Tạp chí Toán học *Journal of Mathematics* **22**(3 & 4)(1994), 120-123

A Short Communication

RINGS FOR WHICH DIRECT SUMS OF CERTAIN CS-MODULES ARE CS

DINH VAN HUYNH *and* BRUNO J. MÜLLER

Let K be an arbitrary field. Put $K_n = K$ and form the direct product

$$R = \prod_{n=1}^{\infty} K_n. \tag{1}$$

Then R is a commutative von Neumann regular self-injective ring. Hence every simple R-module is injective. Moreover, it is easy to see that every uniform R-module is simple. Therefore, every direct sum of uniform R-modules is quasi-injective, in particular it is a CS-module. From this it is natural to ask the question: Which rings R have the property that any direct sum of uniform right R-modules is CS?

We consider this and some related questions on rings and modules.

1. All rings are assumed to be associative rings with identity and all modules are unitary. Let M_R be a right R-module, where R is a ring. Then M is called a CS module if every submodule of M is essential in a direct summand of M (see [2] and [6]). By this definition, any uniform module is CS. For a given module M_R we denote by $\sigma[M]$ the full subcategory of Mod-R whose objects are submodules of M-generated modules (see [7]).

The main result of this section is

Theorem 1. *For a module M_R the following conditions are equivalent:*

a) Every direct sum of uniform modules in $\sigma[M]$ is CS.

b) Every uniform module in $\sigma[M]$ has composition length ≤ 2. Moreover, in this case, the Jacobson radical $J(M)$ of M is a V-module, i.e. each simple right R-module is $J(M)$-injective. In case $M = R$, $J(R)^2 = 0$.

One might ask the question, whether a module of Theorem 1 always has non-zero socle. The answer is 'no'. For this take the ring R in (1). Since R is not semisimple, by [5, Corollary 3] there is a proper ideal I of R such that $\overline{R} = R/I$ has zero socle. But since $\overline{R}$ is von Neumann regular, every simple $\overline{R}$-module is injective and therefore any uniform $\overline{R}$-module is simple. It follows that in Mod-$\overline{R}$, every direct sum of uniform modules is quasi-injective while $\mathrm{Soc}(\overline{R}) = 0$.

2. Following the previous remark we now consider the question, when a module M is semiartinian.

Theorem 2. *Let M be a module. If*

c) Every direct sum of an M-injective module and uniform modules in $\sigma[M]$ is CS, then M is semiartinian.

Proposition 3. *If M satisfies the condition (c) and each submodule of M is CS, then M is a direct sum of uniform modules with composition length ≤ 2.*

It seems to be not true, that for any module M of Proposition 3, every module in $\sigma[M]$ is CS. However, for $M = R$, a combination of Proposition 3 and the consideration in [2, §10] yields the following:

Corollary 4. *For a ring R the following conditions are equivalent:*

i) R_R satisfies (c) and each right ideal of R is CS.

ii) Each right R-module is the direct sum of uniform modules with composition length ≤ 2.

iii) Each right R-module is CS.

iv) R is right and left artinian, right and left serial with $J(R)^2 = 0$.

v) Every cyclic right R-module is a direct sum of an injective module and a semisimple module.

vi) The left-handed versions of (i), (ii), (iii) and (v).

3. Clearly, a module M of Proposition 3 has locally finite length, i.e. any finitely generated submodule of M has finite composition length. From this, a question arises: Is a module of Theorem 2 necessarily locally artinian or locally noetherian? The answer is 'no'. To give the answer we need some other concepts.

A module N in $\sigma[M]$ is called M-singular if there exists a module $A \in \sigma[M]$ with an essential submodule B such that $N \simeq A/B$. Hence M-singular modules are contained in $\sigma[M]$. As shown in [7], any module H in $\sigma[M]$ contains a unique maximal M-singular submodule $Z_M(H)$. If $Z_M(H) = 0$, we say that H is M-nonsingular. A module M is called an SI-module if every M-singular module is M-injective.

Theorem 5. *For an M-nonsingular module M the following conditions are equivalent:*

i) Every direct sum of an M-injective module and uniform modules in $\sigma[M]$ is CS.

ii) M is SI and every uniform module in $\sigma[M]$ has composition length ≤ 2.

In [3] there is an example of a von Neumann regular commutative SI-ring R such that $R/\mathrm{Soc}(R)$ is semisimple, however R is not noetherian. Since this ring R has the property (ii) of Theorem 5 for Mod-R, it provides a negative answer to the above question.

4. In this section we consider the corresponding property for Mod-R (i.e. $M = R$). In this case Theorems 1, 2 give interesting properties for R. For example, if R is a ring satisfying (a) of Theorem 1 and R has finite right uniform dimension, then R is right artinian.

Theorem 6. *For a right non-singular ring R the following conditions are equivalent:*

i) Every direct sum of injective right R-modules and uniform right R-modules is CS.

ii) R is right artinian and any uniform module has composition length ≤ 2.

iii) Every direct sum of CS right R-modules is CS.

Proposition 7. *If every direct sum of quasi-injective right R-modules is CS, then R is semiprimary and $J(R)^2 = 0$.*

It would be interesting to consider the converse of Proposition 7. It is also unknown whether or not a ring of Proposition 7 is right artinian. Furthermore, it might be interesting to consider rings R, such that direct sums of injective right R-modules are CS. For investigation related to these questions we refer to [1], [4].

REFERENCES

1. N. V. Dung and P. F. Smith, *Rings over which certain modules are CS*, J. Pure and Appl. Algebra, to appear.
2. N. V. Dung, D. V. Huynh, P. F. Smith and R. Wisbauer, *Extending modules*, Pitman, London (1994), to appear.
3. K. R. Goodearl, *Singular torsion and splitting properties*, Amer. Math. Soc., Memoirs **124** (1972).
4. D. V. Huynh, *A right countably sigma-CS ring with ACC or DCC on projective principal right ideals is left Artinian and QF-3*, Proc. Amer. Math. Soc., to appear.
5. D. V. Huynh and N. V. Sanh, *A right continuous right weakly SI-ring is semisimple*, to appear.
6. S. H. Mohamed and B. J. Müller, *Continuous and discrete modules*, London Math. Soc. Lecture Note Series no. 147, Cambridge Univ. Press, 1990.
7. R. Wisbauer, *Foundations on module and ring theory*, Gordon and Breach, Reading, 1991.

Institute of Mathematics
P.O. Box 631, Bo Ho
10 000 Hanoi, Vietnam

Department of Mathematics and Statistics
McMaster University
Hamilton, Ontario
Canada L8S 4K1

Received May 18, 1994