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EMPLOYMENT OF CONICAL ALGORITHM AND OUTER APPROXIMATION METHOD IN D.C. PROGRAMMING

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Abstract. *In this article, we show a way to solve a class of d.c. programs by a branch and bound algorithm which is a combination of the conical algorithm with an outer approximation method, and was originally established for concave minimization problems (cf. Horst, Thoai, Benson [2]). Some questions about the convergence of the resulting algorithm are discussed, and computational results on test problems are then reported.*

Key words. D.c. functions, d.c. programming, global optimization, conical algorithm, outer approximation.

1. INTRODUCTION

In the theory of global optimization d.c. programming plays an interesting and important part because of its theoretical aspect as well as of a broad field of application. A function is called d.c. if it can be represented as the difference of two finite convex functions. Frequently, mathematical programming problems dealing with d.c. functions are called d.c. programming problems. It is well-known that the set of d.c. functions on a compact convex set of $\mathbb{R}^n$ is dense in the set of continuous functions on this set. Therefore, in principle, every continuous function can be approximated by a d.c. function with any desired precision. Of course, finding the explicite d.c. representation of a function is in general a hard problem, however, it points out in literature that the class of applicable d.c. functions is quite comprehensive. For a collection of typical properties and practical applications of d.c. programming we refer to the recent works by Horst and Tuy [1] and [7] and references given there.

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It is the purpose of this article to discuss methods for solving d.c. programming problems. In global optimization there are two classes of algorithms which are extensively applied and belong to the most promising tools in many cases. The one is called *conical algorithm* being of the branch and bound type in which polyhedral cones are used for the branching process, and the other, called *outer approximation*, is due to the idea of successive approximation of the feasible set by a sequence of polyhedral convex sets containing it. In many situations a suitable combination between conical algorithm and outer approximation principle led to efficient procedures. In this article, we show, on the one hand, a way to solve a class of d.c. programs by a branch and bound algorithm which is combination of the connical algorithm with an outer approximation method, and was originally established for convave minimization problems (cf. Horst, Thoai, Benson [2]), and, on the other hand, we discuss some questions about the convergence of the resulting algorithm.

Let us consider a d.c. programming problem of the form

$$(P) \qquad \min \{f(x) - g(x) : x \in F\}, \tag{1.1}$$

where f and g are finite convex functions and F is a convex set of $\mathbb{R}^n$.

Throughout this paper we shall use the following assumption

(A) The convex set F is compact with a non-empty interior and a point $x^0 \in \operatorname{int} F$ is available.

By using an additional variable, t, Problem (P) can be transformed into an equivalent form,

$$(P1) \qquad \min \{\varphi(z) = \varphi(x,t) = t - g(x) : x \in F,\ f(x) - t \leq 0\}. \tag{1.2}$$

Problem $(P1)$ deals with the minimization of a concave function over a convex set of $\mathbb{R}^{n+1}$. It is known in optimization theory that from an optimal solution $z^* = (x^*, t_*)$ of Problem $(P1)$ it follows immediately that x^* is an optimal solution of the original d.c. Program (P). Moreover, although the feasible sets of $(P1)$ is unbounded, it can be shown that the optimal solution set of problems $(P1)$ is bounded. We will see later that for establishing algorithms for problems $(P1)$ the boundedness of the optimal solution set shall be exploited.

The article is organized as follows. In Section 2, a basic conical algorithm for solving Problem $(P1)$ is presented. Details on an implementation of the basic algorithm are given in Sections 3-5. Some questions about the convergence of the algorithm are discussed in Section 6, and finally, illustrative examples and computational results are reported in Section 7.

2. A CONICAL ALGORITHM FOR SOLVING PROBLEM $(P1)$

The main idea of applying the concept of conical algorithms to Problem $(P1)$ can be briefly expressed as follows.

Let K and P be a convex polyhedral cone and a convex polyhedral set of $\mathbb{R}^{n+1}$, respectively, such that K contains the set of all optimal solutions and P contains the feasible set, D, of Problem $(P1)$. Based on the sets D, K, P and the objective function $t - g(x)$, we determine an upper bound, γ, of the optimal value of Problem $(P1)$ and a lower bound, μ, of $\varphi(z) = t - g(x)$ over the set $P \cap K$ which yields simultaneously a lower bound of $\varphi(z)$ over $D \cap K$, and a feasible point $z = (x,t) \in D$ satisfying $\varphi(z) = t - g(x) = \gamma$. If it holds $\gamma = \mu$, then we are done: the point $z = (x,t)$ is obviously an optimal solution of Problem $(P1)$ with the optimal value γ. Otherwise, we divide the cone K into a finite number of convex polyhedral subcones $K_1, \ldots, K_r$ and construct a convex polyhedral set $\overline{P}$ such that $P \supseteq \overline{P} \supseteq D$. For each $i = 1, \ldots, r$ a lower bound, μ_i, of φ over $K_i \cap \overline{P}$ is computed, and we obtain a new lower bound, $\overline{\mu}$, of φ over $K \cap D$ by setting $\overline{\mu} = \min\{\mu_i : i = 1, \ldots, r\}$. Throughout the bound estimation, new feasible points can be generated, among those a new (better) upper bound of the optimal value is computed. The procedure is continued by this way until an upper bound is found that coinsides with a lower bound over $K \cap D$.

The following algorithm is based on the idea formulated above.

Algorithm 1.

Initialization.

Construct a cone K, a set P as described above;

Set $\gamma \leftarrow f(x^0) - g(x^0)$ (x^0 from Assumption (A));

Choose $z = (x,t) \in \arg\gamma$, i.e. $z \in D$ and $\varphi(z) = t - g(x) = f(x) - g(x) = \gamma$;

Compute lower bound $\mu = \mu(K)$ of φ over $K \cap P$;

Set $\mathcal{K} \leftarrow \{K\}$, $stop \leftarrow false$, $k \leftarrow 1$.

while $stop = false$ **do**

if $\mu \geq \gamma$ **then**

$stop \leftarrow true$ ($z = (x,t)$ is optimal solution and γ is optimal value of Problem $(P1)$).

else

Divide K into r subcones $K_1, \ldots, K_r$;

Construct a polyhedral set $\overline{P}$ such that $P \supseteq \overline{P} \supseteq D$;

Compute lower bounds $\mu(K_i)$ of φ over $K_i \cap \overline{P}$ $(i = 1, \ldots, r)$;

Set $\mathcal{K} \leftarrow \mathcal{K} \setminus \{K\} \cup \{K_1, \ldots, K_r\}$,

$\mu \leftarrow \min \{\mu(K) : K \in \mathcal{K}\}$;

Update γ and $z = (x,t) \in \arg\gamma$ by using all newly generated feasible points;

Choose $K \in \mathcal{K}$ satisfying $\mu(K) = \mu$.

endif

Set $P \leftarrow \overline{P}$, $k \leftarrow k+1$

endwhile

Clearly, Algorithm 1 consists of three basic operations:

a) the construction of a starting cone containing the set of optimal solutions and the division of a cone at each iteration,

b) the estimation of bounds, and

c) the construction of a decreasing sequence of convex polyhedral sets containing the feasible set.

In the following sections these basic operations are discussed in details.

3. CONSTRUCTION OF A STARTING CONE AND CONICAL DIVISION

First, we show how to construct a starting cone containing the set of all optimal solutions of Problem $(P1)$.

From Assumption (A), the feasible set F can be packed in an n-simplex, S_0, of $\mathbb{R}^n$. Several possibilities for constructing such a simplex can be found e.g. in Thoai [8] and Horst and Tuy [1].

Let T_0 be an n-simplex of $\mathbb{R}^{n+1}$ defined by

$$T_0 = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : x \in S_0,\ t = \bar{t}\,\}, \tag{3.1}$$

where

$$\bar{t} = \max\{f(x) : x \in S_0\}. \tag{3.2}$$

(Note that $\bar{t}$ is simply computed by comparing the value of the concave function $t - g(x)$ at the vertices of simplex S_0.)

Further, let

$$z^0 = (x^0, t_0) \quad \text{where } t_0 \text{ is a number satisfying } t_0 > \bar{t}. \tag{3.3}$$

Then we have

Proposition 1. *The cone $K_0 = K(T_0) = K(z^0, T_0) \subset \mathbb{R}^n \times \mathbb{R}$, generated by $n+1$ rays emanating from z^0 and passing through the vertices of the simplex T_0, respectively, contains the set of all optimal solutions of Problem $(P1)$.*

Proof. Since $S_0 \supset F$, it follows by the construction of K_0, that $K_0 \supset \{(x,t) : x \in S_0,\ f(x) - t = 0\} \supset \{(x,t) : x \in F,\ f(x) - t = 0\}$. This implies that K_0 contains the optimal solution set of Problem $(P1)$ since each optimal solution (x^*, t_*) must obviously satisfy $x^* \in F$, $f(x^*) = t_*$ $\square$

The starting cone constructed by Proposition 1 is a polyhedral convex cone having $n+1$ edges. Obviously, this structure is the simplest which a cone in $\mathbb{R}^n \times R$ can have to contain the set of all optimal solutions of Problem $(P1)$. At each iteration of Algorithm 1 a cone is divided into a finite number of subcones. It is natural that the suitable structure of K_0 should be kept for every of subcones generated throughout the algorithm. A simple way to do this is the following classic radial subdivision.

Let K be any cone vertexed at z^0 and having $n+1$ edges which pass through the $n+1$ vertices of an n-simplex $T = [v^1, \ldots, v^{n+1}] \subseteq T_0$, respectively, and let $u \in K$ be a point that does not lie on an edge of K. Further, let v be the intersection point of T and the ray emanating from z^0, passing through u, and let

$$v = \sum_{i=1}^{n+1} \lambda_i v^i, \quad \lambda_i \geq 0 \ (i = 1, \ldots, n+1), \quad \sum_{i=1}^{n+1} \lambda_i = 1.$$

For each $i \in \{1, \ldots, n+1\}$ such that $\lambda_i > 0$ let K_i be the subcone vertexed at z^0 and having $n+1$ edges which are the ray passing v and n edges v^j, $j \neq i$, of K. Then we have

$$K = \bigcup_{\lambda_i > 0} K_i \text{ and } \operatorname{int} K_i \cap \operatorname{int} K_j = \emptyset \text{ for } i \neq j.$$

By this way, the simplex T is divided into r subsimplices and the cone K is accordingly divided into r subcones, $K_1, \ldots, K_r$, where r is the number of positive components of $\lambda \in \mathbb{R}^{n+1}$ (and hence satisfies $2 \leq r \leq n+1$). We say that the collection $\{K_1, \ldots, K_r\}$ forms a radical division of the cone K by using the point u. Each choice of the point u provides a division. A special division called *bisection* is created when v is the middle point of an edge of T with the bigest length.

Recently Horst, Thoai and de Vries [4-6] have proposed a *conical cover* which is based on a *simplicial cover* rather than the radial simplicial division. By using the idea of this conical cover, a cone K as defined above is covered with a collection of convex polyhedral subcones $K_1, \ldots, K_r$ such that $1 \leq r \leq n+2$ and every subconce is vertexed at z^0 and has exactly $n+1$ edges. Moreover, it must satisfy that $K_i \not\subseteq \bigcup_{i=1}^{r} K_i$.

Within a conical branch and bound algorithm, the conical covering technique was developed for improving the efficiency of the first iterations. Actually, at the first iterations of the algorithm, the conical cover can exploit intensively the structure of the problem, and can, therefore, allow immediately deletion of large parts of the feasible set from further consideration. Moreover, it can provide in many cases considerably improved initial upper and lower bounds.

4. LOWER BOUND ESTIMATION

Let $T = [v^1, \ldots, v^{n+1}] \subset T_0$ and $K = K(z^0, T)$ as in the previous section. Further, let P_0 be a polyhedral set containing the feasible set D of Problem $(P1)$ such that $P_0 \cap K_0$ is bounded (such a set can be simply constructed, e.g. by $P_0 = \{(x,t) : x \in S_0,\ t \geq t'\}$, where $t \leq \min\{f(x) : x \in F\}$).

Let P be a polyhedral set satisfying $D \subset P \subset P_0$, and defined by

$$P = \{z = (x,t) \in \mathbb{R}^n \times \mathbb{R} : Az \leq b\},$$

where A and b are matrix and vector, respectively, of appropriate sizes.

The way to determine a lower bound of the objective function over the set $K \cap P$ (which is also a lower bound of the objective function over $K \cap D$ since $P \supset D$) is proposed in Thoai and Tuy [9]. Here we briefly recall this procedure.

To our purpose, we need an important concept called *γ-extension*. The first version of γ-extension was introduced by Tuy [10]. Several modified versions of this concept were established in connection with the development of algorithms for solving certain problems in global optimization (cf. Horst and Tuy [1], Horst et al [4, 6] and references given there). Below we present a new version of the γ-extension concept which generalizes other versions and is suitably applied in many algorithms for concave programming as well as for other classes in global optimization.

Let $\varphi : \mathbb{R}^\nu \to \mathbb{R}$ be a concave function, $z \in \mathbb{R}^\nu$, and let γ and θ_1 be real numbers satisfying $\gamma \leq \varphi(z)$ and $\theta_1 > 0$. A point $\bar{z} \in \mathbb{R}^\nu$ is called γ-extension of

z in direction d (with respect to φ) if

$$\overline{z} = z + \theta d \text{ with } \theta = \min\{\theta_1; \sup\{\lambda : \varphi(z + \lambda d) \geq \gamma\}\}.$$

From the concavity of φ, the existence of γ-extension of any given point z in any given direction d is guaranteed whenever $\gamma \leq \varphi(z)$.

Now assume that an upper bound $\overline{\gamma}$ of the optimal value of problem $(P1)$ is on hand. For each $i = 1, \dots, n+1$ let z^i be the point where the i-th edge of K intersects the boundary ∂D of D (the existence of such intersection points is guaranteed since $z^0 \in \text{int}D$), and let $\overline{z}^i$ be the γ-extension of z^i in direction $(z^i - z^0)$, where

$$\gamma = \min\{\overline{\gamma}, \varphi(z^1), \dots, \varphi(z^{n+1})\}.$$

Denote by Y the matrix with columns $(\overline{z}^1 - z^0), \dots, (\overline{z}^{n+1} - z^0)$, and define

$$c^* = \max\Big\{c(\lambda) = \sum_{i=1}^{n+1} \lambda_i : AY\lambda \leq b - Az^0,\ \lambda \geq 0\Big\}. \tag{4.1}$$

(Note that c^* is finite, since $P \cap K$ is bounded).

Then a lower bound $\mu(K) = \mu(K, P)$ of φ over $K \cap P$ is given by

$$\mu(K) = \mu(K, P) = \begin{cases} \gamma, & \text{if } c^* \leq 1 \\ \min\{\varphi(\hat{z}^1), \dots, \varphi(\hat{z}^{n+1})\}, & \text{else} \end{cases} \tag{4.2}$$

where $\hat{z}^i = c^*(\overline{z}^i - z^0) + z^0 = c^*\overline{z}^i + (1 - c^*)z^0$ for $i = 1, \dots, n+1$.

Remark. Let K' be a cone such that K is generated by a subdivision of K'. Then, clearly, we should set $\mu(K) = \mu(K')$ whenever the number $\mu(K)$ computed by (4.2) is less than $\mu(K')$.

5. CONSTRUCTION OF POLYTOPE $\overline{P}$

At each iteration of the algorithm a polyhedral set $\overline{P}$ is constructed satisfying $P \supseteq \overline{P} \supseteq D$. To our purpose, we assume that D is given by $D = \{(x,t) : \psi(x,t) \leq 0\}$, where $\psi(x,t)$ is a convex function, defined as the maximum of a finite number of convex functions. This operation is performed as follows. Let K be the cone which is divided at the present iteration, and let $\hat{z}$ be the point computed by

$$\hat{z} = Y\lambda^* + z^0, \tag{5.1}$$

where λ^* is an optimal solution of the linear program in (4.1). Note that here we have $c^* = \sum \lambda_i^* > 1$ since otherwise the cone K would be removed from further consideration. Geometrically, $\hat{z}$ is a point of the set P which stands farthest to the hyperplane containing $\bar{z}^1, \ldots, \bar{z}^{n+1}$.

If it holds $\hat{z} \in D$, then we simply set $\overline{P} = P$. Otherwise, compute the point w where the line segment $[z^0, \hat{z}]$ meets the boundary ∂D, and set

$$\overline{P} = P \cap \{z : \ell(z) = (z - \omega)\xi \leq 0\}, \tag{5.2}$$

where ξ is a subgradient at the point w of the convex function defining the convex set D.

It is known from an outer approximation concept (cf. e.g. Horst, Thoai, Tuy [3]) that we have $\hat{z} \notin \overline{P}$ and $\overline{P} \supseteq D$ which implies that $P \supseteq \overline{P} \supseteq D$.

6. CONVERGENCE OF ALGORITHM 1

Let us assign the index k to everything dealt with at iteration k of Algorithm 1.

Being an algorithm of branch and bound type, Algorithm 1 is convergent under the general *consistency condition* that

$$\lim_{q \to \infty} (\gamma_q - \mu_q) = 0, \tag{6.1}$$

for every subsequence $\{K_q\} \subset \{K_k\}$ such that $K_{q+1} \subset K_q$ for all q. (cf. e.g. Horst and Tuy [1]).

We now establish a sufficient condition for (6.1). For this purpose we need some additional notions. For each $q \geq 1$ we denote by $z^{q,i}$, $\bar{z}^{q,i}$, $\hat{z}^{q,i}$, and $\hat{z}^q$ the points z^i, $\bar{z}^i$, $\hat{z}^i$, and $\hat{z}$, respectively, constructed according to the cone K_q as in previous subsections. Further, denote by z^q and $\bar{z}^q$ the points where the line segment $[z^0, \hat{z}^q]$ intersects the simplex $[z^{q,1}, \ldots, z^{q,n+1}]$ and $[\bar{z}^{q,1}, \ldots, \bar{z}^{q,n+1}]$, respectively.

Proposition 2. *Condition (6.1) is fulfilled for every subsequence $\{K_q\} \subset \{K_k\}$ such that $K_{q+1} \subset K_q$ $\forall q$ if it holds*

$$\lim_{q \to \infty} \|\bar{z}^q - \hat{z}^q\| = 0. \tag{6.2}$$

Proof. Since the hyperplanes containing $[\bar{z}^{q,1},\ldots,\bar{z}^{q,n+1}]$ and $[\hat{z}^{q,1},\ldots,\hat{z}^{q,n+1}]$, respectively, are parallel to each other, we have, for each $i = 1,\ldots,n+1$,

$$\frac{\|\bar{z}^{q,i} - \hat{z}^{q,i}\|}{\|\bar{z}^{q,i} - z^0\|} = \frac{\|\bar{z}^q - \hat{z}^q\|}{\|\bar{z}^q - z^0\|}.$$

But $\|\bar{z}^{q,i} - z^0\|$ and $\|\bar{z}^q - z^0\|$ are all bounded and $z^0 \in \operatorname{int} D$, therefore it follows from the continuity of the function φ that $\gamma_q - \mu_q \le \varphi(\bar{z}^{q,i}) - \varphi(\hat{z}^{q,i}) \to 0$ as $q \to \infty$ if $\|\bar{z}^{q,i} - \hat{z}^{q,i}\| \to 0$ as $q \to \infty$, i.e. if (6.2) holds. □

Clearly, Condition (6.2) depends upon the conical division process performed throughout the algorithm. The most useful characterization of a division process is the concept of *exhaustiveness.* A nested subsequence $\{K_q\}$, $K_q \supset K_{q+1}\ \forall q$, is called to be exhaustive if the intersection $\bigcap_{q=1}^{\infty} K_q$ is a ray (a halfline emanating from z^0). A conical division process is called to be exhaustive if every nested subsequence of cones generated throughout the algorithm is exhaustive. A typical example for exhaustive division processes is the bisection process mentioned in Section 2.2. Other classes of exhaustive divisions are discussed in Tuy, Khachaturov and Utkin [12], Horst and Tuy [1], Tuy [11], Horst, Thoai and de Vries [6].

Proposition 3. *Condition (6.2) is fulfilled for every exhaustive division process.*

Proof. Let $\{K_q\}$ be an exhaustive nested subsequence and let Γ be the ray such that $\bigcap_q K_q = \Gamma$. Then the point $\hat{z}^q$ appoaches a point $z^* \in \Gamma$. From an outer approximation procedure it follows that $z^* \in \partial D$ (cf. e.g. Horst, Thoai and Tuy [3], Horst and Tuy [1]), i.e. z^* is the intersection point of Γ and ∂D. On the other hand, all the points $z^{q,i}$ $(i = 1,\ldots,n+1)$ and z^q approach the point z^* as well. Therefore, it follows that $\|\bar{z}^q - \hat{z}^q\| \le \|z^q - \hat{z}^q\| \to 0$ as $q \to \infty$. □

In general, the radial division process described in Section 2.2 is not exhaustive. However, the following proposition shows a case where Condition (6.2) is fulfilled for a radial division.

Proposition 4. *Assume that at each iteration $k \ge 1$, a radial division of the cone K_k is performed by using the point $u^k = \hat{z}^k$. Then Condition (6.2) is fulfilled if for any subsequence $\{K_q\}$ such that each K_q is generated by a division of K_{q-1} we have*

$$\frac{1}{\|eY_q^{-1}\|} \ge a > 0 \quad \forall q, \tag{6.3}$$

where $e = (1,\ldots,1) \in \mathbb{R}^{n+1}$ and Y_q is the matrix with columns $(\bar{z}^{q,1} - z^0),\ldots,(\bar{z}^{q,n+1} - z^0)$.

Proof. For each q let us denote by H_q the hyperplane containing $\overline{z}^{q,1}, \ldots, \overline{z}^{q,n+1}$ and by $d(z, H_q)$ the distance from a point z to H_q, i.e. $d(z, H_q) = \min\{\|z - z'\| : z' \in H_q$. First, we show that

$$\lim_{q \to \infty} d(\overline{z}^q, H_{q+1}) = 0. \tag{6.4}$$

Since $\{\overline{z}^q\}$ is bounded, we can assume, by passing to subsequence if necessary, that $\overline{z}^q \to \overline{z}^*$. But $\overline{z}^{q+1} \in H_{q+1}\ \forall q$, therefore we have $d(\overline{z}^q, H_{q+1}) \leq \|\overline{z}^q - \overline{z}^{q+1}\| \to 0$ as $q \to \infty$.

Since K_{q+1} is generated from K_q by a radial division, we can fix an index j such that $z^{q+1,j}$ is the intersection point of $[z^0, \hat{z}^q]$ and ∂D. As in the proof of Proposition 3, it follows from an outer approximation concept that $\hat{z}^q$ and $z^{q+1,j}$ approach an unique point $\hat{z}^* \in \partial D$. On the other hand we have

$$\|\overline{z}^q - z^{q+1,j}\| \leq \|\overline{z}^q - \overline{z}^{q+1,j}\| = \frac{d(\overline{z}^q, H_{q+1})}{d(z^0, H_{q+1})} \|\overline{z}^{q+1,j} - z^0\| \quad \forall q \geq q_0. \tag{6.5}$$

Since $\|\overline{z}^{q+1,j} - z^0\|$ is bounded, it follows from (6.3) and (6.4) that $\|\overline{z}^q - z^{q+1,j}\| \to 0$ as $q \to \infty$ which implies that $\overline{z}^q$ approaches $\hat{z}^*$, and hence (6.2).

We conclude discussing the convergence of Algorithm 1 with some additional remarks.

a) The consistency condition (6.1) ensures the convergence of Algorithm 1 in the following sense: if the algorithm is infinite, it generates an infinite sequence of feasible points, every accumulation point of which is an optimal solution to Problem $(P1)$.

b) Condition (6.2) is similar to the *normal condition* introduced by Tuy [11] (see also Horst and Tuy [1]), however, there is a slight difference between them.

c) The prismatic algorithm proposed in Horst et al [7] can actually be described as a special implementation of Algorithm 1 where the vertex z^0 of the starting cone is at infinity, i.e. when in (3.3) we choose $t_0 = \infty$. However, in the case that the prismatic algorithms is finite, it only can be shown that the algorithm generates an infinite sequence of infeasible points, every accumulation point of which is an optimal solution to Problem $(P1)$.

7. NUMERICAL EXAMPLES AND COMPUTATIONAL EXPERIMENT

To illustrate Algorithm 1, we consider d.c. programs of the following type

$$\begin{aligned} &\min \{f(x) - g(x)\} \\ \text{s.t.} \quad &h_i(x) \le 0 \ (i = 1, \ldots, m), \\ &c_j \le x_j \le d_j \ (j = 1, \ldots, n), \end{aligned}$$

where f, g, h_i $(i = 1, \ldots, m)$ are all convex functions, and c_j and d_j are real numbers satisfying $-\infty < c_j < d_j < +\infty$ $(j = 1, \ldots, n)$. Assume that a point x^0 satisfying $h_i(x^0) < 0$ $(i = 1, \ldots, m)$ and $c_j < x_j^0 < d_j$ $(j = 1, \ldots, n)$ is available.

Define

$$\varphi(x,t) = t - g(x),$$
$$\psi(x,t) = \max \{f(x) - t, h_1(x), \ldots, h_m(x)\},$$
$$P_1 = \{(x,t) \in \mathbb{R}^{n+1} : (x,t) - (x^0, f(x^0))\xi \le 0,\ c_j \le x_j \le d_j \ (j = 1, \ldots, n)\},$$

where ξ is a subgradient of the convex function ψ at the point $(x^0, f(x^0))$. Then the according problem $(P1)$ is

$$\min \{\varphi(x,t) : (x,t) \in D\},$$

where

$$D = \{(x,t) \in P_1 : \psi(x,t) \le 0\}.$$

In order to determine an ε-optimal solution (x^*, t^*) in a sense that $\varphi(x^*, t^*) - \varepsilon|\varphi(x^*, t^*)| \le \varphi(x,t)\ \forall (x,t) \in D$ with a prescribed number $\varepsilon > 0$, Algorithm 1 is modified in a way that throughout the lower bound estimation according to each cone, all γ-extensions are replaced by $(\gamma - \varepsilon|\gamma|)$-extensions.

Moreover, from practical point of view, we proposed the following conical subdivision rules called "*λ-bisection*" and "*λ-radial division*", respectively. The convergence of the algorithm when using these subdivision rules is not guaranteed, however, computational results turned out that, in most cases, these rules are most promising for implementing the algorithm.

λ-bisection. Let $K = K(T)$, where $T = [v^1, \ldots, v^{n+1}]$ be the cone which is divided at the present iteration, and let $\lambda^* = (\lambda_1^*, \ldots, \lambda_n^*)$ be an optimal solution of the linear program in (4.1) according to K. Then the cone K is divided into two subcones by using the point $v = \frac{1}{2}(v^{i_1} + v^{i_2})$, where i_1 and i_2 are chosen by

$$\lambda_{i_1}^* = \max \{\lambda_i^* : \lambda_i^* > 0, \quad \text{and} \quad \lambda_{i_2}^* = \max \{\lambda_i^* : \lambda_i^* > 0,\ i \ne i_1\}.$$

λ-radial division. Let $\delta > 0$ be a small number, (a suitable choice is, e.g., $\delta = 1/(2n^2)$). If $\max\{\lambda_i^* : \lambda_i > 0\} > \delta$, then a radial conical division is performed. Otherwise, a λ-bisection is performed.

Example 1. Horst et al [7].

$m = 1$, $n = 2$,

$f(x) = 4x_1^4 + 2x_2^2$, $g(x) = 4x_1^2$,

$h_1(x) = x_1^2 - 2x_1 - 2x_2 - 1$, $c_1 = c_2 = -1$, $d_1 = d_2 = 1$.

$x^0 = (0.5, 0.5)$.

A simplex S^0 containing the feasible set of the original d.c. progam can be defined as

$S_0 = [(3,-1),(-1,3),(-1,-1)]$. Hence $\bar{t} = \max\{f(x) : x \in S_0\} = 326$,

$T_0 = [(3,-1,326),(-1,3,326),(-1,-1,326)]$.

Choose $z^0 = (x^0, t_0) = (0.5, 0.5, 1926)$.

$P_1 = \{(x,t) : 2x_1 + 2x_2 - t - 1.25 \leq 0,\ -1 \leq x_1, x_2 \leq 1\}$.

Choose $\varepsilon = 10^{-6}$.

Iteration 1:

$K_1 = K(z^0, T_0)$. Section points of edges of K_1 and ∂D are respectively

(1.000000, 0.200000, 1606.000000),

(0.200000, 1.000000, 1606.000000),

(−0.236068, −0.236068, 1140.860821).

Current best point $(x^1, t_1) = (-0.236068, -0.236068, 1140.860821)$, $\gamma_1 = \varphi(x^1, t_1) = 1140.637908$.

$(\gamma - \varepsilon|\gamma|)$-extensions of section points are respectively

(1.732422, −0.239453, 1137.250000),

(−0.239453, 1.732422, 1137.250000),

(−0.236068, −236068, 1140.860821).

The optimal value of linear problem (4.1) according to K_1 is $c_1^* = 2.458306$. Lower bound $\mu(K_1, P_1) = -62.823479$. Point $\hat{z}^1 = (-1.000000\ -1.000000\ -5.250000) \notin D$.

Cutting function $\ell_1(x,t) = -2.472136x_1 - 2.0x_2 - 1.055728$,

$\overline{P}_1 = P_1 \cap \{(x,t) : \ell_1(x,t) \leq 0\}$.

Cone K_1 is divided into two subcones by a conical λ-bisection.

The procedure terminates after 12 iterations yielding an (10^{-6})-optimal solution

$(x^*, t^*) = (0.697901,\ 0.007651,\ 0.949050)$ with optimal value

$t^* - g(x^*) = f(x^*) - g(x^*) = -0.999214.$

Example 2.

$m = 1,\ n = 3,$

$$f(x) = \sum_{i=1}^{n} x_i \ln \frac{x_i}{\sum_{j=1}^{n} x_j},$$

$$g(x) = \left| x_1 + \sum_{i=2}^{n} \frac{n-1}{n} x_n \right|^{3/2}$$

$$h_1(x) = \Big(\sum_{i=1}^{n-1} (-1)^i x_i - 1.2 \Big)^2 + x_n - 4.4, \quad c_j = 1, \quad d_j = 3 \ (j = 1, \ldots, n).$$

$S^0 = [(7,1,1), (1,7,1), (1,1,7), (1,1,1)]$, $\bar{t} = -3.295837,$

$z^0 = (2, 2, 2, 2396.704163)$

$P_1 = \{(x,t) : -1.098612x_1 - 1.098612x_2 - 1.098612x_3 - t \leq 0,$

$1 \leq x_j \leq 3 \ (j = 1, \ldots, 3)\}.$

$\varepsilon = 10^{-6}$

Iteration 1 :

$(x^1, t_1) = (1, 1, 1, 1196.704163)$, $\gamma_1 = 1193.514916,$

$c^{1*} = 2.021523$, $\mu(K_1, P_1) = -29.133174$, $\hat{z}^1 = (3, 3, 3, -9.887511) \notin D.$

Cutting function $\ell_1(x,t) = 2.4x_1 - 2.4x_2 + x_3 - 2.96,$

$\overline{P}_1 = P_1 \cap \{(x,t) : \ell_1(x,t) \leq 0\}.$

Cone K_1 is divided into two subcones by a conical bisection.

The procedure terminates after 13 iterations yielding an (10^{-6})-optimal solution $(x^*, t^*) = (2.983216,\ 3.0,\ 3.0,\ -9.869040)$ with optimal value $t^* - g(x^*) = f(x^*) - g(x^*) = -26.376708.$

To obtain further computational results, we take the problem type as in Example 2 with, in addition, a system of constraints of the form

$$Ax \leq b,$$

where A is an $(\ell \times n)$-matrix and $b \in \mathbb{R}^\ell$.

Table 1 contains computational results on a large set of test problems, in which all elements of matrix A and vector b are randomly generated. While performing the test, some technical parameters were used: for fixing a point z^0 as in (3.3) we chose $t_0 = \bar{t} + 800n$, and in (4.2) we set $\mu(K) = \gamma$, with $c^* \leq 1 - \sigma$ with $\sigma = 10^{-4}$. The choice of these parameters has effects, of course, on the speed of the algorithm. However, it is worth noting that for all test problems, an ε-optimal solution was found within a number of first iterations which is relatively small in comparison with the total number of iterations needed. For all test problems, conical λ-radial subdivision was used.

The test runs were performed on a DEC/VAX 6000-410 computer.

Table 1. Some computational results

ℓ	n	average number of iterations	ε	average CPU-time [sec.]
0	9	4	1%	0.12
5	9	14	1%	0.38
10	9	30	1%	0.83
20	9	46	1%	1.27
0	15	14	1%	0.52
5	15	21	1%	0.78
10	15	31	1%	1.03
20	15	56	1%	2.67
0	19	13	2%	0.62
5	19	202	2%	10.16
10	19	148	2%	12.42
20	19	702	2%	221.45

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