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A Short Communication

A COMPLETE PROOF OF BEAUVILLE'S CONJECTURE

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Let $f : X \to \mathbf{P}^1$ be a non isotrivial semi-stable fibration over the projective line whose generic fibre is a complex algebraic curve of genus $g \geq 1$. Then Beauville's theorem ([1]) asserts that $s \geq 4$, where s denotes the number of singular fibres on X.

The aim of this note is to give a proof of the following statement conjectured by A. Beauville (*loc. cit.*)

Theorem (Beauville's conjecture). $s \geq 5$ if $g > 1$.

For convenience we shall prove this theorem in a series of several claims below.

1. Xiao's equality ([2]). Denote by ω the relative canonical class of f and assume that $s = 4$. Then

$$\omega^2 = 4(g-1) \tag{1}$$

The idea of proving this equality is to apply the Sakai-Miyaoka inequality in the following form.

2. The Sakai-Miyaoka inequality ([3]). Let $f : X \to B$ be a semi-stable fibration with $g > 1$ and $q = g(B) \geq 1$ then

$$\sum_i 3\left(r_i + 1 - \frac{1}{r_i+1}\right) \leq 3c_2(X) - c_1^2(X) \tag{2}$$

where c_1, c_2 denote Chern's classes of X and the sum is taken over all chains of (-2)-curves on X corresponding to singularities p_i of the type A_{r_i}.

3. The beginning of the proof. Take a cyclic covering of degree $n : B_n \to \mathbf{P}^1$ (totally) branched over s critical points of $\mathbf{P}^1$, i.e., $2g(B_n) - 2 = -2n + (n-1)s$ by the Riemann-Hurwitz formula. Let X_n be the relatively minimal resolution of $X \times_{\mathbf{P}^1} B_n$. Now we apply (2) to X_n

$$\sum_i 3\left(\widetilde{r}_i + 1 - \frac{1}{\widetilde{r}_i + 1}\right) \leq 3c_2(X_n) - c_1^2(X_n) \tag{3}$$

where p_i has the type $A_{\widetilde{r}_i}$ on X_n. By an easy computation we have the following equalities

1) $\widetilde{r}_i + 1 = n(r_i + 1)$,

2) $c_1^2(X_n) = n\omega^2 + 2(ns - 2n - s)(2g - 2)$,

3) $c_2(X_n) = n\delta + (ns - 2n - s)(2g - 2)$,

where

$$\delta :=^{\#} \{\text{double points of singular fibres on } X\}.$$

Thus by substituting these equalities in (3) we obtain

$$s(2g-2) - \frac{3\delta}{n} \leq n\left[(2g-2)(s-2) - \omega^2\right]. \tag{4}$$

4. The end of the proof. Now putting $s = 4$ in (4) and taking into account Xiao's equality (1) one has

$$8(g-1) - \frac{3\delta}{n} \leq 0.$$

Evidently the latter is possible as $n \to \infty$ if and only if $g = 1$. This completes the proof of the theorem.

5. Remarks.

1) It is not difficult to see that (4) combined with Xiao's inequality for the slope of f (cf. [4]) implies Xiao's equality (1) for the case $s = 4$ which actually does not occur in view of our theorem above.

2) The idea used in the proof above enables us to get the following variant of the canonical class inequality. Let $f : X \to B$ be as in **2.** with $g > 1$. Then there exists a universal constant $A = A(q, s) < 2q - 2 + s$ (effectively computed by q and s) such that

$$\omega^2 \leq A(2g-2).$$

A detailed version with other applications will appear somewhere else.

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