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A NOTE ON THE SOLUTION OF A SPECIAL CLASS OF NONCONVEX OPTIMIZATION PROBLEMS *

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Abstract. *The purpose of this paper is to give an explicit linear programming formulation of the following nonconvex optimization problem: Minimize* $c^T z$, *s.t.* $x \in X, y \in Y, z \in S, z_i = x_i y_i$ *for all* $i = 1, \ldots, n$ *and* $\alpha \le d^T y \le \beta$, *where* X, Y *are rectangles in* $\operatorname{int}\mathbb{R}^n_+, \mathbb{R}^n_+$ *respectively;* S *a polyhedral convex set in* $\mathbb{R}^n$; c, d n*-vectors;* α, β *real numbers. The obtained linear programming problem is solved by a suitable relaxation of its constraints.*

Key words. Linear programming problem, nonconvex optimization problem, relaxation algorithm.

1. INTRODUCTION

We shall be concerned with the following nonconvex optimization problem:

$$
\begin{aligned}
(P)\quad & \text{Minimize} \quad c^T z, \\
& \text{subject to} \\
& \qquad z \in S, \\
& \qquad z_i = x_i y_i, \quad i = 1, \ldots, n,\ x \in X,\ y \in Y,\ \alpha \le d^T y \le \beta, \qquad (1)
\end{aligned}
$$

where S is a polyhedral convex set in $\mathbb{R}^n, X = \{x \in \mathbb{R}^n : 0 < a \le x \le A\}, Y = \{y \in \mathbb{R}^n : 0 \le b \le y \le B\}$; a, A, b, B, c, d n-vectors; α, β real numbers and the superscript T denotes transposition.

Problem (P) has some applications in agriculture and was studied in [2] when $d = (1, \ldots, 1)^T$. The presence of the constraints $z_i = x_i y_i$ $(i = 1, \ldots, n)$

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destroys the linearity of the problem and makes it nonconvex with respect to the variables x and y. In fact (P) can be regarded as a jointly constrained bilinear programming problem (see [1]) and has the rank n structure as defined in [4]. In general, solution techniques developed for global optimization (see e.g. [3]), although of interest by their own right, seem to be inefficient for (P).

Exploiting the special structure of the problem, in the sequel we shall show that (P) is equivalent to a linear program of the variable z with the constraint $z \in S$ and other additional constraints on z instead of (1). Also, we shall show that to solve this linear program it is not necessary to generate all its constraints in advance. Solving relaxed linear subproblems, it suffices to generate these constraints one by one, as needed in the course of computation. After obtaining an optimal solution z to the equivalent program, x and y can easily be defined from z by direct computation. Unlike our results, in [2] Problem (P) with $d = (1, \ldots, 1)^T$ was reduced to two (rather than one as in this paper) linear programming problems. The first one is given only by excursion (to define z) and the second one is to define x and y from an optimal solution z of the former. In addition, no special algorithm was proposed in [2] for solving the equivalent linear program, although it has some peculiar features.

After Introduction, we shall give, in Section 2, an explicit linear programming formulation of (P) and a relaxation algorithm will be developed in Section 3.

2. LINEAR PROGRAMMING FORMULATION OF (P)

Let us denote $I^+ = \{i : d_i \geq 0\}$ and $I^- = \{i : d_i < 0\}$. We shall assume that

$$\sum_{i \in I^+} d_i b_i + \sum_{i \in I^-} d_i B_i \leq \beta \quad \text{and} \quad \sum_{i \in I^+} d_i B_i + \sum_{i \in I^-} d_i b_i \geq \alpha. \tag{2}$$

We have the following property.

Lemma 1. *The condition (2) is fulfilled if and only if there exists at least a point $y \in Y$ satisfying $\alpha \leq d^T y \leq \beta$.*

Proof. We first suppose that (2) holds. Denote by $y^{\min}$ the vector having components $b_i (i \in I^+)$ and $B_i (i \in I^-)$, and by $y^{\max}$ the vector with components $B_i (i \in I^+)$ and $b_i (i \in I^-)$. Clearly $y^{\min} \in Y, y^{\max} \in Y$ and from (2) we obtain

$$d^T y^{\min} \leq \beta \quad \text{and} \quad d^T y^{\max} \geq \alpha.$$

If $d^T y^{\min} \geq \alpha$ or $d^T y^{\max} \leq \beta$, then we already have $y^{\min}$ or $y^{\max}$ as required. Otherwise, we have

$$d^T y^{\min} < \alpha \leq \beta < d^T y^{\max} \tag{3}$$

Let $u = (d^T y^{\max} - \beta)/(d^T y^{\max} - d^T y^{\min})$ and $v = (d^T y^{\max} - \alpha)/(d^T y^{\max} - d^T y^{\min})$. It follows from (3) that $0 \leq u \leq v \leq 1$. Take any $\lambda \in [u, v] \subset [0,1]$ and let $y = \lambda y^{\min} + (1-\lambda) y^{\max}$. Since Y is convex, we have $y \in Y$ and upon simple computation, we get $\alpha \leq d^T y \leq \beta$.

Conversely, if we have a point $y \in Y$ satisfying $\alpha \leq d^T y \leq \beta$ then (2) follows immediately from the definition of Y. The proof is complete. □

Let $Z = \{z \in \mathbb{R}^n : \ a_i b_i \leq z_i \leq A_i B_i, i = 1, \ldots, n\}$ and denote by $\mathcal{C}$ the collection of all index subsets $I \subset \{1, \ldots, n\}$ such that $d_i \neq 0$ for all $i \in I$.

Theorem 1. *Problem (P) is equivalent to the following linear program:*

(L) *Minimize $c^T z$, subject to $z \in S \cap Z$ and*

$$\sum_{i \in I^+ \cap I} \frac{d_i}{A_i} z_i + \sum_{i \in I^- \cap I} \frac{d_i}{a_i} z_i + \sum_{i \in I^+ \setminus I} d_i b_i + \sum_{i \in I^- \setminus I} d_i B_i \leq \beta, \tag{4}$$

$$\sum_{i \in I^+ \cap I} \frac{d_i}{a_i} z_i + \sum_{i \in I^- \cap I} \frac{d_i}{A_i} z_i + \sum_{i \in I^+ \setminus I} d_i B_i + \sum_{i \in I^- \setminus I} d_i b_i \geq \alpha, \tag{5}$$

for all $I \in \mathcal{C}$.

Proof. Since (P) and (L) have the same objective function, we need only to show the equivalence of their constraint sets.

Let $z \in S, z_i = x_i y_i (i = 1, \ldots, n), x \in X, y \in Y$ and $\alpha \leq d^T y \leq \beta$. Clearly $z \in Z$. For $i \in I^+$, since $(z_i/A_i) \leq (z_i/x_i) = y_i$ we have $(d_i/A_i) z_i \leq d_i y_i$. For $i \in I^-$, since $(z_i/a_i) \geq (z_i/x_i) = y_i$ we have $(d_i/a_i) z_i \leq d_i y_i$. Hence, for any $I \in \mathcal{C}$ we have

$$\sum_{i \in I^+ \cap I} \frac{d_i}{A_i} z_i + \sum_{i \in I^- \cap I} \frac{d_i}{a_i} z_i + \sum_{i \in I^+ \setminus I} d_i y_i + \sum_{i \in I^- \setminus I} d_i y_i$$

$$\leq \sum_{i \in I^+ \cap I} d_i y_i + \sum_{i \in I^- \cap I} d_i y_i + \sum_{i \in I^+ \setminus I} d_i b_i + \sum_{i \in I^- \setminus I} d_i B_i \leq \beta.$$

This shows that z satisfies all constraints in (4). By an argument similar to the previous one, we can prove that z satisfies also all constraints in (5).

Conversely, suppose now that $z \in S \cap Z$ and z satisfies all constraints in (4), (5). Denote

$$I_1 = \{i \in I^+ : \frac{z_i}{A_i} \geq b_i\}, \quad I_2 = \{i \in I^- : \frac{z_i}{a_i} \leq B_i\}.$$

We define vectors $x^1 \in X$, $y^1 \in Y$ such that $z_i = x_i^1 y_i^1$ $(i = 1, \ldots, n)$ and $d^T y^1 \leq \beta$ as follows:

a1) for $i \in I_1$ set $x_i^1 = A_i$, $y_i^1 = z_i/A_i$;
b1) for $i \in I^+ \setminus I_1$ set $x_i^1 = z_i/b_i$, $y_i^1 = b_i$ (note that $b_i > 0$ for all $i \in I^+ \setminus I_1$);
c1) for $i \in I_2$ set $x_i^1 = a_i$, $y_i^1 = z_i/a_i$;
d1) for $i \in I^- \setminus I_2$ set $x_i^1 = z_i/B_i$, $y_i^1 = B_i$ ($i \in I^- \setminus I_2$ implies $B_i > 0$).

It is easy to verify that $a_i \leq x_i^1 \leq A_i$, $b_i \leq y_i^1 \leq B_i$ and $z_i = x_i^1 y_i^1$ for all $i = 1, \ldots, n$. If $I_1 \cup I_2 = \emptyset$ then from (2) it follows that $\sum_i d_i y_i^1 \leq \beta$. Otherwise, since z satisfies the constraint (4) in particular for $I = \{i \in I_1 \cup I_2 : d_i \neq 0\}$ we have

$$\sum_{i \in I_1} \frac{d_i}{A_i} z_i + \sum_{i \in I_2} \frac{d_i}{a_i} z_i + \sum_{i \in I^+ \setminus I_1} d_i b_i + \sum_{i \in I^- \setminus I_2} d_i B_i \leq \beta$$

or, equivalently, $d^T y^1 \leq \beta$.

Similarly, we can find vectors $x^2 \in X, y^2 \in Y$ such that $z_i = x_i^2 y_i^2 (i = 1, \ldots, n)$ and $d^T y^2 \geq \alpha$. Namely, denoting

$$I_3 = \{i \in I^+ : \frac{z_i}{a_i} \leq B_i\}, \quad I_4 = \{i \in I^- : \frac{z_i}{A_i} \geq b_i\},$$

we set

a2) $x_i^2 = a_i$, $y_i^2 = z_i/a_i$ for $i \in I_3$;
b2) $x_i^2 = z_i/B_i$, $y_i^2 = B_i$ for $i \in I^+ \setminus I_3$ (note that $B_i > 0$ for all $i \in I^+ \setminus I_3$);
c2) $x_i^2 = A_i$, $y_i^2 = z_i/A_i$ for $i \in I_4$;
d2) $x_i^2 = z_i/b_i$, $y_i^2 = b_i$ for $i \in I^- \setminus I_4$ ($i \in I^- \setminus I_4$ implies $b_i > 0$).

Then, by an argument analogous to that used for the proof of Lemma 1 (with y^1, y^2 in place of $y^{\min}, y^{\max}$ respectively), we can show that there exists $\lambda \in [0, 1]$ such that $y = \lambda y^1 + (1 - \lambda) y^2 \in Y$ and $\alpha \leq d^T y \leq \beta$.

To complete the proof it remains to show that with y just obtained there exists $x \in X$ such that $z_i = x_i y_i$ for all $i = 1, \ldots, n$. To do this, observe that for every $k = 1,2 : a_i y_i^k \leq z_i = x_i^k y_i^k \leq A_i y_i^k$ for all $i = 1,\dots,n$ (since $x^k \in X, y^k \geq 0$). It follows that

$$\frac{z_i}{A_i} \leq y_i^k \leq \frac{z_i}{a_i} \text{ and, hence, } \frac{z_i}{A_i} \leq y_i \leq \frac{z_i}{a_i} \text{ for all } i = 1,\dots,n. \tag{6}$$

For each $i = 1,\dots,n$ let

$$x_i = \begin{cases} z_i/y_i & \text{if } y_i \neq 0, \\ \text{any number in } [a_i, A_i] & \text{if } y_i = 0. \end{cases} \tag{7}$$

It follows from (6), (7) that $z_i = x_i y_i$ and $a_i \leq x_i \leq A_i$ for all $i = 1,\dots,n$, i.e. $x = (x_1,\dots,x_n)^T \in X$, as was to be proved. □

3. RELAXATION ALGORITHM

Theorem 1 shows that instead of solving (P) we can solve (L). To solve (L), we shall solve its relaxed problems

(L_k) Minimize $c^T z$ subject to $z \in S \cap Z$ and $z \in D_k$, $k = 1,2,\dots$

Here D_k denotes the set of all z satisfying

$$\sum_{i\in I^+\cap I} \frac{d_i}{A_i} z_i + \sum_{i\in I^-\cap I} \frac{d_i}{a_i} z_i + \sum_{i\in I^+\setminus I} d_i b_i + \sum_{i\in I^-\setminus I} d_i B_i \leq \beta \text{ for all } I \in \mathcal{G}_k, \tag{8}$$

$$\sum_{i\in I^+\cap J} \frac{d_i}{a_i} z_i + \sum_{i\in I^-\cap J} \frac{d_i}{A_i} z_i + \sum_{i\in I^+\setminus J} d_i B_i + \sum_{i\in I^-\setminus J} d_i b_i \geq \alpha \text{ for all } J \in \mathcal{H}_k, \tag{9}$$

where $\mathcal{G}_k$, $\mathcal{H}_k$ are subcollections of $\mathcal{C}$ $(\mathcal{G}_1 = \mathcal{H}_1 = \emptyset)$.

Given any $z \in S \cap Z$ and any $I \in \mathcal{C}$, we denote by $g(I)$ and $h(I)$ the expression on the left side of (4) and (5) respectively, and let

$$K = \{i \in I^+ : d_i > 0 \text{ and } \frac{z_i}{A_i} \geq b_i\} \cup \{i \in I^- : \frac{z_i}{a_i} \leq B_i\}, \tag{10}$$

$$M = \{i \in I^+ : d_i > 0 \text{ and } \frac{z_i}{a_i} \leq B_i\} \cup \{i \in I^- : \frac{z_i}{A_i} \geq b_i\}. \tag{11}$$

Theorem 2. $g(K) = \max\{g(I) : I \in \mathcal{C}\}$ *and* $h(M) = \min\{h(J) : J \in \mathcal{C}\}$.

Proof. Direct computation shows that for all $I \in \mathcal{C}$ we have

$$g(K) - g(I) = \sum_{i \in I^+ \cap (K \setminus I)} d_i\left(\frac{z_i}{A_i} - b_i\right) + \sum_{i \in I^+ \cap (I \setminus K)} d_i\left(b_i - \frac{z_i}{A_i}\right)$$

$$+ \sum_{i \in I^- \cap (K \setminus I)} d_i\left(\frac{z_i}{a_i} - B_i\right) + \sum_{i \in I^- \cap (I \setminus K)} d_i\left(B_i - \frac{z_i}{a_i}\right).$$

From the definition of I^+, I^- and K it follows that each term on the right side of the above equality is non-negative. Thus, $g(K) \geq g(I)$ for all $I \in \mathcal{C}$.

The second assertion is proved by an argument analogous to the previous one. □

We are led to the following

Algorithm. Initialization: Solve

(L_1) Minimize $c^T z$, s.t. $z \in S \cap Z$,

obtaining an optimal solution z^1. Set $k = 1, \mathcal{G}_1 = \mathcal{H}_1 = \emptyset$.

Iteration $k \geq 1$. Define K by (10) and M by (11) with z^k in place of z. If $g(K) \leq \beta$ and $h(M) \geq \alpha$, then stop: z^k is an optimal solution to (L) (for z^k satisfies all constraints of (L) by Theorem 2 and (L_k) is a relaxed problem of (L)). Otherwise, we shall add to (L_k) the new constraint (4) with $I = K$ or (5) with $I = M$ depending on which is more violated by z^k. So we set

$$\mathcal{G}_{k+1} = \mathcal{G}_k \cup \{K\},\ \mathcal{H}_{k+1} = \mathcal{H}_k \text{ if } g(K) - \beta \geq \alpha - h(M) \text{ or}$$

$$\mathcal{G}_{k+1} = \mathcal{G}_k,\ \mathcal{H}_{k+1} = \mathcal{H}_k \cup \{M\} \text{ otherwise.}$$

Solve (L_{k+1}), obtaining an optimal solution z^{k+1} and go to iteration $k+1$.

Theorem 3. *The above algorithm is finite.*

Proof. If Algorithm does not stop at iteration k, then we have $z^k \in S \cap Z$ and $z^k \in D_k \setminus D_{k+1}$ (because z^k violates the new constraint (4) with $I = K$ or (5) with $I = M$). Therefore, no repetition occurs in the sequence

$$z^1, z^2, \ldots, z^k, \ldots$$

generated by the algorithm. Since $\mathcal{C}$ is finite, the algorithm must terminate after finitely many iterations. □

As shown in the proof of the second part of Theorem 1, when having an optimal solution z to (L) it is easy to compute two vectors $x \in X, y \in Y$, $\alpha \leq d^T y \leq \beta$ such that $z_i = x_i y_i$ for all $i = 1, \ldots, n$.

Then, by Theorem 1 (x, y, z) is an optimal solution of the original problem (P).

Remark. If $b_i > 0$ for all $i = 1, \ldots, n$, then by replacing x_i with z_i / y_i Problem (P) can be converted into the following linear programming problem of the variables z and y

$$(Q) \quad \text{Minimize} \quad c^T z,$$
$$\text{subject to}$$
$$z \in S \cap Z, y \in Y, \alpha \leq d^T y \leq \beta,$$
$$z_i - A_i y_i \leq 0 \text{ and } a_i y_i - z_i \leq 0 \text{ for all } i = 1, \ldots, n,$$

and the linear system (4), (5) along with $z \in S \cap Z$ describes the projection of the constraint set of (Q) on the z-space. However, even in this case our method should be applied because the equivalent linear program has the variables reduced by a half and can be solved by a relaxation algorithm. Furthermore, in case $b_i = 0$ for some i, replacing $b_i = 0$ by a sufficiently small positive number may make the perturbed problem unsolvable.

To illustrate the above algorithm we present the following small example:

Minimize $z_1 - 2z_2 + z_3$ $(c_1 = 1, c_2 = -2, c_3 = 1)$,

subject to

$$z \in S = \{z \in R^3 : z_1 + z_2 + z_3 \leq 4, z_1 + z_2 - 2z_3 \geq 4\},$$
$$z_i = x_i y_i, i = 1, 2, 3,$$
$$x \in X = \{x \in R^3 : 1 \leq x_i \leq 2, i = 1, 2, 3\} \ (a_i = 1, A_i = 2),$$
$$y \in Y = \{y \in R^3 : 0 \leq y_i \leq 2, i = 1, 2, 3\} \ (b_i = 0, B_i = 2),$$
$$-2 \leq y_1 - 2y_2 \leq 1 \ (d_1 = 1, d_2 = -2, d_3 = 0, \alpha = -2, \beta = 1).$$

For this problem $I^+ = \{1, 3\}, I^- = \{2\}$ and $Z = \{z \in R^3 : a_i b_i = 0 \leq z_i \leq 4 = A_i B_i, i = 1, 2, 3\}$.

Initialization. Solve

$$(L_1) \quad \text{Minimize } z_1 - 2z_2 + z_3$$
$$\text{subject to}$$
$$z_1 + z_2 + z_3 \leq 4,$$
$$z_1 + z_2 - 2z_3 \geq 4,$$
$$0 \leq z_i \leq 4, i = 1, 2, 3,$$

obtaining the optimal solution $z^1 = (0,4,0)$.

Iteration 1. From (10), (11) with z^1 in place of z we have $K = \{1\}$, $M = \{1,2\}$. From (4) with $I = K$ and from (5) with $I = M$ we obtain

$$g(K) = \frac{d_1}{A_1} z_1^1 + d_3 b_3 + d_2 B_2 = -4 \leq \beta = 1,$$

$$h(M) = \frac{d_1}{a_1} z_1^1 + \frac{d_2}{A_2} z_2^1 + d_3 B_3 = -4 \leq \alpha = -2.$$

So we add to (L_1) the new constraint (5) with $I = M$

$$\frac{d_1}{a_1} z_1 + \frac{d_2}{A_2} z_2 + d_3 B_3 \geq \alpha, \text{ i.e. } z_1 - z_2 \geq -2$$

and solve

(L_2) Minimize $z_1 - 2z_2 + z_3$

subject to

$$z_1 + z_2 + z_3 \leq 4,$$

$$z_1 + z_2 - 2z_3 \geq 4,$$

$$0 \leq z_i \leq 4, i = 1,2,3,$$

$$z_1 - z_2 \geq -2,$$

obtaining $z^2 = (1,3,0)$.

Iteration 2. With z^2 we have

$$K = \{1\},\ g(K) = \frac{d_1}{A_1} z_1^2 + d_3 b_3 + d_2 B_2 = -3.5 \leq \beta = 1,$$

$$M = \{1,2\},\ h(M) = \frac{d_1}{a_1} z_1^2 + \frac{d_2}{A_2} z_2^2 + d_3 B_3 = -2 = \alpha.$$

Stop: $z_{opt} = z^2 = (1,3,0)$ is an optimal solution. To compute x_{opt}, y_{opt} from z_{opt} we first define $y^1 \in Y$ satisfying $d^T y^1 \leq \beta$ (see the proof of Theorem 1):

$$I_1 = \{1,3\}, I_2 = \emptyset, x^1 = (2, 1.5, 2), y^1 = (0.5, 2, 0)$$

and then define $y^2 \in Y$ such that $d^T y^2 \geq \alpha$:

$$I_3 = \{1,3\}, I_4 = \{2\}, x^2 = (1,2,1), y^2 = (1, 1.5, 0).$$

Since $d^T y^2 = -2 = \alpha \leq \beta = 1$ we conclude that

$$x_{opt} = (1,2,1),\ y_{opt} = (1, 1.5, 0),\ z_{opt} = (1,3,0)$$

is an optimal solution and the optimal function value is -5.

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