Tạp chí Toán học *Journal of Mathematics* **22***(1 & 2)(1994), 55-70*

ON QUADRATIC BUNDLES OF OPERATORS, CONNECTED WITH STABILITY PROBLEMS *

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Abstract. *There is a class of mechanical problems which can be reduced to the study of the quadratic bundle of operators*

$$L(\lambda) = \lambda^2 I + \lambda(C + B) + A$$

where A, B, C are operators (bounded or unbounded), acting in the Hilbert space $\mathcal{H}$.

The purpose of this paper is to obtain sufficient conditions for operators A, B, C, under which the number of eigenvalues of the bundle $L(\lambda)$, lying in the half-plane $\mathrm{Re}\,\lambda > 0$, equals to the number of negative eigenvalues of the operator A.

The application of the obtained results to a stability problem will be considered.

1. INTRODUCTION

Let $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}_3$ be Hilbert spaces. Let $\mathcal{C}(\mathcal{H}_1, \mathcal{H}_2)$ (resp., $\mathcal{C}_\infty(\mathcal{H}_1, \mathcal{H}_2)$) denote the set of all linear closed operators, acting from $\mathcal{H}_1$ to $\mathcal{H}_2$ (resp., compact operators, acting from $\mathcal{H}_1$ to $\mathcal{H}_2$). Instead of $\mathcal{C}(\mathcal{H}, \mathcal{H})$ and $\mathcal{C}_\infty(\mathcal{H}, \mathcal{H})$ we will write $\mathcal{C}(\mathcal{H})$ and $\mathcal{C}_\infty(\mathcal{H})$ resp.. Further, $I_\mathcal{H}$ is the unit operator in $\mathcal{H}$ and $(.,.)_\mathcal{H}$, $\|.\|$ are the scalar product and norm in $\mathcal{H}$. Late on, under a word "operator" we will understand a linear operator with everywhere dense domain of definition in the corresponding Hilbert space. Let $A \in \mathcal{C}(\mathcal{H}_1, \mathcal{H}_2)$, then $\mathcal{D}(A) \subseteq \mathcal{H}_1$ denotes the definition domain of A, and ker A denotes the kernel of the the operator A.

Let $L(\lambda)$ be an operator-function defined on a subset $\mathcal{G}$ of the complex plane $\mathcal{C}$ with values in the set of closed operators having domain and range in the Hilbert space $\mathcal{H}$.

* This work is supported in part by the National Basic Research Program in Natural Sciences, Vietnam

The set of all λ in $\mathcal{G}$ for which $L(\lambda)$ is one to one and surjective is called the resolvent set of L. The set of all λ in $\mathcal{G}$ for which $L(\lambda)$ (resp., $A - \lambda I$) is not one to one and surjective is called the spectrum of L (of A) and is denoted by $\sigma(L)$ (resp., $\sigma(A)$).

Let $\mathbf{H} = \mathcal{H} \oplus \mathcal{H}$ denote the direct sum of two Hilbert spaces $\mathcal{H}$ and $\mathcal{W}_2^k[0,1]$, $k = 1, 2, \dots$ denote Sobolev spaces (see [8]).

Let A be a self-adjoint operator with discrete spectrum (i.e. there exists λ_0 such that the $(A - \lambda_0 I)^{-1}$ is a compact operator). The number λ_0 will be called an eigenvalue of the quadratic bundle $L(\lambda)$ if the equation $L(\lambda_0)y = 0$ has nontrivial solution $y_0 \in \mathcal{H}$. In this case y_0 is called an eigenvector, corresponding to eigenvalue λ_0.

Let $N(A)$ (resp., $N(L)$) denote the number of negative eigenvalues of the operator A (resp., the number of eigenvalues of the operator-function $L(\lambda)$, lying in the right half-plane $\operatorname{Re}\lambda > 0$).

Let T_1 and T be operators from $\mathcal{C}(\mathcal{H})$.

Definition 1.1. The operator $T_1 \in \mathcal{C}(\mathcal{H})$ is called relatively bounded with respect to $T \in \mathcal{C}(\mathcal{H})$ (or briefly, T_1 is T-bounded), if $\mathcal{D}(T) \subseteq \mathcal{D}(T_1)$ and there exist nonnegative numbers a and b such that

$$\|T_1 y\|^2 \leq a\|y\|^2 + b\|Ty\|^2, \quad \forall y \in \mathcal{D}(T). \tag{1.1}$$

The greatest lower bound of all possible constants b_0 in (1.1) is called the relative bound of T_1 with respect to T or simply the T-bound of T_1.

Let M, N be any two closed linear manifolds of $\mathcal{H}$. We denote by S_M the unit sphere of M (the set of all $u \in M$ with $\|u\| = 1$). We set

$$\delta(M, N) = \sup_{u \in S_M} \operatorname{dist}(u, N)$$

where

$$\operatorname{dist}(u, M) = \inf_{v \in M} \|u - v\|$$

and

$$\hat{\delta}(M, N) = \max\{\delta(M, N), \delta(N, M)\}.$$

$\hat{\delta}(M, N)$ will be called the *gap* between M, N.

Let $A_+ \in \mathcal{C}(\mathcal{H})$ denote a positive-definite operator, $A_- \in \mathcal{C}(\mathcal{H})$ denote a symmetric and nonnegative operator. We have the following simple lemma:

Lemma 1.1. *Let A_+ be a positive definite operator, A_- be a symmetric and positive operator. Also, assume A_- is A_+ - bounded with the A_+ - bound less than 1. Then the operator $A = A_+ - A_-$ is self-adjoint and semi-bounded.*

Proof. By the theorem of Rellich [5] the operator $A = A_+ - A_-$ is self-adjoint. Since A_- is A_+ - bounded, by the theorem of Heinz [5] it follows that there exist constants $\beta < 0$, $0 < \beta' < 1$ such that

$$|(A_-y, y)| \leq (-\beta)\|y\|^2 + \beta'|(A_+y, y|, \quad \forall y \in \mathcal{D}(A_+). \tag{1.2}$$

By conditions of Lemma 1.1

$$(A_-y, y) \geq 0, \quad (A_+y, y) \geq 0. \tag{1.3}$$

From (1.2) and (1.3), it follows

$$\big((A_+ - A_-)y, y\big) \geq \beta(y, y).$$

That means the operator $A = A_+ - A_-$ is semi-bounded lower and Lemma 1.1 is proved.

2. FORMULATION AND PROOF OF MAIN RESULTS

The main result of this paper is the following:

Theorem 2.1. *Assume that for the bundle $L(\lambda) = \lambda^2 I + \lambda(C + B) + A$, where A, B, $C \in \mathcal{C}(\mathcal{H})$, $\mathcal{D}(L(\lambda)) = \mathcal{D}(A)$, the following conditions are satisfied:*

1) $A = A_+ - A_-$, where A_+, A_- satisfy all the conditions, which are described in Lemma 1.1. Also, assume $\ker A = 0$ and A_+^{-1} and $A_-A_+^{-1}$ are compact operators;

2) The operator C is a positive definite and A-bounded;

3) The operator B is antisymmetric and A-bounded;

4) The operator A_+ is A-bounded;

5) The spectrum of the bundle $\mathcal{L}(\lambda, \tau) = \lambda^2 I + \lambda(\tau C + (1-\tau)A_+ + \tau B) + A$, where $0 \leq \tau \leq 1$ is discrete, consisting of isolated eigenvalues with finite algebraic multiplicity and does not have limit points on the right half-plane $\mathrm{Re}\,(\lambda) > 0$.

Then the spectrum $\sigma(L(\lambda))$ of the bundle $L(\lambda)$ does not intersect the imaginary axis and $N(L) = N(A)$, where $N(A)$ is the number of negative eigenvalues of operator A.

From the conditions of Theorem 2.1 and Lemma 1.1, it follows that the operator A is self-adjoint and semi-bounded lower. In particular, there exists constant $\beta \leq 0$ such that

$$A \geq \beta I. \tag{2.4}$$

Also, since A_+^{-1}, $A_- A_+^{-1}$ are compact operators, the spectrum A is discrete [4]. Therefore A has only a finite number of negative eigenvalues.

From the second condition of Theorem 2.1 and Lemma 1.1, it follows that C is A-bounded, so there exist positive numbers p, q such that

$$\|Cy\|^2 \leq p\|Ay\|^2 + q\|y\|^2, \quad \forall y \in \mathcal{D}(A) \subseteq \mathcal{D}(C). \tag{2.5}$$

From the third condition of Theorem 2.1, it follows that B is an A-bounded operator, so there exist numbers $\tilde{p}$, $\tilde{q}$ such that

$$\|By\|^2 \leq \tilde{p}\|Ay\|^2 + \tilde{q}\|y\|^2, \quad y \in \mathcal{D}(B) \subseteq \mathcal{D}(C). \tag{2.6}$$

Since A_- is A_+-bounded and A_+ is A-bounded, A_- is also A-bounded, that means there exist positive numbers $\hat{p}_1$, $\hat{q}_1$, $\hat{p}_2$, $\hat{q}_2$ such that

$$\|A_+y\|^2 \leq \hat{p}_1\|Ay\|^2 + \hat{q}_1\|y\|^2, \quad y \in \mathcal{D}(A) = \mathcal{D}(A_+), \tag{2.7}$$

$$\|A_-y\|^2 \leq \hat{p}_2\|Ay\|^2 + \hat{q}_2\|y\|^2, \quad y \in \mathcal{D}(B) \subseteq \mathcal{D}(C). \tag{2.8}$$

Definition 2.1. The domain Ω of the complex plane C bounded by a simple-connected, closed, rectifiable curve will be called normal for the bundle $L(\lambda)$ if

1) $\partial\Omega \cap \sigma(L(\lambda)) = \emptyset$.

2) The whole spectrum of the bundle $L(\lambda)$ inside Ω consists of a finite number of isolated eigenvalues with finite algebraic multiplicity.

Now we shall prove some lemmas.

Lemma 2.1. *Let the conditions of Theorem 2.1 be satisfied, then the bundle $\mathcal{L}(\lambda, \tau)$ has not eigenvalues on the imaginary axis.*

Proof. Assume to the contrary that $\lambda_0 = i\varsigma_0$ is an eigenvalue of the bundle $\mathcal{L}(\lambda, \tau)$ and y_0 is the corresponding eigenvector. Then we have

$$(\mathcal{L}(\lambda, \tau)y_0, y_0) = 0.$$

It follows
$$\operatorname{Im}\left(\mathcal{L}(\lambda,\tau)y_0,y_0\right)=0$$
or
$$\varsigma[\tau(Cy_0,y_0)+(1-\tau)(A_+y_0,y_0)]=0. \tag{2.9}$$
But since $C\geq\alpha I$, $\alpha>0$ and A_+ is positive definite, the numbers τ, $1-\tau$ are nonnegative and are not simultaneously equal to zero, therefore from (2.9), it follows $\varsigma_0=0$ and Lemma 2.1 is proved.

Corollary 2.1. *Let the conditions of Theorem 2.1 be satisfied, then the spectrum $\sigma(L(\lambda))$ does not intersect the imaginary axis.*

Proof. In fact, by putting $\tau=1$ into the bundle $\mathcal{L}(\lambda,\tau)$, we obtain the corollary.

Lemma 2.2. *Let the conditions of Theorem 2.1 be satisfied, then there exists a simple connected, bounded, rectifiable curve γ, enclosing all eigenvalues of the bundle $\mathcal{L}(\lambda,\tau)$, lying in the right half-plane* $\operatorname{Re}\lambda>0$.

Proof. Let λ_0 be an eigenvalue of the bundle $\mathcal{L}(\lambda,\tau)$, lying in the right half-plane $\operatorname{Re}\lambda>0$, and y_0 the corresponding vector to λ_0. Without loss of generality, we suppose that $\|y_0\|=1$. Then we have
$$\begin{aligned}0=\operatorname{Re}\left(\mathcal{L}(\lambda_0,\tau)y_0,y_0\right)&=(x^2-z^2)+x\tau(Cy_0,y_0)+x(1-\tau)(A_+y_0,y_0)\\&\quad+zi\tau(By_0,y_0)+(Ay_0,y_0),\end{aligned} \tag{2.10}$$
$$\begin{aligned}0=\operatorname{Im}\left(\mathcal{L}(\lambda_0,\tau)y_0,y_0\right)&=2xz+x\tau i(By_0,y_0)+z\tau(Cy_0,y_0)\\&\quad+z(1-\tau)(A_+y_0,y_0)\end{aligned} \tag{2.11}$$
where $\lambda_0=x+iz$, $x>0$.

We put $c=(Cy_0,y_0)$, $b=i(By_0,y_0)$, $a=(Ay_0,y_0)$, $a_+=(A_+y_0,y_0)$. Then from the conditions of Theorem 2.1 and (2.4), it follows $c>0$, $a\geq\beta$, $a_+>0$ and b is a real number. From (2.11), it follows
$$z=\frac{x\tau b}{2x+\tau c+(1-\tau)a_+}\,. \tag{2.12}$$
Substituting (2.12) into (2.10) we obtain
$$x^2-\left(\frac{x\tau b}{2x+\tau c+(1-\tau)a_+}\right)^2+\tau b\frac{x\tau b}{2x+\tau c+(1-\tau)a_+}+a=0. \tag{2.13}$$

Using the conditions $c > 0$, $a \geq \beta$, $\tau \geq 0$, $1 - \tau \geq 0$, by simple calculations, we have

$$x^2 + a \leq 0, \tag{2.14}$$

which means

$$x \leq \sqrt{|\beta|}. \tag{2.15}$$

From (2.13), it also follows that

$$z^2 = \frac{x^2\tau^2b^2}{(2x + \tau c + (1-\tau)a_+)^2} \leq -\beta$$

or

$$|z| \leq \sqrt{|\beta|}. \tag{2.16}$$

From (2.15) and (2.16), it follows that all eigenvalues of the bundle, lying in the right half-plane $\operatorname{Re}\lambda > 0$ will locate inside the rectangle $0 < x < \sqrt{|\beta|} + 1$, $-\sqrt{|\beta|} - 1 < x < \sqrt{|\beta|} + 1$. Lemma 2.2 guarantees that bundle's $\mathcal{L}(\lambda, \tau)$ eigenvalues lying in the right half-plane $\operatorname{Re}\lambda > 0$ never cross on left half-plane through imaginary axis for all values τ, $0 \leq \tau \leq 1$. This completes the proof of Lemma 2.2.

From Lemma 2.2, it follows two consequences.

Corollary 2.2. *Let the conditions of Theorem 2.1 be satisfied and assume* $A \geq \beta I$, *where* $\beta > 0$, *then the bundle* $\mathcal{L}(\lambda, \tau)$ *has not eigenvalues in the right half-plane* $\operatorname{Re}\lambda > 0$.

Proof. Assume that $\beta > 0$ and the bundle $\mathcal{L}(\lambda, \tau)$ has eigenvalue $\lambda = x + iz$ with $x > 0$. Repeating the above argument, from (2.14) we get

$$x^2 + a \leq 0$$

or

$$x^2 \leq -a \leq -\beta < 0,$$

a contradiction.

Corollary 2.3. *Let the conditions of Theorem 2.1 be satisfied, then the bundle* $\mathcal{L}(\lambda, \tau)$ *has only a finite number of eigenvalues, in the right half-plane which can be enclosed by a simple connected, bounded, closed, rectifiable curve.*

Proof. Obvious.

We connect the bundle $\mathcal{L}(\lambda,\tau)$ with its linearized bundle

$$\mathcal{G}_\tau(\lambda)=\begin{Bmatrix} -(\tau C+(1-\tau)A_+ +\tau B) & I \\ -A & 0 \end{Bmatrix} - \lambda \begin{Bmatrix} I & 0 \\ 0 & I \end{Bmatrix}.$$

where $\mathcal{D}(\mathcal{G}_\tau(0))=\mathcal{D}(A)\oplus\mathcal{H}$.

It is not difficult to prove that $\sigma(\mathcal{L}(\lambda,\tau))\equiv\sigma(\mathcal{G}_\tau(0))$, moreover the geometric and algebraic multiplicities coincide.

From Lemma 2.1, it follows that $\sigma(\mathcal{G}_\tau(0))$ may be separated into two isolated parts by the curve γ : $\sigma'(\mathcal{G}_\tau(0))$ lies in the right half-plane, $\sigma''(\mathcal{G}_\tau(0))$ lies in the left half-plane, moreover $\sigma'(\mathcal{G}_\tau(0))$ is the finite part of the spectrum, by Corollary 2.2.

Let M be a subspace, corresponding to the finite part of the spectrum $\sigma'(\mathcal{G}_\tau(0))$, i.e. $M=P\mathcal{H}$, where

$$P=-\frac{1}{2\pi i}\int_\gamma R(\varsigma)d\varsigma, \qquad R(\varsigma)=(\mathcal{G}_\tau(0)-\lambda I)^{-1}.$$

Now we shall prove the following

Proposition 2.1. *Let the conditions of Theorem 2.1 be satisfied. Then the equality $N(\mathcal{L}(\lambda,1))=N(\mathcal{L}(\lambda,0))$ holds.*

Proof. First, we shall prove that, if the domain Ω bounded by the curve γ is normal for $\mathcal{G}_{\tau_0}(0)$ then there exists neighbourhood $|\tau-\tau_0|<\varepsilon$ such that in this neighbourhood, Ω remains normal for $\mathcal{G}_\tau(0)$ and the number of eigenvalues lying inside Ω remains as previous.

By Theorem 3.16 from [5] there exists $\delta>0$ on $\mathcal{G}_{\tau_0}$ and γ such that for all τ, satisfying $\hat{\delta}(\mathcal{G}_\tau,\mathcal{G}_{\tau_0})<\delta$, we have $\dim M(\mathcal{G}_{\tau_0})=\dim M(\mathcal{G}_\tau)$, where we write $\mathcal{G}_\tau$ and $\mathcal{G}_{\tau_0}$ instead of $\mathcal{G}_\tau(0)$ and $\mathcal{G}_{\tau_0}(0)$, for simplicity.

By the definition

$$\hat{\delta}(\mathcal{G}_\tau,\mathcal{G}_{\tau_0})=\max\,[\delta(\mathcal{G}_\tau,\mathcal{G}_{\tau_0}),\delta(\mathcal{G}_{\tau_0},\mathcal{G}_\tau)],$$

where

$$\delta(\mathcal{G}_\tau,\mathcal{G}_{\tau_0})=\sup_{\tilde{u}\in S_{\Gamma(\mathcal{G}_\tau(0))}}\ \inf_{\tilde{v}\in S_{\Gamma(\mathcal{G}_{\tau_0}(0))}}\|\tilde{u}-\tilde{v}\|_{\mathbf{H}},\quad \tilde{u}=\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix},\quad \tilde{v}=\begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix}.$$

We have

$$\delta(\mathcal{G}_\tau, \mathcal{G}_{\tau_0}) = \sup_{\tilde{u}\in S_{\Gamma(\mathcal{G}_\tau(0))}} \inf_{\tilde{v}\in S_{\Gamma(\mathcal{G}_{\tau_0}(0))}} \{\|u_1 - v_1\|^2 + \|u_2 - v_2\|^2 + \| - (\tau C$$
$$+ (1-\tau)A_+ \tau B)u_1 + u_2 + (\tau_0 C + (1-\tau_0)A_+ + \tau_0 B)v_1 - v_2\|^2 + \| - Au_1$$
$$+ Av_1\|^2\}^{1/2} \le \sup_{\tilde{u}\in S_{\Gamma(\mathcal{G}_\tau(0))}} \{3|\tau - \tau_0|^2(\|Cu_1\|^2\|A_+u_1\|^2 + \|Bu_1\|^2)\}^{1/2}. \tag{2.17}$$

where $\Gamma(\mathcal{G}_\tau(0))$ denotes the graph of the operator $\mathcal{G}_\tau(0)$ and $\Gamma(\mathcal{G}_{\tau_0}(0))$ denotes the graph of the operator $\mathcal{G}_{\tau_0}(0)$ (we put $u_1 = v_1$, $u_2 = v_2$ and using the inequality $\|Cu_1 - A_+u_1 + Bu_1\|^2 \le 3(\|Cu_1\|^2 + \|A_+u_1\|^2 + \|Bu_1\|^2)$).

The condition $u \in S_{\Gamma(\mathcal{G}_\tau(0))}$ means

$$\|u_1\|^2 + \|u_2\|^2 + \| - (\tau C + (1-\tau)A_+ + \tau B)u_1 + u_2\|^2 + \|Au_1\|^2 = 1. \tag{2.18}$$

From (2.18) it follows

$$\|u_1\|^2 \le 1, \quad \|Au_1\|^2 \le 1. \tag{2.19}$$

From (2.5) and (2.19), it follows

$$\|Cu_1\|^2 \le (p+q). \tag{2.20}$$

Similarly from (2.6), (2.7) and (2.19) one gets

$$\|By\|^2 \le \tilde{p} + \tilde{q}, \quad \|A_+y\|^2 \le \hat{p}_1 + \hat{q}_1. \tag{2.21}$$

Therefore, by (2.17), (2.20), (2.21), we have

$$\delta(\mathcal{G}_\tau, \mathcal{G}_{\tau_0}) \le \{3|\tau - \tau_0|^2(p + q + \tilde{p} + \tilde{q} + \hat{p}_1 + \hat{q}_1)\}^{1/2}. \tag{2.22}$$

Putting $a = \sqrt{3}(p + q + \tilde{p} + \tilde{q} + \hat{p}_1 + \hat{q}_1)^{1/2}$, we can rewrite (2.22) as

$$\delta(\mathcal{G}_\tau, \mathcal{G}_{\tau_0}) \le a|\tau - \tau_0|. \tag{2.23}$$

Similarly, we have

$$\delta(\mathcal{G}_{\tau_0}, \mathcal{G}_\tau) \le a|\tau - \tau_0|. \tag{2.24}$$

Hence

$$\hat{\delta}(\mathcal{G}_\tau, \mathcal{G}_{\tau_0}) \le a|\tau - \tau_0|.$$

For $|\tau - \tau_0|$ sufficiently small, $a|\tau - \tau_0| \leq \delta$, where δ is a number that mentioned in Theorem 3.16 (see [5]). According to this theorem, for $\tau \in \left(\tau_0 - \frac{a}{\delta}, \tau_0 + \frac{a}{\delta}\right)$, we have $\dim M(\mathcal{G}_{\tau 0}) = \dim M(\mathcal{G}_\tau)$. It follows that the domain bounded by the curve γ is either normal range or is not normal range for the bundle $\mathcal{L}(\lambda, \tau)$ for all τ, $\tau \in \left(\tau_0 - \frac{a}{\delta}, \tau_0 + \frac{a}{\delta}\right)$ depending on the condition : either indicated range is or is not normal for $\tau = \tau_0$. That means, range bounded by the curve γ either is normal for the bundle $\mathcal{L}(\lambda, \tau)$ for all $\tau \in [0, 1]$, or is not normal for the all $\tau \in [0, 1]$. But range bounded by the curve γ is normal for the bundle $\mathcal{L}(\lambda, \tau)$ for all $\tau \in [0, 1]$ (see Corollary 2.3). So from Theorem 3.16 [5] we obtain an equality $N(\mathcal{L}(\lambda, 1)) = N(\mathcal{L}(\lambda, 0))$. Proposition 2.1 is proved.

We introduce the following bundle : $L_1(\lambda, \eta) = \lambda^2 I + \lambda(A + \eta A_-) + A$. From Proposition 2.1 it follows

Corollary 2.4. *Let the conditions of Theorem 2.1 be satisfied, then the equality* $N(L(\lambda)) = N(L_1(\lambda, 1))$ *holds.*

Proof. In fact $\mathcal{L}(\lambda, 0) = L_1(\lambda, 1)$, $\mathcal{L}(\lambda, 1) = L(\lambda)$. From Proposition 2.1 we obtain the equality $N(L(\lambda)) = N(L_1(\lambda, 1))$.

Lemma 2.3. *Let the conditions of Theorem 2.1 be satisfied, then for all* η, $0 \leq \eta \leq 1$, *the bundle's spectrum* $L_1(\lambda, \eta)$ *lying in the right half-plane* $\operatorname{Re}\lambda > 0$ *is real.*

Proof. Suppose that $\lambda_0 = x + iz$, $x > 0$ is an eigenvalue of bundle $L_1(\lambda, \eta)$ and y_0 is the corresponding eigenvector with $\|y_0\| = 1$. By the assumptions

$$\operatorname{Re}(L_1(\lambda_0, \eta)y_0, y_0) = 0, \quad \operatorname{Im}(L_1(\lambda_0, \eta)y_0, y_0) = 0,$$

it follows

$$x^2 - z^2 + x((A + \eta A_-)y_0, y_0) + (Ay_0, y_0) = 0, \tag{2.25}$$

$$2xz + zx((A + \eta A_-)y_0, y_0) = 0. \tag{2.26}$$

Suppose that $z \neq 0$. Then (2.26) implies

$$((A + \eta A_-)y_0, y_0) = -2x < 0, \tag{2.27}$$

and (2.25) and (2.27) yield

$$(Ay_0, y_0) = x^2 + z^2. \tag{2.28}$$

Therefore

$$((A+\eta A_-)y_0, y_0) > 0, \tag{2.29}$$

contradicting (2.27). It follows that $z = 0$ and the proof is complete.

Lemma 2.4. *Let the conditions of Theorem 2.1 be satisfied. Assume also that λ_0 is an eigenvalue of the bundle $L_1(\lambda_0, \eta)$, $\lambda_0 > 0$ and y_0 is the corresponding eigenvector. Then y_0 has not an adjoint vector.*

Proof. By condition of the lemma we have

$$L_1(\lambda_0, \eta) y_0 = 0,$$

which means

$$\lambda_0^2 I + \lambda(A+\eta A_-)y_0 + Ay_0 = 0 \tag{2.30}$$

Assume that y_1 is an adjoint vector to y_0. That means

$$\frac{\partial L_1}{\partial \lambda}\Big|_{\lambda=\lambda_0} y_0 + L_1(\lambda_0, \eta) y_1 = 0$$

or

$$(2\lambda_0 I + (A+\eta A_-))y_0 + \lambda_0^2 y_1 + \lambda_0(A+\eta A_-)y_1 + Ay_1 = 0. \tag{2.31}$$

By multiplying the both parts of (2.31) by y_0 and using the fact that λ_0 is real and A_- is a self-adjoint operator we obtain, by (2.30),

$$2\lambda_0(y_0, y_0) + ((A+\eta A_-)y_0, y_0) = 0. \tag{2.32}$$

From (2.30) and (2.32) it follows

$$(Ay_0, y_0) = \lambda_0^2(y_0, y_0). \tag{2.33}$$

From (2.30) and (2.33) we obtain

$$\lambda_0^2(y_0, y_0) + \lambda_0^3(y_0, y_0) + \lambda_0 \eta A_-(y_0, y_0) + \lambda_0^2(y_0, y_0) = 0.$$

Since $\lambda_0 > 0$, $\eta \geq 0$, $(A_- y_0, y_0) \geq 0$, the latest equality is impossible proving the lemma.

Proposition 2.2. *Let the conditions of Theorem 2.1 be satisfied. Then the inequality $N(L_1(\lambda,1)) \leq N(L_1(\lambda,0))$ holds.*

Proof. We consider the bundle $L_1(\lambda,\eta) = \lambda^2 I + \lambda(A+\eta A_-) + A$. Let $\eta \in (0,1)$ and λ_0 be an eigenvalue of the bundle $L_1(\lambda,\eta)$, λ_0 lying in the right half-plane. By Lemma 2.4, λ_0 lies on the positive semi-axis, so $\lambda_0 > 0$. Assume that y_0 is the corresponding eigenvector. By Lemma 2.4, y_0 has not an adjoint vector. Then by [3] for small perturbation $|\eta-\eta_0|$, we shall have the following formulas for bundle's $L_1(\lambda,\eta)$ eigenvalue $\lambda(\eta)$ and eigenvector $y(\eta)$:

$$\lambda(\eta) = \lambda_0 + \sum_{j=1}^{\infty} (\eta-\eta_0)^j \lambda_j, \tag{2.34}$$

$$y(\eta) = y_0 + \sum_{j=1}^{\infty} (\eta-\eta_0)^j y_j. \tag{2.35}$$

By substituting (2.34) and (2.35) into the equation $L_1(\lambda,\eta)y = 0$, we obtain the following identies

$$\lambda_0^2 I + \lambda_0(A+\eta A_-)y_0 + Ay_0 = 0, \tag{2.36}$$

$$\lambda_0^2 y_1 + 2\lambda_0\lambda_1 y_0 + \lambda_0 A y_1 + \lambda_0 A y_0 + \lambda_0\eta_0 A_- y_1 + \lambda_1 A y_0 + \lambda_1\eta_0 A_- y_0 + Ay_1 = 0. \tag{2.37}$$

By multiplying both parts of (2.37) by y_0 and using (2.36) we obtain

$$\lambda_1[2\lambda_0(y_0,y_0) + (Ay_0,y_0) + \eta_0(A_-y_0,y_0)] = -\lambda_0(A_-y_0,y_0). \tag{2.38}$$

From (2.36) it follows

$$\lambda_0^2(y_0,y_0) + \lambda_0((A+\eta_0 A_-)y_0,y_0) + (Ay_0,y_0) = 0.$$

Therefore

$$(Ay_0,y_0) = \frac{-\lambda_0^2(y_0,y_0) - \lambda_0\eta_0(A_-y_0,y_0)}{\lambda_0+1}. \tag{2.39}$$

From (2.38) and (2.39) it follows

$$\lambda_1 = \frac{-\lambda_0(\lambda_0+1)(A_-y_0,y_0)}{\lambda_0^2(y_0,y_0) + 2\lambda_0(y_0,y_0) + \eta_0(A_-y_0,y_0)}. \tag{2.40}$$

But A_- is a nonnegative and symmetric operator, $\lambda_0 > 0$, $\eta_0 > 0$, so from (2.40), we get $\lambda_1 \leq 0$. This shows that, when η_0 increasing to $\eta_0+\varepsilon$, (where ε is a small positive number) eigenvalues of the bundle $L_1(\lambda,\eta)$ lying in the positive semi-axis move to the left. From this fact we obtain an inequality $N(L_1(\lambda,1)) \leq N(L_1(\lambda,0))$. Proposition 2.2 is proved.

From the inequality (2.8) and inequality Heinz [5] it follows that there exist positive numbers $\tilde{p}_3$ and $\tilde{q}_3$ such that

$$|(A_-y,y)| \leq \tilde{p}_3|(Ay,y)| + \tilde{q}_3\|y\|^2, \tag{2.41}$$

Since $(A_-y,y) \geq 0$, (2.41) can be rewritten as follows

$$(A_-y,y) \leq \tilde{p}_3|(Ay,y)| + \tilde{q}_3\|y\|^2. \tag{2.42}$$

Lemma 2.5. *Let the conditions of Theorem 2.1 be satisfied. Then the inequality*

$$(A_-y,y) \leq \tilde{p}_3(Ay,y) + (\tilde{q}_3 + 2|\beta|)\|y\|^2, \quad y \in \mathcal{D}(A) \tag{2.43}$$

holds.

Proof. First, we shall prove that

$$|(Ay,y)| \leq (Ay,y) + 2|\beta|\|y\|^2, \quad y \in \mathcal{D}(A). \tag{2.44}$$

To this end, it suffices to prove that

$$(Ay,y) \leq (Ay,y) + 2|\beta|\|y\|^2, \quad \forall y \in \mathcal{D}(A) \tag{2.45}$$

and

$$-(Ay,y) \leq (Ay,y) + 2|\beta|\|y\|^2. \tag{2.46}$$

The inequality (2.46) is obvious. The inequality (2.45) is equivalent to the following inequality

$$(Ay,y) \geq -|\beta|\|y\|^2.$$

But the above mentioned inequality is the same as (2.4). Therefore, from (2.42) and (2.44), we get (2.43) and Lemma 2.5 is proved.

We introduce the following bundle

$$L_2(\lambda,\varsigma) = \lambda^2 I + A + \lambda[A(1+(1-\varsigma))\tilde{p}_3 + (1-\varsigma)(\tilde{q}_3 + 2|\beta|\tilde{p}_3)I + \varsigma A_-].$$

Remark that

$$L_2(\lambda,1) = L_1(\lambda,1), \quad L_2(\lambda,0) = \lambda^2 I + A + \lambda[A(1+\tilde{p}_3) + (\tilde{q}_3 + 2|\beta|\tilde{p}_3)I].$$

Similarly, we obtain following lemmas

Lemma 2.6. *Let the conditions of Theorem 2.1 be satisfied, then for all* ς, $0 \le \varsigma \le 1$, *the spectrum of the bundle* $L_2(\lambda,\varsigma)$, *lying in the right half-plane* $\mathrm{Re}\,\lambda > 0$ *is real.*

Lemma 2.7. *Let the conditions of Theorem 2.1 be satisfied. Assume that* λ_0 *is an eigenvalue of the bundle* $L_2(\lambda_0,\varsigma)$, $\lambda_0 > 0$, *and* y_0 *is the corresponding eigenvector. Then* y_0 *has not an adjoint vector.*

On the basis of Lemmas 2.5 - 2.7 we can prove the following statement:

Proposition 2.3. *Let the conditions of Theorem 2.1 be satisfied, then the inequality* $N(L_2(\lambda,0)) \le N(L_2(\lambda,1))$ *holds.*

It is easy to see that $L_1(\lambda,0) = \lambda^2 I + A$, $L_1(\lambda,1) = \lambda^2 I + \lambda(A + A_-) + A$, $L_2(\lambda,1) = L_1(\lambda,1)$, $L_2(\lambda,0) = \lambda^2 I + A + \lambda[A(1+\tilde{p}_3) + (\tilde{q}_3 + 2|\beta|\tilde{p}_3)I]$.

From Propositions 2.2 and 2.3 it follows

$$N(L_2(\lambda,0)) \le N(L_2(\lambda,1)) = N(L_1(\lambda,1)) \le N(L_1(\lambda,0)).$$

It is not difficult to see that

$$N(L_1(\lambda,0)) = N(A), \tag{2.47}$$
$$N(L_2(\lambda,0)) = N(A). \tag{2.48}$$

Proof of Theorem 2.1. By Proposition 2.1 $N(\mathcal{L}(\lambda,1)) = N(\mathcal{L}(\lambda,0))$, where $\mathcal{L}(\lambda,\tau) = \lambda^2 I + \lambda(\tau C + (1-\tau)A_+ + \tau B) + A$. We have $\mathcal{L}(\lambda,1) = L(\lambda)$, $\mathcal{L}(\lambda,0) = L_1(\lambda,1) = L_2(\lambda,1)$. But $N(L_1(\lambda,0)) = N(L_2(\lambda,0)) = N(A)$. Thus the first part of Theorem 2.1 is proved. The assertion that the bundle $L(\lambda)$ has not eigenvalue on imaginary axis follows from Corollary 2.1. The proof of the theorem is complete.

3. APPLICATION TO STABILITY PROBLEM

Let us consider a system consisting of an uniform pipe of length L, flexural rigidity, EI, and mass per unit length ρ, conveying fluid of mass, m, per unit length at a constant velocity U. Let ν be coefficient of internal friction and c be coefficient of external friction. The equation of motion for the vibration of small amplitude can be expressed as follows (see [2], [7])

$$EI\frac{\partial^4 y}{\partial x^4} + \nu\frac{\partial^5 y}{\partial x^4 \partial t} + c\frac{\partial y}{\partial t} + mU^2\frac{\partial^2 y}{\partial x^2} + 2mU\frac{\partial^2 y}{\partial x \partial t} + (m+\rho)\frac{\partial^2 y}{\partial x^2} = 0. \qquad (3.1)$$

The initial conditions (at $t = 0$) are assumed to be

$$y = G(x),$$
$$\frac{\partial y}{\partial x} = H(x).$$

The boundary conditions (at $x = 0$ and $x = L$) for various support configurations are simply-supported ends, i.e.

$$y = 0,$$
$$\frac{\partial^2 y}{\partial x^2} = 0.$$

For convenience, we introduce the following dimensionless coordinates and parameters:

$$\varsigma = \frac{x}{L}\,,\ \eta = \frac{y}{L}\,,\ \tau = \Big(\frac{EI}{m+\rho}\Big)^{1/2}\frac{t}{L^2}\,,\ v = \Big(\frac{m}{EI}\Big)^{1/2} UL,\ \beta = \Big(\frac{m}{m+\rho}\Big)^{1/2}. \qquad (3.2)$$

Substituting equation (3.1) into equation (3.2) yields the equation of motion as follows

$$\frac{\partial^4 \eta}{\partial \varsigma^4} + \alpha\frac{\partial^5 \eta}{\partial \varsigma^4 \partial \tau} + k\frac{\partial \eta}{\partial \tau} + v^2\frac{\partial^2 \eta}{\partial \varsigma^2} + 2\beta v\frac{\partial^2 \eta}{\partial \varsigma \partial \tau} + \frac{\partial^2 \eta}{\partial \tau^2} = 0, \qquad (3.3)$$

where

$$\alpha = \frac{\nu}{L^2}\Big[\frac{EI}{m+\rho}\Big]^{1/2}, \quad k = cL^2\frac{1}{[EI(m+\rho)]^{1/2}}.$$

The dimensionless initial conditions at $\tau = 0$ are

$$\eta = g(\varsigma),$$
$$\frac{\partial \eta}{\partial \tau} = h(\eta). \qquad (3.4)$$

The dimensionless boundary conditions at $\varsigma = 0$ and $\varsigma = 1$ become

$$\eta = 0, \qquad \frac{\partial^2 \eta}{\partial \varsigma^2} = 0. \tag{3.5}$$

The parameters α, k, β, v are nonnegative. We consider the case when $\alpha + k > 0$. In order to investigate solution's stability in the Liapunov's sense for equations (3.3)-(3.4) we put $\eta(\varsigma, \tau) = e^{\lambda\tau} y(\varsigma)$ into the equation (3.3) and study spectrum's location of the following spectral problem

$$y^{iv} + v^2 y'' + \lambda(\alpha y^{iv} + 2\beta v y' + ky) + \lambda^2 y = 0, \tag{3.6}$$

$$y(0) = y(1) = y''(0) = y''(1) = 0. \tag{3.7}$$

If the whole spectrum of Problems (3.6), (3.7) is located in the left half-plane then the solution of Equations (3.3)-(3.5) will be stabilized by Liapunov. If there exists a number λ such that $\operatorname{Re}\lambda > 0$ then Equation's (3.3)-(3.5) solution will be unstabilized.

We introduce the Hilbert space $\mathcal{H} = \mathcal{L}_2[0,1]$ and operators acting in $\mathcal{H}$ as follows:

$$A_+ y = y^{iv}, \quad A_- y = v^2 y'', \quad B = 2\beta v y', \quad y \in \mathcal{D}(A_+),$$

where

$$\mathcal{D}(A_+) = \{y | y \in W_2^4,\ y(0) = y(1) = y''(0) = y''(1) = 0\}.$$

It is easy to verify that A_+, A_-, B satisfy all the conditions in Theorem 2.1. Therefore, we obtain the following result.

Theorem 3.1. *Suppose that $\alpha + k > 0$, $v \neq \pi n$, then the spectrum of the problem (3.6), (3.7) does not intersect the imaginary axis and we have $N(P) = \left[\frac{v}{\pi}\right]$, where $N(P)$ is the number of nonnegative eigenvalues of spectral problem (3.6), (3.7).*

Corollary 3.1. *For $v > \pi$ and $v \neq \pi n$, the problem (3.3)-(3.5) is unstabilized.*

Some results of this paper have been published in [1].

Acknowledgement. The author wishes to thank Prof. A. G. Kostyuchenko and Prof. A. S. Bratus for the discussions concerning this paper.

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Received April 20, 1993
Revised November 29, 1993