is Σ-CS if and only if R is right nonsingular and R_R is Σ-CS, and

(2) if a left nonsingular ring R contains no infinite set of orthogonal idempotents then ${}_RR$ countably Σ-CS implies that ${}_RR$ is Σ-CS.

First we state, without proof, a well known lemma.

Lemma 25.

(i) Let R be a subring of a ring S such that S is a flat left R-module. Then

any flat left S-module is a flat left R-module.

(ii) Let R be a left (or right) semihereditary ring. Then any submodule of a flat left R-module is flat.

Theorem 26. *The following statements are equivalent for a left nonsingular ring R with maximal left quotient ring Q.*

(i) ${}_RR$ is Σ-CS.

(ii) R is left hereditary left Artinian and the left R-module Q is flat.

(iii) R is left perfect and right semihereditary and the left R-module Q is flat.

Proof.

(i) $\Rightarrow$ (ii). By Proposition 1 and Lemma 23, R is left hereditary and left perfect, and the nonsingular left R-module Q is projective, whence flat. Let X be the direct product of any non-empty collection of projective left Q-modules. Then X is a nonsingular left R-module and hence X is a projective left R-module. Let F be a free left Q-module and let $\varphi : F \to X$ be a Q-epimorphism. Then φ is an R-epimorphism, so there exists an R-homomorphism $\theta : X \to F$ such that $\varphi\theta = 1_F$. But θ is also a Q-homomorphism (Lemma 5). Thus X is Q-projective. It follows that every direct product of projective left Q-modules is projective. By Lemma 23, Q is left perfect. But Q is regular by Proposition 6, and hence Q is semiprime Artinian.

In particular, the left Q-module Q has finite uniform dimension. It follows that the left R-module R has finite uniform dimension. But R is a left hereditary ring. Thus R is left Noetherian, by [44, Theorem 2.1 Corollary 1]. Because, R is left perfect, R is left Artinian.

(ii) $\Rightarrow$ (iii). Clear.

(iii) $\Rightarrow$ (i). Let Y be any nonsingular left R-module. Recall that the ring Q is regular and hence the left Q-module $E(Y)$ is flat. By Lemma 25(i), $E(Y)$ is a flat left R-module. But R is right semihereditary and so, by Lemma 25 (ii), Y is a flat left R-module. By Lemma 23, the left R-module Y is projective.

Theorem 27. *The following statements are equivalent for a ring R.*

(i) The ring R is left nonsingular and ${}_RR$ is Σ-CS.

(ii) The ring R is right nonsingular and R_R is Σ-CS.

(iii) The ring R is (left and right) hereditary Artinian and the maximal left and right quotient rings of R coincide.

Proof.

(i) $\Rightarrow$ (iii). Suppose that R is a left nonsingular ring such that the left R-module R is Σ-CS. By Theorem 18, R is right semihereditary and hence right nonsingular. Let Q denote the maximal left quotient ring of R. By Theorem 26, R is left hereditary left Artinian and the left R-module Q is flat. Moreover, by the proof of Theorem 26 the ring Q is semiprime Artinian. Now R is an essential submodule of the *right* R-module Q by Theorem 18 (i) $\Leftrightarrow$ (iii). Applying Lemma 6 (ii) we have that Q is also the maximal right quotient ring of R. Clearly R is left coherent and hence Q_R is flat (Corollary 16). Moreover, R is left semihereditary and right perfect. By Theorem 26 R_R is Σ-CS and hence (iii) follows.

(iii) $\Rightarrow$ (i). By Corollary 16 and Theorem 26.

(ii) $\Leftrightarrow$ (iii). By symmetry.

Theorem 28. *Let R be a left nonsingular ring which does not contain an infinite set of orthogonal idempotents. Suppose further that ${}_RR$ is countably Σ-CS. Then ${}_RR$ is Σ-CS.*

Proof. Note first that because R is CS with no infinite sets of orthogonal idempotents, R is a (finite) direct sum of uniform left ideals and hence R is a left Goldie ring. By Theorem 18 and Lemma 20, R is a left hereditary left Artinian ring and the maximal left quotient ring Q of R is a flat left R-module. By Theorem 26, ${}_RR$ is Σ-CS.

For any module M, the *second singular submodule* $Z_2(M)$ is defined as follows:

$$Z_2(M)/Z(M) = Z(M/Z(M)).$$

Lemma 29. *Let R be a ring and M an R-module. Then M is CS if and only if $M = Z_2(M) \oplus N$ for some submodule N such that $Z_2(M)$ and N are both CS and $Z_2(M)$ is N-injective.*

Proof. See [28, Theorem 1].

Using Lemma 29 we can now prove the following corollary to Theorem 28.

Corollary 30. *Let R be a ring which does not contain an infinite set of orthogonal idempotents. Suppose further that ${}_RR$ is countably Σ-CS. Then every nonsingular left R-module is CS.*

Proof. Let $Z = Z_2({}_RR)$ and let S denote the ring R/Z. Note that the R-module R is CS. By Lemma 29, there exists a left ideal L of R such that $R = Z \oplus L$.

Now S is a left nonsingular ring with no infinite set of orthogonal idempotents and ${}_SS$ is countably Σ-CS. By Theorem 28, ${}_SS$ is Σ-CS. If M is a nonsingular left R-module then $ZM = 0$, and M is a nonsingular left S-module. By Proposition 1, M is a CS-module.

Let R be any ring and M any R-module. A submodule L of M is called *small (in M)* if, whenever N is a submodule of M such that $M = L + N$, then $M = N$. The module M will be called a *dual CS-module* if for every submodule H of M there exists a direct summand K of M such that $K \subseteq N$ and H/K is small in M/K.

We saw in Proposition 1 that a ring R has the property that every projective module is CS if and only if every module is a direct sum of a projective module and a singular module. Recall that a module M is singular if and only if there exists a projective module P and an epimorphism $\varphi : P \to M$ such that ker φ is essential in P. Dually, a module M is called *small* if there exists an injective module E and a monomorphism $\theta : M \to E$ such that im θ is small in E.

The next result is due to Oshiro. The proof is somewhat long and technical, and is therefore omitted.

Theorem 31. *The following statements are equivalent for a ring R.*

(i) Every projective left R-module is a CS-module.

(ii) Every injective right R-module is a dual CS-module.

(iii) Every left R-module is a direct sum of a projective module and a singular module.

(iv) Every right R-module is a direct sum of an injective module and a small module.

(v) (a) R satisfies ACC on left annihilators, and

(b) every left R-module is singular or contains a non-zero projective direct summand.

(vi) (a) R is right Artinian, and

(b) every right R-module is small or contains a non-zero injective submodule.

Recall that our concern is with the general question of when every module in a given class **C** of modules is CS. So far we have considered different specific classes **C**, but we now will prove a result for a general class **C**. In order to prove this theorem, we shall require a number of preparatory lemmas. The first lemma follows easily from [39, Theorem 1].

Lemma 32. *Let R by any ring and let M be a finitely generated R-module such that every quotient of every cyclic submodule of M is CS. Then every quotient of M has finite uniform dimension.*

Proof. By [39, Theorem 1], every quotient of every cyclic submodule of M has finite uniform dimension. By an easy induction on the number of generators of M, we obtain that M has finite uniform dimension. Then apply the same argument to the quotients of M.

Our next lemma extends a result of Okado [34], which states that any CS-module over a left Noetherian ring is a direct sum of uniform modules. A non-empty family $A_i : i \in I\}$ of submodules of a module M is called a *local direct summand* of M if $\sum_I A_i$ is direct and $\sum_F A_i$ is a direct summand of M for any finite subset $F \subseteq I$. A module M is called *locally Noetherian* if every finitely generated submodule of M is Noetherian.

Lemma 33. *Let M be a locally Noetherian CS-module. Then every local direct summand of M is a direct summand, and M is a direct sum of uniform modules.*

Proof. Let $m \in M$. Let $\ell(m) = \{r \in R : rm = 0\}$. Then $R/\ell(m) \cong Rm$, so that $R/\ell(m)$ is Noetherian. Now apply [32, Theorem 2.17 and Proposition 2.18].

If the direct sum of two modules is quasi-continuous, then these modules are relatively injective (see, for example, [32, Proposition 2.10]). This fails for a CS direct sum, but we have the following which is still very useful when dealing with CS-modules. It could be deduced from [3, Lemma 8], but we give a short direct proof here for completeness.

Lemma 34. *Let A and B be uniform modules with local endomorphism rings such that $M = A \oplus B$ is CS. Let C be a submodule of A and let $\theta : C \to B$ be a homomorphism. Then the following hold.*

(i) If θ cannot be extended to a homomorphism from A to B, then θ is a monomorphism and B is embedded in A.

(ii) If any monomorphism $\varphi : B \to A$ is an isomorphism, then B is A-injective.

(iii) If B is not embedded in A, then B is A-injective.

Proof.

(i) Suppose θ cannot be extended to A. Let

$$U = \{(x, -\theta(x)) \ : \ x \in C\} \subseteq A \oplus B.$$

Then U is a submodule of M and clearly $U \cap B = 0$. Since M is CS, there is a direct summand U^* of M such that U is essential in U^*. By the Krull-Schmidt-Azumaya Theorem (see, for example, [2, Corollary 12.7]), we have $M = U^* \oplus A$ or $M = U^* \oplus B$.

Suppose that $M = U^* \oplus B$. Let $\pi : U^* \oplus B \to B$ be the projection. Then it is easy to see that $\pi|_A$ extends $\theta : C \to B$, a contradiction. Thus $M = U^* \oplus A$ which implies that $\theta(x) \neq 0$ for $x \neq 0$, i.e. θ is a monomorphism. Since $U^* \cap B = 0$, clearly B is embedded in A.

(ii) As in the proof of (i), given any homomorphism $\theta : C \to B$ with $C \subseteq A$, suppose that $M = U^* \oplus A$. Let $\psi : U^* \oplus A \to A$ be the projection. Then clearly $\psi|_B$ is a monomorphism (because U is essential in U^*), hence an isomorphism by the hypothesis. It follows easily that $M = U^* \oplus B$, so that, as in (i), θ can be extended to a homomorphism from A to B. It follows that B is A-injective.

(iii) Immediate by (i).

Corollary 35. *Let M be a uniserial module with unique composition series $M \supset U \supset V \supset 0$. Then $M \oplus (U/V)$ is not a CS-module.*

Proof. Clearly M and U/V have local endomorphism rings. Suppose that $M \oplus (U/V)$ is CS. Let $\pi : U \to U/V$ be the canonical homomorphism. Since π is not a monomorphism, by Lemma 34(i), π can be extended to a homomorphism $\varphi : M \to U/V$. Since U/V is simple, $\ker \varphi = U$ or M, a contradiction.

This corollary shows that the direct sum of a uniserial module of length 3 and a simple module need not be CS. However, the direct sum of a module of length 2 and a simple module is always CS. In fact, the following more general result holds. The proof uses some techniques from Kamal-Müller [29]. Recall that a family $\{M_i \ ; i \in I\}$ of modules is called *locally semi-T-nilpotent* if, for any countable set of non-isomorphisms

$$\{f_n \ : \ M_{i(n)} \to M_{i(n+1)}\}$$

with $i(n) \neq i(m)$ in I, for $n \neq m$, and for any $x \in M_i$, there exists k (depending on x) such that $f_k \dots f_2 f_1(x) = 0$ (see [23, p.174]).

For a module M of finite length, the composition length of M is denoted by length M.

Lemma 36. *Let a module $M = \oplus_{i\in I} M_i$ be a direct sum of submodules M_i ($i \in I$), each of composition length at most 2. Suppose further, that M_j is M_k-injective for all $j, k \in I$ with M_j, M_k both of length 2. Then M is a CS-module.*

Proof. First we show that every maximal uniform submodule of M is a direct summand of M. Let D be any maximal uniform submodule of M. Let $0 \neq x \in D$. Then there exists a finite subset I' of I such that $x \in \oplus_{i\in I'} M_i$. Since Rx is essential in D, it is easy to see that D can be embedded in $\oplus_{i\in I'} M_i$, and hence D is finitely generated. Thus there exists a positive integer n and $i(j) \in I$ ($1 \leq j \leq n$) such that

$$D \subseteq M_{i(1)} \oplus \cdots \oplus M_{i(n)} = N,$$

and we choose n minimal.

For each $1 \leq j \leq n$, let $\pi_j : N \to M_{i(j)}$ denote the projection. Since $\cap_{1\leq j\leq n} \ker(\pi_j|_D) = 0$ and D is uniform, without loss of generality, we can suppose that $\ker(\pi_j|_D) = 0$ and hence $D \cap (M_{i(2)} \oplus \cdots \oplus M_{i(n)}) = 0$. Thus D can be embedded in $M_{i(1)}$, so D is simple or has length 2.

Suppose first that D has length 2. Then $\pi_1(D) = M_{i(1)}$ and hence $N = D \oplus M_{i(2)} \oplus \cdots \oplus M_{i(n)}$. Now suppose that D is simple. By the choice of n, $\pi_j(D) \neq 0$ ($1 \leq j \leq n$). Suppose that there exists $1 \leq k \leq n$ such that $M_{i(k)}$ is simple. Then $\pi_k(D) = M_{i(k)}$, and hence $N = D \oplus \{\oplus_{j\neq k} M_{i(j)}\}$. Otherwise, length $M_{i(j)} = 2$ ($1 \leq j \leq n$), and, by hypothesis, N is N-injective, and hence CS [32, Proposition 2.1]. Thus, D is a direct summand of N, and hence also of M.

Now we claim that any closed submodule C of M contains a nonzero uniform direct summand of M. Indeed, there is a nonzero uniform submodule K in C. Then K has a maximal essential extension K' in C. Clearly K' is a closed submodule of C, and since C is a closed submodule of M, K' is a closed submodule of M (see [9, Proposition 2.2]). Because K' is uniform, K' is a direct summand of M, by the above argument.

Now let A be any closed submodule of M. By Zorn's Lemma, there exists a maximal local direct summand $\{A_\alpha : \alpha \in \Omega\}$ of M such that $A_\alpha \subseteq A$ and A_α is uniform for all $\alpha \in \Omega$. Since End M_i is local and length $M_i \leq 2$ for each $i \in I$, the family $\{M_i : i \in I\}$ is locally semi-T-nilpotent by [24, Lemma 12], and hence every local direct summand of M is a direct summand (see [23, Theorem 7.3.15]). Thus $\oplus_{\alpha\in\Omega} A_\alpha$ is a direct summand of M. Now $A = (\oplus_\Omega A_\alpha) \oplus B$ for some submodule B of A. If $B \neq 0$, again by [9, Proposition 2.2], B is a closed submodule of M, hence B contains a nonzero uniform direct summand A' of M.

Then $\{\{A_\alpha : \alpha \in \Omega\}, A'\}$ is a local direct summand of M, which contradicts the maximality of $\{A_\alpha : \alpha \in \Omega\}$. Thus $B = 0$, and $A = \oplus_\Omega A_\alpha$ is a direct summand of M. Therefore M is a CS-module, and the proof is complete.

For any module M, we denote by $\sigma[M]$ the full subcategory of R-Mod whose objects are all submodules of M-generated modules. In other words, $N \in \sigma[M]$ if and only if N is a submodule of a quotient of a direct sum of copies of M. It is well known that $\sigma[M]$ is a locally finitely generated Grothendieck category (see, for example, [51]).

Let **C** be a Grothendieck category. A short exact sequence

$$0 \to X \to Y \to Z \to 0$$

in **C** is called a *pure sequence* when the induced morphism $p : \mathrm{Hom}_{\mathbf{C}}(F, Y) \to \mathrm{Hom}_{\mathbf{C}}(F, Z)$ is an epimorphism for every finitely presented object F of **C**. In this case X is called a *pure subobject* of Y. An object E of **C** is called *pure-injective* when it has the injectivity property with respect to all pure sequences in **C**. A locally finitely presented Grothendieck category **C** is called *pure semisimple* if each of its objects is pure-injective. It is well known that **C** is a pure semisimple category if and only if every object of **C** is a coproduct of indecomposable Noetherian subobjects with local endomorphism rings (see [45]). The next lemma gives, for a given module M, a necessary condition for the category $\sigma[M]$ to be a pure semisimple category. (In this case, M is called a *pure semisimple* module (see [51])).

Lemma 37. *Let M be a module such that for each module X in $\sigma[M]$, every local direct summand of X is a direct summand of X. Then $\sigma[M]$ is a pure semisimple category.*

Proof. By a result of Simson [45, Theorem 1.9], a locally finitely presented Grothendieck category **C** is pure semisimple if and only if the direct sum of any family of pure-injective objects in **C** is pure-injective. Note first that, by hypothesis, $\sigma[M]$ is locally finitely presented; for, every module in $\sigma[M]$ is a direct sum of indecomposable modules, hence $\sigma[M]$ is locally Noetherian and hence locally finitely presented. Let $\{A_i : i \in I\}$ be any family of pure-injective objects in $\sigma[M]$, and let $A = \oplus_{i \in I} A_i$. We claim that A is also a pure-injective object in $\sigma[M]$.

Let P be the (categorical) direct product of $\{A_i : i \in I\}$ in $\sigma[M]$. Since $\sigma[M]$ is a Grothendieck category, P always exists and in fact is the largest submodule of the usual direct product $\Pi_{i \in I} A_i$ (in R-Mod) which belongs to $\sigma[M]$ (see, for example, [51, 15.1, 13.5]). A standard argument shows that P is also a pure-injective object in $\sigma[M]$. Clearly P contains A as a local direct summand. By hypothesis, A is a direct summand of P, hence A is pure-injective in $\sigma[M]$. Thus $\sigma[M]$ is a pure semisimple category, by [45].

We are now in a position to prove the following result. Note that if $\mathbf{C}$ is any class of R-modules which is closed under direct sums, quotients and submodules and X is the direct sum of an isomorphic copy of all cyclic modules in $\mathbf{C}$ then $\mathbf{C} = \sigma[X]$. Given a class $\mathbf{C}$ of R-modules, an R-module M will be called $\mathbf{C}$-*injective* if M is X-injective for each module X in $\mathbf{C}$.

Theorem 38. *Let R be any ring and let $\mathbf{C}$ be any class of R-modules which is closed under direct sums, quotients and submodules. Then the following statements are equivalent.*

(i) Every module M in $\mathbf{C}$ is a CS-module.

(ii) Every module M in $\mathbf{C}$ has a module decomposition $M = \oplus_{i \in I} M_i$, where each module M_i has length 2 and is $\mathbf{C}$-injective or M_i is simple.

(iii) Every (cyclic) module M in $\mathbf{C}$ is a direct sum of a $\mathbf{C}$-injective module and a semisimple module.

Proof.

(i) $\Rightarrow$ (ii). Suppose that every module M in $\mathbf{C}$ is a CS-module. We proceed in two main steps.

Step 1. First we will prove that every finitely generated module M in $\mathbf{C}$ is Noetherian.

Let $M \in \mathbf{C}$, M finitely generated. Suppose first that $\mathrm{Soc}(M)$ is 0. By Lemma 32, M is a finite direct sum of uniform modules, so without loss of generality, we may assume that M is uniform. By the definition of $\sigma[M]$, clearly $\sigma[M] \subseteq \mathbf{C}$. Let $\hat{M}$ be the injective hull of M in $\sigma[M]$; then $\hat{M} \in \sigma[M]$, $\hat{M}$ is quasi-injective and M is essential in $\hat{M}$ (see, for example, [51, 17.9]).

Let T be any simple module which is a quotient of a submodule of $\hat{M}$. Then $T \in \mathbf{C}$ and $\hat{M} \oplus T$ is CS by hypothesis. We have that $\mathrm{End}\,\hat{M}$ and $\mathrm{End}\,T$ are local, and since $\mathrm{Soc}(\hat{M}) = 0$, T is not embedded in $\hat{M}$. Thus, by Lemma 34, T is $\hat{M}$-injective. It follows that $\hat{M}$ is a V-module, so M is also a V-module. By Lemma 32, every quotient of M has finite uniform dimension. Thus M is Noetherian by [27, Corollary 3].

Now let M be any finitely generated module in $\mathbf{C}$, and suppose that M is not Noetherian. By Lemma 32, M is a finite direct sum of uniform modules. Without loss of generality, we suppose that M is uniform. By the above argument, M has a nonzero simple socle A_1. Again, if $\mathrm{Soc}(M/A_1) = 0$, then M/A_1, is Noetherian, hence M is Noetherian, a contradiction. Let A_2 be a submodule of M such that

$A_2/A_1 = \mathrm{Soc}(M/A_1)$. Then $A_2 \neq A_1$ and, by Lemma 32, M/A_1 has finite uniform dimension, so A_2 is of finite length. By induction, we obtain a strictly ascending sequence

$$0 = A_0 \subset A_1 \subset A_2 \subset \cdots \subset A_n \subset A_{n+1} \subset \cdots \subseteq M$$

with $A_{n+1}/A_n = \mathrm{Soc}(M/A_n)$ $(n \geq 0)$. Set $A = U_{n\geq 1}A_n$; then because each A_n is of finite length, A is locally Noetherian. Then every module $L \in \sigma[A]$ is locally Noetherian (see [51, 27.3]) and CS, so that, by Lemma 33, every local direct summand of L is also a direct summand. By Lemma 37, $\sigma[A]$ is a pure semisimple category, so every module in $\sigma[A]$ is a direct sum of Noetherian modules (see [45] or [51, 53.4, 53.5]). Thus, because A is uniform, A must be Noetherian. But in this case the ascending chain $0 \subset A_1 \subset A_2 \subset \cdots \subseteq A$ cannot be infinite, a contradiction. This shows that M is Noetherian.

Step 2. We will show next that every module M in $\mathbf{C}$ is a direct sum of modules of length at most 2.

Let $M \in \mathbf{C}$ and consider the category $\sigma[M] \subseteq \mathbf{C}$. By Step 1, every module $N \in \sigma[M]$ is locally Noetherian and CS, hence, by Lemma 33, every local direct summand of N is also a direct summand. Thus by Lemma 37, $\sigma[M]$ is a pure semisimple category. Every module in $\sigma[M]$ is a direct sum of indecomposable Noetherian modules with local endomorphism rings (see [45]). Since an indecomposable CS-module is uniform, it follows that $M = \oplus_{i\in I}M_i$, where each M_i is a uniform Noetherian module with End M_i local.

Next we show that every uniform module $L \in A$ is quasi-injective. By the above argument, End L is local. Let N be a countable direct sum of copies of L, i.e. $N = \oplus_{i\geq 1}L_i$, with $L_i \cong L$ for all i. Since N is locally Noetherian and CS, every local direct summand in N is a direct summand, by Lemma 33. Also, because End L_i is local for each i, the family $\{L_i : i \geq 1\}$ is locally semi-T-nilpotent (see [23, Theorem 7.3.15] or [32, Theorem 2.25]). Let $\theta : L \to L$ be any monomorphism. Suppose that θ is not an isomorphism. By the local semi-T-nilpotency of $\{L_i : i \geq 1\}$, it follows easily that, for any $x \in L$, there is a positive integer n such that $\theta^n(x) = 0$, which implies that $x = 0$, a contradiction. Thus any monomorphism $\theta : L \to L$ is an isomorphism. Since $L \oplus L$ is CS, by Lemma 34(ii) it follows that L is quasi-injective.

Now we show that the uniform module L in A is uniserial. Let A and B be any submodules of L. Since A and B have local endomorphism rings and the external direct sum $A \oplus B$ is CS, either B is A-injective or B is embedded in A, by Lemma 34(iii). If B is A-injective, then since $C = A \cap B$ is an essential submodule of A, the identity map in C can be extended to a monomorphism from A to B. Thus either A is embedded in B or B is embedded in A.

We may assume that there is a monomorphism $\varphi : A \to B$. Since L is quasi-injective, φ can be extended to a homomorphism $\psi : L \to L$, and clearly ψ is an isomorphism (see above). But A is a quasi-injective essential submodule of L, so it is well-known that $\xi(A) \subseteq A$ for any homomorphism $\xi : L \to L$. In particular, this implies $A = \psi(A) = \varphi(A) \subseteq B$. Hence L is uniserial.

We claim now that length $L \leq 2$. Since L is uniform, we know that L is Noetherian. Suppose that L is not simple, then L contains a non-zero maximal submodule L_1. If L_1 is not simple, then L_1 contains a non-zero maximal submodule L_2. Let L_3 be a (possibly zero) maximal submodule of L_2. Then L/L_3 is a uniserial module of length 3, and, by Lemma 35, the direct sum $(L/L_3) \oplus (L_1/L_2)$ is not a CS-module, a contradiction. This shows that L_1 is simple, and so length $L \leq 2$.

Thus we have shown that every module M in $\mathbf{C}$ is a direct sum of modules of length at most 2. To complete the proof of the theorem, it remains to show that if $T \in \mathbf{C}$ and length $T = 2$, then T is X-injective for any $X \in \mathbf{C}$. Indeed, X has a decomposition $X = \oplus_{\alpha \in \Omega} X_\alpha$, with length $X_\alpha \leq 2$, for each $\alpha \in \Omega$. If X_α is simple, clearly T is X_α-injective. If X_α has length 2, we consider the CS-module $T \oplus X_\alpha$. Let F and G be any direct summands of $T \oplus X_\alpha$ with $F \cap G = 0$, then by the Krull-Schmidt theorem, F and G have length 2, hence $F \oplus G = T \oplus X_\alpha$. It follows that $T \oplus X_\alpha$ is a quasi-continuous module, hence T is X_α-injective (see [32, Proposition 2.10]). Therefore T is X-injective (see, for example, [32, Proposition 1.5]), which completes the proof of (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (iii). For every module M in $\mathbf{C}$, $M = (\oplus_{i \in I} N_i) \oplus S$, where each N_i ($i \in I$) has length 2 and is $\mathbf{C}$-injective and S is semisimple. Let $N = \oplus_{i \in I} N_i$. Then for every module $X \in \mathbf{C}$, since X is locally Noetherian, N is X-injective (see, for example, [32, Theorem 1.11]). Thus N is $\mathbf{C}$-injective.

(iii) $\Rightarrow$ (i). Suppose that every cyclic module in $\mathbf{C}$ has the form $N \oplus S$, where N is $\mathbf{C}$-injective and S is semisimple. Let M be any cyclic $\mathbf{C}$-module. Every quotient of a cyclic submodule of M is a direct sum of a quasi-injective module and a simisimple module of finite length. Thus, by [26, Theorem 1.3], M has finite uniform dimension. In particular, $M = M_1 \oplus \cdots \oplus M_n$, where each M_i is cyclic indecomposable, hence each M_i is simple or uniform quasi-injective. If, for $1 \leq i \leq n$, M_i is not simple, then any cyclic proper submodule C_i in M_i is simple because C_i cannot be M_i-injective. Thus each M_i ($1 \leq i \leq n$) is either simple or of length 2, and M is Noetherian. It follows that every cyclic module in $\mathbf{C}$ is Noetherian, hence clearly every module in $\mathbf{C}$ is locally Noetherian.

Now let K be any module in $\mathbf{C}$. By Zorn's Lemma, there exists a maximal family of independent modules L_α ($\alpha \in \Omega$) in K such that L_α is $\mathbf{C}$-injective. Then $L = \oplus_{\alpha \in \Omega} L_\alpha$ is $\mathbf{C}$-injective (see [32, Theorem 1.11]). Thus $K = L \oplus T$ for some submodule T of K. Take any cyclic submodule D of T; then $D = N \oplus S$, where N is C-injective and S is semisimple. By the maximality of the family $\{L_\alpha : \alpha \in \Omega\}$, we observe that $N = 0$, whence T is semisimple. Since L is locally Noetherian and quasi-injective, by Lemma 33, $L = \oplus_{i \in I} H_i$ with each H_i $(i \in I)$ uniform. By a similar argument as above, if H_i is not simple, then every cyclic proper submodule of H_i is simple, so H_i has length 2. In this case clearly H_i is cyclic and H_i is C-injective. Thus K is a CS-module by Lemma 36.

Since the class of all singular modules over a ring R is closed under direct sums, quotients and submodules (see, for example, [20, Proposition 1.22]), Theorem 38 gives immediately the following result.

Corollary 39. *Let R be any ring. Then the following statements are equivalent.*

(i) Every singular R-module is CS.

(ii) Every singular R-module M has a decomposition $M = \oplus_{i \in I} M_i$, where each M_i is simple, or M_i has length 2 and is X-injective for each singular R-module X.

(iii) Every (cyclic) singular left R-module M has a decomposition $M = N \oplus S$, where N is X-injective for all singular left modules X, and S is semisimple.

If R is a left nonsingular ring, then the class of all singular (left) R-modules is closed under essential extensions (see, for example, [20, Proposition 1.23]). In this case, if N is a singular R-module such that N is X-injective for every singular R-module X, then N is $E(N)$-injective, and it follows that $N = E(N)$, i.e. N is an injective module. This gives at once the following result.

Corollary 40. *Let R be a left nonsingular ring. Then the following statements are equivalent.*

(a) Every singular left R-module is CS.

(b) Every singular left R-module is a direct sum of an injective module and a semisimple module.

(c) For every essential left ideal K of R, $R/K = N \oplus S$ where N is an injective module and S a semisimple module.

Example 41. Let K be any field and let n be a positive integer. Let $T_n(K)$ denote the ring of all upper triangular $n \times n$ matrices with entries in K. Then $T_n(K)$ is a (left and right) hereditary (left and right) Artinian (left and right) serial ring for every positive integer n. If $n = 2$ then $T_n(K)$ is a (left and right) SI-ring. If $n = 3$ then $T_n(K)$ is not a left SI-ring but every singular left module is CS.

Proof. It is well known that $T_n(K)$ is hereditary Artinian serial, and hence, in particular, nonsingular. If $n = 2$ then $T_n(K)$ is an SI-ring by [19, Theorem 3.11]. Now suppose that $n = 3$. Let $R = T_3(K)$ and let S denote the left socle of R. Then S consists of all matrices in R with last two rows zero, and S is obviously an essential left ideal of R. Note that $R/S \cong T_2(K)$. Thus R is not a left SI-ring, by [19, Theorem 3.11]. However, R/S is an Artinian serial ring with $J(R/S)^2 = 0$. By [25, Theorem 2.6], every (left) (R/S)-module is a direct sum of an injective module and a semisimple module and thus every (R/S)-module is a CS-module by Theorem 38. It follows that every singular R-module is CS.

Now we shall study rings for which every finitely generated (left) module is CS. By [39, Corollary 1], over such rings every finitely generated (left) module is a direct sum of uniform modules. As was remarked in [39, p. 345], the ring $\mathbf{Z}$ of integers is an example to show that the converse is false. Our aim is to give ideal-theoretic characterizations of rings whose finitely generated left modules are CS.

For the next result, we recall that a ring R with Jacobson radical $J(R)$ is called *semiregular* if $R/J(R)$ is a von Neumann regular ring and idempotents can be lifted over $J(R)$.

Theorem 42. *The following conditions are equivalent for a ring R with Jacobson radical J.*

(i) $E(_RR)$ is projective and every 2-generated left R-module is CS.

(ii) R is semiregular and every 2-generated left R-module is CS.

(iii) Every left R-module is CS.

(iv) R is (left and right) Artinian serial and $J^2 = 0$.

(v) The right-handed versions of (i), (ii) and (iii).

Proof.

(i) $\Rightarrow$ (ii). Since every cyclic left R-module is CS, by Lemma 32, there exists a complete family of orthogonal idempotents $e_1, \ldots, e_n$ of R such that each R-module Re_i is uniform. Consider the injective hull $E(Re_i)$ of Re_i; then $E(Re_i)$ is indecomposable injective and projective, hence there is an idempotent f_i of R such that $E(Re_i) \cong Rf_i$ (see, for example, [16, Theorem 20.15]). Because End (Rf_i) is local, Rf_i contains a unique maximal submodule (namely Jf_i). Thus every quotient of Rf_i is indecomposable and CS, so is uniform. Clearly Rf_i is a uniserial module. Since Re_i is isomorphic to a submodule of Rf_i, Re_i is also uniserial. Therefore R is a left serial ring, so it is well known that R is semiperfect (see, for example, [51, 55.3]). In particular R is semiregular.

(ii) $\Rightarrow$ (iv). By Lemma 32, R/J has finite uniform dimension, so R/J is semisimple. Since the idempotents are lifted over J, R is semiperfect. Thus there are orthogonal idempotents $e_1, \ldots, e_n$ of R such that each Re_i has local endomorphism ring. Since each quotient of Re_i is CS, we can apply the same argument as in the proof of (i) $\Rightarrow$ (ii) to show that each Re_i is uniserial. Thus R is a left serial ring.

Now we claim that each Re_i has nonzero socle. Suppose, on the contrary, that $\mathrm{Soc}(Re_i) = 0$ for some $i \geq 1$. Take any simple R-module U; then the 2-generated module $Re_i \oplus U$ is CS. Since U cannot be embedded in Re_i, U is Re_i-injective by Lemma 34. Thus Re_i is a V-module, so in particular Re_i has zero radical (see, for example, [51, 23.1]). Thus $Je_i = 0$ and Re_i is simple, a contradiction. Therefore $\mathrm{Soc}(Re_i) \neq 0$ for each i, so R has finitely generated essential left socle. For any two-sided ideal K of R, R/K is also a left serial ring, and it is easy to check that every 2-generated left (R/K)-module is CS, so by the above argument, R/K has finitely generated essential left socle. Then by a result of Beachy [4], it follows that R is left Artinian.

Suppose that $J^2 \neq 0$. Then there is a positive integer j such that $J^2 e_j \neq 0$. Since Re_j is uniserial, we have a composition series

$$Re_j \supset Je_j \supset J^2 e_j \supset J^3 e_j \supset \ldots$$

Then $Re_j/J^3 e_j$ is uniserial of length 3 and so, by Lemma 35, $(Re_j/J^3 e_j) \oplus (Je_j/J^2 e_j)$ is not CS, a contradiction. Thus $J^2 = 0$, and hence all $Re_1, \ldots, Re_n$ have length ≤ 2. If length $Re_k =$ length $Re_t = 2$, then since $Re_k \oplus Re_t$ is CS, it is easy to see that $Re_k \oplus Re_t$ is quasi-continuous (see the last part of the proof of Theorem 38 (i) $\Rightarrow$ (ii)), and it follows that Re_k and Re_t are relatively injective (see [32, Proposition 2.10]). Thus, it is clear that if Re_k has length 2, then Re_k is $(Re_1 \oplus \cdots \oplus Re_n)$-injective, i.e. Re_k is an injective R-module. Now by [25, Theorem 2.6] it follows that R is (left and right) Artinian serial (with $J^2 = 0$).

(iv) $\Rightarrow$ (iii). If R is (left and right) Artinian serial with $J^2 = 0$, then every R-module is a direct sum of an injective module and a semisimple module (see [25, Theorem 2.6]). Thus, by Theorem 38, every R-module is CS.

(iii) $\Rightarrow$ (i). This follows easily by Theorem 38, but we give here a short direct proof. By Lemma 32, $R = \oplus Re_i$, where each Re_i is uniform. Since every free R-module is CS, we know by Proposition 1 that every R-module is a direct sum of a projective module and a singular module. In particular, $E(Re_i)$ must be projective or singular. But $E(Re_i)$ contains the projective submodule Re_i, so clearly $E(Re_i)$ is not singular. Thus $E(Re_i)$ is projective for each i, which implies that $E({}_RR)$ is projective.

(v) $\Leftrightarrow$ (iv). Since (iv) is left-right symmetric, clearly (iv) is equivalent to the right-handed versions of (i), (ii) and (iii). This completes the proof of Theorem 42.

The ring $T_3(K)$ of upper triangular 3×3 matrices over a field K has the property that every singular module is CS (see above). However, Theorem 42 shows that not every $T_3(K)$-module is CS.

Recall that a ring R is called *left SI* if every singular left R-module is injective. If R is a domain which is left SI, then R is called a *left SI-domain.* Examples of non-Artinian simple left and right Noetherian left and right SI-domains can be found in [12]. However, it is still unknown whether a simple left Noetherian left SI-domain must be right SI.

Theorem 43. *The following statements are equivalent for a ring R.*

(i) R is left nonsingular and every 2-generated left R-module is CS.

(ii) $R = R_1 \oplus \cdots \oplus R_n$ is a direct sum of rings R_i $(1 \leq i \leq n)$, each Morita equivalent to an upper triangular 2×2 matrix ring over a division ring, or to a simple (left and right) Noetherian (left and right) SI domain.

Moreover, in this case, every finitely generated left (and right) R-module is CS.

Proof.

(i) $\Rightarrow$ (ii). Let L be any cyclic singular left R-module, and consider $M = L \oplus_R R$. Then M is a CS-module by hypothesis. Since R is left nonsingular, we have $Z_2(M) = Z(M) = L$, so by Lemma 29, L is R-injective. Thus every cyclic singular left R-module is injective, so by [39, Corollary 5] every singular left R-module is injective, i.e. R is a left SI-ring.

By [19, Theorem 3.11], there is a ring direct decomposition $R = R_0 \oplus B_1 \oplus \cdots \oplus B_m$, such that R_0/S_0 is semisimple, where S_0 is the left socle of the ring R_0, and each B_j $(1 \leq j \leq m)$ is a simple left Noetherian left SI-ring. Since every cyclic left R-module is CS, R has finite left uniform dimension by Lemma 32, whence R_0 is left Artinian. Since every 2-generated R_0-module is CS, by Theorem 42 (ii) $\Leftrightarrow$ (iv), R_0 is left and right Artinian serial and $J(R_0)^2 = 0$. Note that R_0 is left nonsingular, and all nonsingular left R_0-modules are projective, hence $R_0 = A_1 \oplus \cdots \oplus A_n$, where each A_i is Morita equivalent to a full upper triangular matrix ring T_i over a division ring D_i (see [20, Theorem 5.28]). Each indecomposable module over R_0 (and hence A_i) has length ≤ 2, thus by [20, Proposition 5.25, Theorem 5.27], it follows easily that each T_i is an upper triangular 2×2 matrix ring over D_i.

Now we consider the simple left Noetherian rings B_j $(1 \leq j \leq m)$. Choose any $1 \leq j \leq m$. Since B_j is left nonsingular and $B_j \oplus B_j$ is CS, it is easy to see that every 2-generated nonsingular left B_j-module is projective. Hence, by the proof of Theorem 5.3 in [31], the left classical quotient ring Q_j of B_j is also the right classical quotient ring of B_j, thus B_j is also a right Goldie ring ([10, Theorem 1.28]). Also, since B_j is left hereditary left Noetherian, B_j is right semihereditary ([10, Corollary 8.19]). Since B_j is left SI, B_j/K_j is semisimple for every essential left ideal K_j of B_j ([19, Proposition 3.1]), thus B_j is right Noetherian (see [12]). Sin B_j is Morita equivalent to a simple left SI-domain F_j ([19, Theorem 3.11]), it follows that F_j is right Noetherian and hence F_j is right SI by [12, Theorem 6.26].

(ii) $\Rightarrow$ (i). Suppose that $R = R_1 \oplus \cdots \oplus R_n$ where the R_i's are as in (ii). Since the property of being a left SI-ring is a Morita invariant, each R_i $(1 \leq i \leq n)$ is (two-sided) SI. Hence R is (two-sided) SI. In particular, R is left and right hereditary ([19, Proposition 3.3]). Since every (two-sided) hereditary Noetherian prime ring is CS (see [9, Proposition 6.8]), it follows that each R_i $(1 \leq i \leq n)$ is left and right CS. Hence R is (two-sided) CS.

Now let M be any finitely generated left R-module. Then $Z(M)$ is injective, so $M = Z(M) \oplus N$, and clearly N is nonsingular finitely generated. Since R is hereditary CS, by [11, Theorem 4.1], every nonsingular finitely generated left R-module is projective, hence it follows easily that every nonsingular finitely generated left R-module is CS (Proposition 1). Therefore, we have that N is a CS-module. Thus, by Lemma 29, M is CS. By symmetry, we can show that every finitely generated nonsingular right R-module is CS. This improves Theorem 43.

Corollary 44. *Let R be a commutative ring. Then the following statements are equivalent.*

(i) Every 2-generated R-module is CS.

(ii) Every R-module is CS.

(iii) $R = R_1 \oplus \cdots \oplus R_n$ is a direct sum of rings R_i $(1 \leq i \leq n)$, each a QF ring of length 2 or a field.

Proof.

(i) $\Rightarrow$ (ii). Denote by $N(R)$ the prime radical of R, and let $\overline{R} = R/N(R)$. Then $\overline{R}$ is a semiprime commutative ring, hence $\overline{R}$ is nonsingular (see, for example, [10, Lemma 1.3]). Clearly every 2-generated $\overline{R}$-module is CS. By the proof of (i) $\Rightarrow$ (ii) in Theorem 43, $\overline{R}$ is an SI-ring. Because $\overline{R}$ is commutative, it follows by [19, Theorem 3.9] that $\overline{R}$ is von Neumann regular. But every cyclic R-module is CS, so by Lemma 32, $\overline{R}$ has finite uniform dimension, hence $\overline{R}$ is semisimple.

It follows that $N(R) = J(R)$, and since the idempotents lift over $N(R)$, R is semiperfect. Therefore, by Theorem 42, every R-module is CS.

(ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (i) are immediate by Theorem 42.

BACKGROUND

Any finitely generated Abelian group (i.e. $\mathbf{Z}$-module) is a direct sum of a torsion group and a torsion-free group, i.e. for any finitely generated Abelian group A, the torsion subgroup T is a direct summand of A. Rotman [43] proved that a commutative domain R has the property that the torsion submodule of every R-module is a direct summand if and only if R is a field. In an attempt to extend Rotman's result, Cateforis and Sandomierski [6, 7] characterized commutative rings R with the property that every R-module is a direct sum of a singular module and a nonsingular module; in this case, the ring R is said to have the *splitting property* (or SP, for short). It turns out that these rings are precisely the commutative rings R for which every singular module is injective, i.e. SI-rings. We also should note here the work of Alin and Dickson [1] involving derived functors.

Goodearl [19] characterized general rings for which every singular left module is injective and called such rings *left SI-rings*. Clearly every left SI-ring has SP. Another class of rings with SP is provided by the rings R for which every nonsingular left R-module is projective. Again there is a characterization by Goodearl [19] when R is left nonsingular. For another account of this work see [20]. In both these cases of rings with SP, the rings are left hereditary. In general, if a ring R has SP for its left modules then the left global dimension of R is at most 2, but can be 2 (see [46], [19]). For a recent account of developments in this area see [47].

The relevance of these investigations to when certain projective modules are CS is given in Proposition 1. Proposition 3 is due to Chatters and Khuri [10, Theorem 2.1]. They characterize in [11, Theorem 4.1] when a left nonsingular ring R whose identity element is a sum of orthogonal primitive idempotents has the property that ${}_RR$ is finitely Σ-CS, and show that this is the case precisely when R is a left and right semihereditary left and right CS-ring, equivalently, R is a left and right semihereditary ring with a two-sided classical quotient ring which is (left and right) Artinian hereditary serial. In particular, for such a ring R, R_R is finitely Σ-CS. Proposition 6(i) is due to Gabriel [17, p.418 Théorèm 1]. Lemma 9 and Corollary 10 are due to Tercan [48], but based on work of Chatters and Khuri [11]. Theorem 18, and much of the material leading up to it, can be found in [20] (see [20, Theorems 3.9 and 5.18] in particular), and is based on [5, Theorem

2.3], [19, Theorem 2.5], [40, Théorèm 3.1] and [41, Théorème 2.7]. Theorem 21 is taken mostly from [20, Theorem 3.12] which is based on [5, Theorem 2.1], [13, Proposition 3] and [19, Corollary 2.6].

Cateforis, Sandomierski and Goodearl are also responsible for Theorems 26 and 27. Theorem 26 is taken from [20, Theorem 5.21] (see also [8, Proposition 3.2] and [19, Theorem 2.11]). Theorem 27 is [20, Theorem 5.23] which is derived from [8, Theorem 3.1, Proposition 3.2].

The work on when, for a given ring R, the left R-module R is countably Σ-CS is motivated by the corresponding results for injective modules (see [15], [16]). Lemma 20, Theorem 28 and Corollary 30 can be found in [13]. In view of Theorem 28, it is natural to ask if it is the case that rings R which do not contain an infinite set of orthogonal idempotents and for which ${}_RR$ is countably Σ-CS have the property that ${}_RR$ is Σ-CS. In other words, is Theorem 28 true without the hypothesis that R be left nonsingular. Corollary 30 represents a first step in this particular direction.

Harada and his students have studied CS-modules in detail. They use the term "extending module" where we have followed [9] and used the term "CS-module". For Harada (and others), dual CS-modules are called "lifting modules". Theorem 31 is due to Oshiro [35, 36], and these papers of Oshiro with their bibliographies give a good introduction to the work of Harada and his school on CS-modules. Leonard [30] considered small modules, and proved that a module M is small if and only if M is small in some module M', which contains it.

The rest of the discussion is taken from [14]. Corollaries 39 and 40 answer the question raised in [39, p. 351] of characterizing those rings whose singular rings are CS. Goodearl [19] characterized left SI-rings and Rizvi and Yousif [42] showed that a ring R has the property that all its singular modules are quasi-continuous if and only if all singular R-modules are semisimple. It follows by [19, Proposition 3.1] that a ring R is a left SI-ring if and only if R is left nonsingular and every singular R-module is quasi-continuous.

Now consider Theorems 42 and 43. A ring R is semiprime Artinian if and only if every cyclic module is injective ([37], [38]). In [39, Proposition 2 and Corollary 9], information is obtained about rings with the properties that every cyclic module is quasi-continuous or continuous. Now suppose that R is a ring such that every 2-generator left module is quasi-continuous. Let M be any cyclic R-module. Then, by hypothesis, $M \oplus R$ is quasi-continuous, and hence M is an injective R-module ([32, Proposition 2.10]). By the result of Osofsky mentioned above, it follows that R is semiprime Artinian. On the other hand, we do not know in general the structure of rings for which every finitely generated module is CS. If, for a given ring R, every cyclic left R-module is CS then every cyclic left R-module is a direct sum of uniform modules. It would be interesting to know when the converse is true. For example, it is certainly true for commutative rings ([10, Corollary 6.6]).

Theorem 43 (see also Theorem 27) should be compared with Goodearl's theorem which states that the following statements are equivalent for a ring R:

(i) R is left nonsingular and ${}_RR$ is Σ-CS,

(ii) R is left nonsingular and every nonsingular left R-module is projective,

(iii) $R = R_1 \oplus \cdots \oplus R_n$ is a direct sum of rings R_i $(1 \leq i \leq n)$, each Morita equivalent to an upper triangular matrix ring over a division ring, (see Proposition 1 and [20, Theorem 5.28] or [19, Theorem 2.15]).

Let R be any ring. An R-module M is called *Goldie torsion* if $M = Z_2(M)$. In [42, Theorem 3.10], Rizvi and Yousif prove that the ring R is left SI if and only if every torsion R-module is quasi-continuous. If R is left nonsingular then every torsion module is singular. Thus , for any field K, the ring $T_3(K)$ has the property that every torsion module is CS but $T_3(K)$ is not an SI-ring. Moreover, let p be any prime in $\mathbf{Z}$ and let $S = \mathbf{Z}/\mathbf{Z}p^2$. Then the ring T of all upper triangular 2×2 matrices with entries in S has the property that every singular module is CS, but the T-module T is torsion but not CS.

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Department of Mathematics
University of Glasgow
Glasgow G12 8QW
Scotland U. K.

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