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ON THE LAW OF LARGE NUMBERS OF HSU - ROBBINS TYPE IN NON-COMMUTATIVE PROBABILITY

NGUYEN VAN QUANG

Abstract. *Let A be a von Neumann algebra with a trivial state and $\tilde{A}$ be the *- algebra of all measurable operators in Segal - Nelson's sense. The aim of this paper is to investigate laws of large numbers of Hsu - Robbins type for sequences and 2- dimensional arrays of operators in $\tilde{A}$ in general case. Our results extend some results in [3] and [5].*

1. INTRODUCTION AND NOTATION

In the classical probability, the Hsu - Robbins law of large numbers is studied by many authors (see e.g. [1], [2], [5]). But to the best of our knowledge, in non-commutative probability, this law is investigated only by Jajte in a special case (see [4] or [3]). The purpose of this paper is to extend the result of Jajte to the general case. Moreover some results for 2-dimensional arrays are considered.

Let us begin with some notations and definitions. Throughout this paper, A will denote a von Neumann algebra with a tracial state τ, $\tilde{A}$- the *-algebra of all measurable operators in Segal - Nelson's sense (see [9]).

Let A_1 and A_2 be two von Neumann subalgebras of A. We say that A_1 and A_2 are independent if

$$\tau(xy) = \tau(x)\tau(y) \quad \text{for all} \quad x \in A_1,\ y \in A_2.$$

Two elements x, y in $\tilde{A}$ are said to be independent if the von Neumann algebras $W^*(x)$ and $W^*(y)$ generated by x and y, respectively, are independent. A sequence (x_n) of elements in $\tilde{A}$ is said to be successively independent if, for every n, the von Neumann algebra $W^*(x)$ generated by x_n is independent of the von Neumann algebra $W^*(x_1, ..., x_m)$ generated by the elements $x_1, ..., x_m$ for $m < n$.

A family $(x_\lambda)_{\lambda\in\Lambda}$ is said to be strongly independent if the von Neumann algebra $W^*(x_\lambda, \lambda \in \Lambda_1)$ generated by the family $(x_\lambda)_{\lambda\in\Lambda_1}$ is independent of the von Neumann algebra $W^*(x_\lambda, \lambda \in \Lambda_2)$ generated by the family $(x_\lambda)_{\lambda\in\Lambda_2}$, for any two disjoint subsets Λ_1 and Λ_2 of Λ.

Let now x and y be two self-adjoint elements of $\tilde{A}$. We say that x and y are identically distributed if $\tau(e_\triangle(x)) = \tau(e_\triangle(y))$ for every Borel subset $\triangle \subset R$ (where $e_\triangle(x)$ is the spectral projection of element x corresponding to Borel subset $\triangle \subset R$).

For other notations and definitions of the theory of von Neumann algebra and non-commutative probability the reader is referred to [4], [9].

2. HSU - ROBBINS LAW OF LARGE NUMBERS FOR SEQUENCES

In the sequel we will need the following lemmas

Lemma 2.1. *Let x be a self-adjoint element of $\tilde{A}$. Then for each $a \in R$*

$$\tau(|x+a|^r) \le C_r(\tau(|x|^r) + |a|^r)$$

where

$$C_r = \begin{cases} 1 & \text{if } r \le 1, \\ 2^{r-1} & \text{if } r > 1. \end{cases}$$

Proof. Suppose that the spectral representation of self-adjoint element $x \in \tilde{A}$ is

$$x = \int_{-\infty}^{+\infty} \lambda e(d\lambda).$$

Then, from the elementary inequality

$$|\alpha+\beta|^r \le C_r(|\alpha|^r + |\beta|^r), \quad \alpha, \beta \in R.$$

we get

$$\tau(|x+a|^r) = \int_{-\infty}^{+\infty} |\lambda + a|^r \tau(e(d\lambda))$$

$$\le C_r \int_{-\infty}^{+\infty} (|\lambda|^r + |a|^r)\tau(e(d\lambda)) = C_r(\tau|x|^r + |a|^r),$$

proving the assertion.

The following lemma is a slight generalization of Lemma 1.1 in [6] (with $g(x) = x^2$) and it can be proved in the same way.

Lemma 2.2 (Tchebyshev's inequality). *Let x be an element of $\tilde{A}, g : R^+ \longrightarrow R^+$ is a nondecreasing Borel function. Then for each $\varepsilon > 0$*

$$\tau(e_{[\varepsilon,\infty)}(|x|)) \leq \frac{\tau(g(|x|))}{g(\varepsilon)}.$$

We now prove the main result of this section.

Theorem 2.3. *Let (x_n) be a successively independent sequence of self-adjoint elements of $\tilde{A}$ with $\tau(x_n) = 0 \quad \forall n \in N$. Suppose that (t_k) is a sequence of positive real numbers and (n_k) is a strictly increasing sequence of positive integers. If*

$$i) \quad \sum_{k=1}^{\infty} t_k n_k^{-4} \sum_{i=1}^{n_k} \tau(|y_i|^4) < \infty,$$

$$ii) \quad \sum_{k=1}^{\infty} t_k n_k^{-4} \sum_{i=2}^{n_k} \tau(|\bar{x}_i - \tau(\bar{x}_i)|^2) \sum_{j=1}^{i-1} \tau(|\bar{x}_j - \tau(\bar{x}_j)|^2) < \infty,$$

$$iii) \quad \sum_{k=1}^{\infty} t_k n_k^{-4} \left(\sum_{i=1}^{n_k} \tau(\bar{x}_i) \right)^4 < \infty,$$

$$iv) \quad \sum_{k=1}^{\infty} t_k \sum_{i=1}^{n_k} \tau\left(e_{[n_k,\infty)}(|x_i|)\right) < \infty,$$

where

$$\bar{x}_i = x_i e_{[0,n_k)}(|x_i|), \quad 1 \leq i \leq n_k \ ; \quad y_i = \bar{x}_i - \tau(\bar{x}_i).$$

Then

$$\sum_{k=1}^{\infty} t_k \tau\left(e_{[\varepsilon,\infty)} \left(\left| \frac{1}{n_k} \sum_{i=1}^{n_k} x_i \right| \right) \right) < \infty \qquad (2.1)$$

for any given $\varepsilon > 0$.

Proof. Put

$$S_{n_k} = \sum_{i=1}^{n_k} x_i, \quad \bar{S}_{n_k} = \sum_{i=1}^{n_k} \bar{x}_i.$$

Then, using the similar technique as in Theorem 2.2 of [7], we have, for an arbitrary $\gamma > 0$

$$e_{[2\gamma,\infty)}(|S_{n_k}|) \wedge e_{[0,\gamma)}(|S_{n_k}|) \wedge \Big(\bigwedge_{i=1}^{n_k} e_{[0,n_k)}(|x_i|)\Big) = 0$$

This implies

$$e_{[2\gamma,\infty)}(|S_{n_k}|) \leq e_{[\gamma,\infty)}\left(|\bar{S}_{n_k}|\right) \vee \Big(\bigvee_{i=1}^{n_k} e_{[n_k,\infty)}(|x_i|)\Big).$$

From the positivity of trace and Tchebyshev's inequality with $g(x) = x^4$ we obtain

$$\begin{aligned}\tau\left(e_{[2\gamma,\infty)}(|S_{n_k}|)\right) &\leq \tau\left(e_{[\gamma,\infty)}\left(|\bar{S}_{n_k}|\right)\right) + \sum_{i=1}^{n_k} \tau\left(e_{[n_k,\infty)}(|x_{n_k}|)\right) \\ &\leq \gamma^{-4}\tau(|S_{n_k}|^4) + \sum_{i=1}^{n_k} \tau\left(e_{[n_k,\infty)}(|x_i|)\right).\end{aligned}$$

Using (2.3) and Lemma 2.1 we can get the following estimate

$$\begin{aligned}\tau\big(e_{[2\gamma,\infty)}(|S_{n_k}|)\big) &\leq \gamma^{-4}\tau\Big(\big|(\bar{S}_{n_k} - \tau(\bar{S}_{n_k})) + \tau(\bar{S}_{n_k})\big|^4\Big) + \sum_{i=1}^{n_k} \tau\big(e_{[n_k,\infty)}(|x_i|)\big) \\ &\leq 2^3\gamma^{-4}\left[\tau(|\bar{S}_{n_k} - \tau(\bar{S}_{n_k})|^4) + (\tau(\bar{S}_{n_k}))^4\right] + \sum_{i=1}^{n_k} \tau\big(e_{[n_k,\infty)}(|x_i|)\big). \end{aligned} \tag{2.4}$$

On the other hand, using the equality $\tau(y_i y_j) = \tau(y_j y_i) \quad \forall i,j \in N$ and the successive independence of the sequence (x_n) we obtain

$$\begin{aligned}\tau(|\bar{S}_{n_k} - \tau(\bar{S}_{n_k})|^4) &= \tau\left(\left|\sum_{i=1}^{n_k} y_i\right|^4\right) = \sum_{i=1}^{n_k} \tau(|y_i|^4) + 6\sum_{i=2}^{n_k} \tau(|y_i|^2) \sum_{j=1}^{i-1} \tau(|y_j|^2) \\ &+ 12\sum_{i=1}^{n_k} \tau(|y_i|^2) \sum_{\substack{j=2 \\ j\neq i}}^{n_k} \tau(y_j) \sum_{\substack{k=1 \\ k\neq i}}^{j-1} \tau(y_k) + 4\sum_{i=2}^{n_k} \tau(y_i) \sum_{j=1}^{i-1} \tau(y_i^3) \\ &+ 4\sum_{i=2}^{n_k} \tau(y_i^3) \sum_{j=1}^{i-1} \tau(y_j) + 24\sum_{i=4}^{n_k} \tau(y_i) \sum_{j=3}^{i-1} \tau(y_j) \sum_{l=2}^{j-1} \tau(y_l) \sum_{s=1}^{k-1} \tau(y_s) \\ &= \sum_{i=1}^{n_k} \tau(|y_i|^4) + 6\sum_{i=2}^{n_k} \tau(|y_i|^2) \sum_{j=1}^{i-1} \tau(|y_j|^2).\end{aligned}$$

Now, for given $\varepsilon > 0$, we put $\gamma = n_k\varepsilon/2$. Then from (2.4) and (2.5) we get

$$\tau\left(e_{[\varepsilon,\infty)}\left|\frac{1}{n_k}\sum_{i=1}^{n_k} x_i\right|\right) = \tau\left(e_{[n_k\varepsilon,\infty)}(|S_{n_k}|)\right)$$
$$\leq 16\varepsilon^{-4}n_k^{-4}\tau(|\bar{S}_{n_k}|^4) + \sum_{i=1}^{n_k}\tau\left(e_{[n_k,\infty)}(|x_i|)\right)$$
$$\leq 2^7\varepsilon^{-4}n_k^{-4}\left[\tau(|\bar{S}_{n_k} - (\bar{S}_{n_k})^4|) + (\tau(S_{n_k}))^4\right] + \sum_{i=1}^{n_k}\tau\left(e_{[n_k,\infty)}(|x_i|)\right)$$
$$= 2^7\varepsilon^{-4}n_k^{-4}\left[\sum_{i=1}^{n_k}\tau(|y_i|^4) + 6\sum_{i=2}^{n_k}\tau(|y_i|^2)\sum_{j=1}^{i-1}\tau(|y_j|^2) + \tau(\bar{S}_{n_k})^4\right]$$
$$+ \sum_{i=1}^{n_k}\tau\left(e_{[n_k,\infty)}(|x_i|)\right) \tag{2.6}$$

which proves (2.1) and completes the proof.

For successively independent sequences of self-adjoint identically distributed operators, we get the following

Corollary 2.4 ([3]). *Let (x_n) be successively independent sequence of self-adjoint identically distributed elements of $\tilde{A}$ with $\tau(x_1) = 0$ and $\tau(|x_1|^t) < \infty$ for some $t : 1 < t < 2$. Then*

$$\sum_{k=1}^{\infty} k^{t-2}\tau\left(e_{[\varepsilon,\infty)}\left(\left|\frac{1}{k}\sum_{i=1}^{k} x_i\right|\right)\right) < \infty$$

for any given $\varepsilon > 0$.

Proof. In Theorem 2.3, put $t_k = k^{t-2}$, $n_k = k$. It is clear that the conditions (i)-(iv) are satisfied and the proof is complete.

3. HSU - ROBBINS LAW OF LARGE NUMBERS FOR 2-DIMENSIONAL ARRAYS

The main result of this section is the following theorem

Theorem 3.1. *Let $(x_{m,n}, (m,n) \in N^2)$ be a strongly independent double sequence of self-adjoint elements of $\tilde{A}$ with $\tau(x_{m,n}) = 0$, $\forall(m,n) \in N^2$. Suppose* *that* $(t_{k,l},\quad (k,l)\in N^2)$ *is a double sequence of positive real numbers and let* (m_k) (n_l) *be strictly increasing sequences of positive integers. If*

$$i)\ \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} t_{k,l}(m_k n_l)^{-4}\left(\sum_{i=1}^{m_k}\sum_{j=1}^{n_k}\tau(|y_{i,j}|^4)\right)<\infty,$$

$$ii)\ \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} t_{k,l}(m_k n_l)^{-4}\left[\sum_{i=1}^{m_k}\left(\sum_{j=2}^{n_k}\tau(|\bar{x}_{i,j}-\tau(\bar{x}_{i,j})|^2)\sum_{v=1}^{j-1}\tau(|\bar{x}_{i,v}-\tau(\bar{x}_{i,v})|^2)\right)\right.$$
$$\left.+\sum_{i=2}^{m_k}\left(\sum_{j=1}^{n_k}\tau(|\bar{x}_{i,j}-\tau(\bar{x}_{i,j})|^2)\right)\sum_{u=1}^{i-1}\left(\sum_{v=1}^{n_k}\tau(|\bar{x}_{u,v}-\tau(\bar{x}_{u,v})|^2)\right)\right]<\infty,$$

$$iii)\ \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} t_{k,l}(m_k n_l)^{-4}\left(\sum_{i=1}^{m_k}\sum_{j=1}^{n_k}\tau(x_{i,j})\right)^4<\infty,$$

$$iv)\ \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} t_{k,l}\sum_{i=1}^{m_k}\sum_{j=1}^{n_l}\tau\left(e_{[m_k n_l,\infty)}(|x_{i,j}|)\right)<\infty,$$

where

$$\bar{x}_{i,j}=x_{i,j}e_{[0,m_k n_l)},$$
$$y_{i,j}=\bar{x}_{i,j}-\tau(\bar{x}_{i,j}).$$

Then, for any given $\varepsilon>0$,

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} t_{k,l}\,\tau\left(e_{[\varepsilon,\infty)}\left(\left|\frac{1}{m_k n_l}\sum_{i=1}^{m_k}\sum_{j=1}^{n_k}x_{i,j}\right|\right)\right)<\infty.$$

Proof. Put

$$S_{k,l}=\sum_{i=1}^{k}\sum_{j=1}^{l}x_{i,j}\,;\quad \bar{S}_{k,l}=\sum_{i=1}^{k}\sum_{j=1}^{l}\bar{x}_{i,j}.$$

Then by the same way as in Theorem 2.3 we obtain

$$\tau(e_{[2\varepsilon,\infty)}(|S_{m_k,n_l}|))\le 2^3\gamma^{-4}\left[\tau(|\bar{S}_{m_k,n_l}-\tau(\bar{S}_{m_k,n_l})|^4)+(\tau(\bar{S}_{m_k,n_l}))^4\right]$$
$$+\sum_{i=1}^{m_k}\sum_{j=1}^{n_l}\tau(e_{[m_k n_l,\infty)}(|x_{i,j}|)). \qquad (3.1)$$

On the other hand, using the equality $\tau(y_i y_j) = \tau(y_j y_i)$ and the strong independence of $(x_{m,n})$ we get

$$
\begin{aligned}
\tau(|\bar{S}_{m_k,n_l} - \tau(S_{m_k,n_l})|^4) &= \tau\left(\left|\sum_{i=1}^{m_k}\sum_{j=1}^{n_l} y_{i,j}\right|^4\right) \\
&= \sum_{i=1}^{m_k}\left(\sum_{j=1}^{n_l}\tau(|y_{i,j}|^4)\right) + 6\sum_{i=1}^{m_k}\left(\sum_{j=2}^{n_l}\tau(|y_{i,j}|^2)\sum_{u=1}^{j-1}\tau(|y_{i,u}|^2)\right) \\
&+ 12\sum_{i=1}^{m_k}\left(\sum_{j=1}^{m_l}\tau(|y_{i,j}|^2)\sum_{\substack{u=2\\u\neq j}}^{n_l}\tau(y_{i,u})\sum_{\substack{v=1\\v\neq j}}^{u-1}\tau(y_{i,v})\right) + 4\sum_{i=1}^{m_k}\left(\sum_{j=2}^{n_l}\tau(y_{i,j})\sum_{u=1}^{j-1}\tau(y_{i,j}^3)\right) \\
&+ 4\sum_{i=1}^{m_k}\left(\sum_{j=1}^{n_k}\tau(y_{i,j}^3)\sum_{u=1}^{j-1}\tau(y_{i,u})\right) \\
&+ 24\sum_{i=1}^{m_k}\left(\sum_{j=4}^{n_l}\tau(y_{i,j})\sum_{u=3}^{j-1}\tau(y_{i,u})\sum_{v=2}^{u-1}\tau(y_{i,v})\sum_{s=1}^{v-1}\tau(y_{i,s})\right) \\
&+ 6\sum_{i=2}^{m_k}\left(\sum_{j=1}^{n_l}\tau(|y_{i,j}|^2)\right)\sum_{u=1}^{i-1}\left(\sum_{v=1}^{r}\tau(|y_{u,v}|^2)\right) \\
&+ 12\sum_{i=2}^{m_k}\left(\sum_{j=1}^{n_l}\tau(|y_{i,j}|^2)\right)\sum_{u=1}^{i-1}\left(\sum_{v=2}^{n_k}\tau(y_{u,v})\sum_{r=1}^{v-1}\tau(y_{u,r})\right) \\
&+ 12\sum_{i=2}^{m_k}\left(\sum_{j=2}^{n_l}\tau(y_{i,j})\sum_{s=1}^{j-1}\tau(y_{i,s})\right)\sum_{u=1}^{i-1}\left(\sum_{v=1}^{n_l}\tau(|y_{u,v}|^2)\right) \\
&+ 24\sum_{i=2}^{m_k}\left(\sum_{j=2}^{n_l}\tau(y_{i,j})\sum_{s=1}^{j-1}\tau(y_{i,s})\right)\sum_{u=1}^{i-1}\left(\sum_{v=2}^{n_l}\tau(y_{u,v})\sum_{r=1}^{v-1}\tau(y_{u,r})\right) \\
&+ 12\sum_{i=1}^{m_k}\left(\sum_{j=1}^{n_l}\tau(y_{i,j})\right)^2\sum_{\substack{u=2\\u\neq i}}^{m_k}\left(\sum_{i=1}^{n_l}\tau(y_{u,j})\right)\sum_{\substack{v=1\\v\neq i}}^{u-1}\left(\sum_{j=1}^{n_l}\tau(y_{v,j})\right) \\
&+ 4\sum_{i=2}^{m_k}\left(\sum_{i=1}^{n_l}\tau(y_{i,j})\right)\sum_{u=1}^{i-1}\left(\sum_{j=1}^{n_l}\tau(y_{u,j})\right)^3 \\
&+ 4\sum_{i=2}^{m_k}\left(\sum_{j=1}^{n_l}\tau(y_{i,j})\right)^3\sum_{u=1}^{i-1}\left(\sum_{j=1}^{n_l}\tau(y_{u,j})\right)
\end{aligned}
$$

$$+24\sum_{i=4}^{m_k}\Big(\sum_{j=1}^{n_l}\tau(y_{i,j})\Big)\sum_{u=3}^{i-1}\Big(\sum_{j=1}^{n_l}\tau(y_{u,j})\Big)\sum_{v=2}^{u-1}\Big(\sum_{j=2}^{n_l}\tau(y_{v,j})\Big)\sum_{s=1}^{v-1}\Big(\sum_{j=1}^{n_l}\tau(y_{s,j})\Big)$$

$$=\sum_{i=1}^{m_k}\sum_{j=1}^{n_l}\tau(|y_{i,j}|^4)+6\sum_{i=1}^{m_k}\Big(\sum_{j=2}^{n_l}\tau(|y_{i,j}|^2)\sum_{v=1}^{j-1}\tau(|y_{u,v}|^2)\Big)$$

$$+6\sum_{i=2}^{m_k}\Big(\sum_{j=1}^{n_l}\tau(|y_{i,j}|^2)\Big)\sum_{u=1}^{i-1}\Big(\sum_{v=1}^{n_l}\tau(|y_{u,v}|^2)\Big). \tag{3.2}$$

Now, for a given $\varepsilon > 0$, we put $\gamma = m_k n_l \varepsilon/2$ then from (3.1) and (3.2) we obtain

$$\tau\Big(e_{[\varepsilon,\infty)}\Big(\Big|\frac{1}{m_k n_l}\sum_{i=1}^{m_k}\sum_{j=1}^{n_l}x_{i,j}\Big|\Big)\Big)=\tau\left(e_{[m_k n_l\varepsilon,\infty)}(|S_{m_k,n_l}|)\right)$$

$$=2^7\varepsilon^{-4}(m_k n_l)^{-4}\Big[\sum_{i=1}^{m_k}\sum_{j=1}^{n_l}\tau(|y_{i,j}|^4)+6\sum_{i=1}^{m_k}\Big(\sum_{j=2}^{n_l}\tau(|y_{i,j}|^2)\sum_{v=1}^{j-1}\tau(|y_{i,v}|^2)\Big)$$

$$+6\sum_{i=2}^{m_k}\sum_{j=1}^{n_l}\tau(|y_{i,j}|^2)\sum_{u=1}^{i-1}\Big(\sum_{v=1}^{n_l}\tau(|y_{u,v}|^2)+\tau(\bar{S}_{m_k,n_l})\Big)^4\Big]$$

$$+\sum_{i=1}^{m_k}\sum_{j=1}^{n_l}\tau\left(e_{[m_k n_l,\infty)}(|x_{i,j}|)\right), \tag{3.3}$$

completing the proof.

The following result can be proved by the same techniques as in Corollary 2 of Section 3 in [5] (applying Theorem 3.1), so we omit the proof.

Corollary 3.2. *Let $(x_{m,n},\quad (m,n)\in N^2)$ be a strongly independent double sequence of self-adjoint identically distributed elements of $\tilde{A}$ with $\tau(x_{1,1}) = 0$ and $\tau(|x_{1,1}|^2\log^+|x_{1,1}|) < \infty$. Then*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\tau\Big(e_{[\varepsilon,\infty)}\Big(\Big|\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}x_{i,j}\Big|\Big)\Big)<\infty.$$

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Department of Mathematics
Pedagogical Colledge of Nghe An
Nghe An, Vietnam

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